MATHEMATICS

(w.e.f.2019 – 20)

SECOND YEAR

Intermediate Vocational Bridge Course

Paper II : MATHEMATICS

STATE INSTITUTE OF VOCATIONAL EDUCATION

BOARD OF INTERMEDIATE EDUCATION, A.P.

Text Book Development Committee

Paper-II Mathematics

AUTHOR

Dr. Gudimella V R K Sagar,
M. Sc., M. Phil, Ph.D.
Lecturer in Mathematics
Government Polytechnic, Ponnur
GUNTUR

EDITOR

Dr. Bavanari Satyanarayana
M. Sc., Ph. D
Associate Professor in Mathematics
Department of Mathematics
Acharya Nagarjuna University
Nagarjuna Nagar – 522 510

MATHEMATICS

TEXT BOOK DEVELOPMENT COMMITTEE

S.No	Name	Designation	Signature
1	**Dr. Bavanari Satyanarayana** **Associate Professor in Mathematics** **Department of Mathematics** **Acharya Nagarjuna University** **Nagarjuna Nagar – 522 510**	Editor	
2	**Dr. Gudimella V R K Sagar,** **Lecturer in Mathematics** **Government Polytechnic, Ponnur** **GUNTUR**	Author	

Smt. B.UDAYA LAKSHMI, I.A.S.

Commissioner & Secretary

Intermediate Education

ANDHRA PRADESH

GUNTUR

S.I.V.E Co – Ordinating Committee

Sri. Podili Yerraiah, M.Sc., B.Ed.
Professor
State Institute of Vocational Education
Commissioner of Intermediate Education, Guntur

Sri. B. Nageswara B.Com, B.L.,
Joint Secretary (Vocational)
Board of Intermediate Education, Vijayawada

Sri. Dr.G.V.S.R. Murthy, M.SC., Ph.D.
Lecturer
State Institute of Vocational Education
Commissioner of Intermediate Education, Guntur

DTP

Ch.V.L.R.Vijay Kumar M.Sc.,

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

Second Year - Paper – II (w.e.f. 2019-2020)

MATHEMATICS SYLLABUS

TOTAL PERIODS - 150

***

1. COMPLEX NUMBERS

Introduction:

In the earlier classes we have learnt the properties of real numbers and studied certain operations on real numbers like addition, subtraction, multiplication and division. We have also learnt solving linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2+1=0$ has no real solution since the square of every real number is non-negative. This suggests that we need to extend the real number system to a larger system, so that we can account for the solutions of the equation $x^2=-1$. If this is done, it would be help solving the equation $ax^2+bx+c=0$ for the case $b^2-4ac<0,$ which is not possible number system.

It was Euler (1707-1783) that identified a root for the quadratic equation $x^2+1=0$ with the symbol $i.$ Euler called it an imaginary root and termed the numbers of the form $a+ib,\ a,b\in \mathrm{R}$ as complex numbers. However, we define complex number in a more general way following the definitions given by Hamilton (1805-1865) as an ordered pair of real numbers (a,b) and study their representation, equality and some algebraic operations on the set of complex numbers. We shall also discuss some concepts like conjugacy, modulus and amplitude of a complex number. Finally we give a geometrical representation of a complex number and introduce the Argand plane and Argand diagram to make out the sum, difference, product and quotient of two complex numbers geometrically.

1.1 Complex numbers as an ordered pair of elementary operations:

In this section, we shall give a general definition of a complex number and introduce certain algebraic operations on the set of complex numbers to develop the algebraic structure of complex numbers.

1.1.1 Definition (Complex number):

A complex number is an ordered pair of real numbers $(a,b).$ The set of complex numbers is denoted by $\mathbb{R}=\{(a,b)|a\in\mathbb{R},\ b\in\mathbb{R}\}=\mathbb{R}\times\mathbb{R}.$

1.1.2 Definition (Equality of complex numbers):

Two complex numbers $z_1=(a,b)$ and $z_2=(c,d)$ are said to be equal if $a=c$ and $b=d.$

1.1.3 Definitions:

(i) **Addition of complex numbers**

If $z_1=(a,b)$ and $z_2=(c,d),$ we define $z_1+z_2=(a,b)+(c,d)$ to be the complex number $(a+c,b+d).$

(ii) **Negative of a complex number**

The negative of any complex number $z=(a,b)$ denoted by $-z$ is defined as $-z=(-a,-b)$.

(iii) **Difference of complex numbers**

If $z_1=(a,b)$ and $z_2=(c,d)$, we define $z_1-z_2=(a,b)-(c,d)$ to be the complex number $(a-c,b-d)$.

(iv) **Difference of complex numbers**

If $z_1=(a,b)$ and $z_2=(c,d)$, then we define their product by $z_1.z_2=(a,b).(c,d)=(ac-bd,ad+bc)$.

1.1.4 Examples:

(i) If $z_1=(2,3)$ and $z_2=(-6,5)$, then $z_1+z_2=(2-6,3+5)=(-4,8)$

(ii) If $z_1=(6,3)$ and $z_2=(2,-1)$, then $z_1-z_2=(6-2,3-(-1))=(4,4)$

(iii) If $z_1=(1,2)$ and $z_2=(3,-4)$, then $z_1.z_2=((1)(3)-(2)(-4),(1)(-4)+(2)(3))$

$$=(3+8,-4+6)=(11,2)$$

1.1.5 Note: From the definition of addition of two complex numbers, it is clear that $\alpha\in\mathbb{C},\beta\in\mathbb{C}\Rightarrow\alpha+\beta\in\mathbb{C}$. In a similar manner $\alpha.\beta\in\mathbb{C}$. Hence the operations addition and multiplication are both binary operations on $\mathbb{C}$. We denote $\alpha.\beta$ by $\alpha\beta$ also.

We now prove the following laws of addition on $\mathbb{C}$.

1.1.6 Theorem:

(i) In $\mathbb{C}$, addition is associative $i.e., \alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$, for all $\alpha,\beta,\gamma\in\mathbb{C}$

(ii) In $\mathbb{C}$, additive identity exists and is unique $i.e.,$ there exists unique $\tau\in\mathbb{C}$ such that $\alpha+\tau=\tau+\alpha=\alpha$, for all $\alpha\in\mathbb{C}$. This element τ is called the additive identity.

(iii) In $\mathbb{C}$, additive inverse exists and is unique $i.e.,$ there exists unique $\alpha'\in\mathbb{C}$ such that $\alpha+\alpha'=\alpha'+\alpha=\tau$, for all $\alpha\in\mathbb{C}$. This element α' is called the additive inverse of α.

(iv) In $\mathbb{C}$, addition is commutative $i.e.,$ $\alpha+\beta=\beta+\alpha$, for all $\alpha,\beta\in\mathbb{C}$.

Proof: (i) Let $\alpha=(a,b),\beta=(c,d),\gamma=(e,f)\in\mathbb{C}$, where $a,b,c,d,e,f\in\mathbb{R}$.

Now $(\alpha+\beta)+\gamma=\big((a,b)+(c,d)\big)+(e,f)=(a+c,b+d)+(e,f)$

$$=\big((a+c)+e,(b+d)+f\big)=\big((a+(c+e),b+(d+f)\big)$$

$$= (a,b) + (c+e, d+f) = (a,b) + \big((c,d) + (e,f)\big) = \alpha + (\beta + \gamma)$$

i.e., In $\mathbb{C}$, addition is associative

(ii) Let $\alpha = (a,b), \tau = (0,0) \in \mathbb{C}$, where $0, a, b \in \mathbb{R}$.

Now $\alpha + \tau = (a,b) + (0,0) = (a+0, b+0) = (a,b) = \alpha$

Also $\tau + \alpha = (0,0) + (a,b) = (0+a, 0+b) = (a,b) = \alpha$

i.e., there exists unique $\tau \in \mathbb{C}$ such that $\alpha + \tau = \tau + \alpha = \alpha$, for all $\alpha \in \mathbb{C}$.

In $\mathbb{C}$, additive identity exists and is unique.

(iii) Let $\alpha = (a,b), \alpha' = (-a,-b) \in \mathbb{C}$, where $a, b, -a, -b \in \mathbb{R}$.

Now $\alpha + \alpha' = (a,b) + (-a,-b) = (a-a, b-b) = (0,0) = \tau$

Also $\alpha' + \alpha = (-a,-b) + (a,b) = (-a+a, -b+b) = (0,0) = \tau$

i.e., there exists unique $\alpha' \in \mathbb{C}$ such that $\alpha + \alpha' = \alpha' + \alpha = \tau$, for all $\alpha \in \mathbb{C}$.

In $\mathbb{C}$, additive inverse of α exists and is unique.

(iv) Let $\alpha = (a,b), \beta = (c,d) \in \mathbb{C}$, where $a, b, c, d \in \mathbb{R}$.

Now $\alpha + \beta = (a,b) + (c,d) = (a+c, b+d)$

$$= (c+a, d+b) = (c,d) + (a,b) = \beta + \alpha.$$

i.e., In $\mathbb{C}$, addition is commutative.

1.1.7 Note: The additive identity in $\mathbb{C}$ is denoted by 0. The additive inverse of $\alpha \in \mathbb{C}$ is denoted by $-\alpha$. We shall now establish the following theorem with multiplication as binary operation.

1.1.8 Theorem:

(i) In $\mathbb{C}$, multiplication is associative *i.e.,* $\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma$, for all $\alpha, \beta, \gamma \in \mathbb{C}$

(ii) In $\mathbb{C}$, multiplicative identity exists and is unique *i.e.,* there exists unique $\tau \in \mathbb{C}$ such that $\alpha.\tau = \tau.\alpha = \alpha$, for all $\alpha \in \mathbb{C}$. This element τ is called the multiplicative identity.

(iii) In $\mathbb{C}$, multiplicative inverse exists and is unique *i.e.,* there exists unique $\alpha' \in \mathbb{C}$ such that $\alpha.\alpha' = \alpha'.\alpha = \tau$, for all $\alpha \in \mathbb{C}$. This element α' is called the multiplicative inverse of α.

(iv) In $\mathbb{C}$, multiplication is commutative *i.e.,* $\alpha.\beta = \beta.\alpha$, for all $\alpha, \beta \in \mathbb{C}$.

(v) In $\mathbb{C}$, distributive laws holds *i.e.*, $\alpha.(\beta+\gamma)=\alpha.\beta+\alpha.\gamma$, $(\alpha+\beta).\gamma=\alpha.\gamma+\beta.\gamma$, for all $\alpha,\beta,\gamma\in\mathbb{C}$.

Proof: (i) Let $\alpha=(a,b),\beta=(c,d),\gamma=(e,f)\in\mathbb{C}$, where $a,b,c,d,e,f\in\mathbb{R}$.

Now $(\alpha.\beta).\gamma=\big((a,b).(c,d)\big).(e,f)=(ac-bd,ad+bc).(e,f)$

$$=\big((ac-bd)e-(ad+bc)f,(ac-bd)f+(ad+bc)e\big)$$

$$=\big(a(ce-df)-b(cf+de),a(cf+de)+b(ce-df)\big)$$

$$=(a,b).\big((c,d).(e,f)\big)=\alpha.(\beta.\gamma)$$

i.e., In $\mathbb{C}$, multiplication is associative

(ii) Let $\alpha=(a,b),\tau=(1,0)\in\mathbb{C}$, where $0,a,b\in\mathbb{R}$.

Now $\alpha.\tau=(a,b)+(1,0)=(a.1-b.0,a.0+b.1)=(a,b)=\alpha$

Also $\tau.\alpha=(1,0)+(a,b)=(1.a-0.b,1.b+0.a)=(a,b)=\alpha$

i.e., there exists unique $\tau\in\mathbb{C}$ such that $\alpha.\tau=\tau.\alpha=\alpha$, for all $\alpha\in\mathbb{C}$.

In $\mathbb{C}$, multiplicative identity exists and is unique.

(iii) Let $\alpha=(a,b),\alpha'=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)\in\mathbb{C}$, where $a,b,\in\mathbb{R}$.

Now $\alpha.\alpha'=(a,b).\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=\left(\frac{a.a-b(-b)}{a^2+b^2},\frac{a(-b)+b(a)}{a^2+b^2}\right)=(1,0)=\tau$

Also $\alpha.\alpha'=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right).(a,b)=\left(\frac{a.a-b(-b)}{a^2+b^2},\frac{a(b)+(-b)a}{a^2+b^2}\right)=(1,0)=\tau$

i.e., there exists unique $\alpha'\in\mathbb{C}$ such that $\alpha.\alpha'=\alpha'.\alpha=\tau$, for all $\alpha\in\mathbb{C}$.

In $\mathbb{C}$, multiplicative inverse of α exists and is unique.

(iv) Let $\alpha=(a,b),\beta=(c,d)\in\mathbb{C}$, where $a,b,c,d\in\mathbb{R}$.

Now $\alpha.\beta=(a,b).(c,d)=(ac-bd,ad+bc)=(ca-db,da+cb)$

$$=(c,d).(a,b)=\beta.\alpha$$

i.e., In $\mathbb{C}$, multiplication is commutative.

(v) Let $\alpha=(a,b),\beta=(c,d),\gamma=(e,f)\in\mathbb{C}$, where $a,b,c,d,e,f\in\mathbb{R}$.

Now $\alpha.(\beta+\gamma)=(a,b).\big((c,d)+(e,f)\big)=(a,b).(c+e,d+f)$

$$=\big(a(c+e)-b(d+f),a(d+f)+b(c+e)\big)$$

$$=\big(ac+ae-bd-bf,ad+af+bc+be\big)$$

$$=\big(ac-bd,ad+bc\big)+\big(ae-bf,af+be\big)$$

$$=(a,b).(c,d)+(a,b).(e,f)=\alpha.\beta+\alpha.\gamma$$

Similarly we can prove $(\alpha+\beta).\gamma=\alpha.\gamma+\beta.\gamma$

i.e., In C, distributive laws holds

1.1.9 Note: From theorems 1.1.6 and 1.1.8 it follows that the set of all complex numbers is a commutative ring with unity under the operations of addition and multiplication. This concept will be dealt with in higher classes.

Now we can define the concept of division by non-zero complex numbers. We establish through the following theorem.

1.1.10 Theorem:

If $\alpha,\beta\in\mathbb{C}$ and $\beta\neq(0,0)$ then there exists unique $z\in\mathbb{C}$ such that $\alpha=\beta z.$

Proof: Let $\alpha=(a,b),\beta=(c,d)\in\mathbb{C},$ where $a,b,c,d\in\mathbb{R}.$

Since $\beta\neq(0,0),$ we have either $c\neq0$ or $d\neq0$ and hence $c^2+d^2\neq0.$

Let $z=\left(\dfrac{ac+bd}{c^2+d^2},\dfrac{bc-ad}{c^2+d^2}\right).$ Then $z\in\mathbb{C}.$

Now $\beta.z=(c,d).\left(\dfrac{ac+bd}{c^2+d^2},\dfrac{bc-ad}{c^2+d^2}\right)=\left(\dfrac{ac^2+bcd-bcd+ad^2}{c^2+d^2},\dfrac{bc^2-adc+adc+bd^2}{c^2+d^2}\right)$

$$=\left(\frac{ac^2+ad^2}{c^2+d^2},\frac{bc^2+bd^2}{c^2+d^2}\right)=(a,b)=\alpha.$$

Let $(x,y)\in\mathbb{C}$ such that $\alpha=\beta.(x,y)$

Then $a=cx-dy$ and $b=dx+cy$

Hence $x=\dfrac{ac+bd}{c^2+d^2}$ and $y=\dfrac{bc-ad}{c^2+d^2}.$

Hence $(x,y)=z.$

1.1.11 Note: If $\beta \neq (0,0)$, then the unique element $z \in \mathbb{C}$ such that $\beta z = \eta$, η being the multiplicative identity $(1,0)$ in $\mathbb{C}$, is called the multiplicative inverse of β. it is denoted by β^{-1}. Note that if $\beta = (a,b) \neq (0,0)$, then $\beta^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right)$.

1.1.12 Definition: For any $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \beta \neq (0,0)$, the unique $z \in \mathbb{C}$ satisfying $\alpha = \beta z$ denoted α / β. From this we can define the division of $\alpha = (a,b)$ by $\beta = (c,d)$ as $z = \frac{\alpha}{\beta} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2} \right)$.

1.1.13 Example: If $\alpha = (2,5), \beta = (-1,4)$ then $z = \frac{\alpha}{\beta} = \left(\frac{-2+20}{1+16}, \frac{-5-8}{1+16} \right) = \left(\frac{18}{17}, -\frac{13}{17} \right)$.

1.1.14 Definition: Let $z \in \mathbb{C}$, we take $z^1 = z$. We define z^n inductively for any $z \neq 0$ as follows: $\alpha = (a,b)$ by $\beta = (c,d)$ as $z^n = \begin{cases} z & if \ n=1 \\ (0,1) & if \ n=0 \\ z^{n-1}.z & if \ n>0 \\ \left(z^{-1}\right)^{-n} & if \ n<0 \end{cases}$

1.2 Expressing the complex numbers in the form $a+ib$:

Let us recall by a complex number we mean an ordered pair (a,b) of real numbers. For any real number a, $(a,0)$ is a complex number. Also $(a,0)+(b,0)=(a+b,0)$ and $(a,0).(b,0)=(ab,0)$ for any $a,b \in \mathbb{R}$.

This situation helps us in identifying a real number a with the complex number $(a,0)$. With this identification, addition and multiplication of real numbers are same as those of the corresponding complex numbers. For simplicity, we denote $(a,0)$ by a and the complex number $(0,1)$ by i.

Then, we find that $i.i = (0,1).(0,1) = (-1,0) = -1$

Also for any complex number (a,b), we have

$(a,b) = (a,0)+(0,b) = (a,0)+\left[(b,0).(0,1)\right] = a+ib$

Under the above identification, the additive identity $0=(0,0)$ in $\mathbb{C}$ coincides with the additive identity 0 in $\mathbb{R}$. Similarly, the multiplicative identity $1=(1,0)$ in $\mathbb{C}$ coincides with the multiplicative identity 1 in $\mathbb{R}$.

Hence any complex number (a,b) can be represented as $a+ib$ or $a+bi$ where a and b are real numbers and i is a complex number such that $i^2=-1$.(that is, i is a root of the equation $x^2+1=0$)

1.2.1 Real and imaginary parts of a complex number z :

Let $z = a+ib; a,b \in \mathbb{R}$, then a is called a real part of z and b is called an imaginary part of z. These are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively. When $\mathrm{Re}(z)=0$, $\mathrm{Im}(z) \neq 0$, z is called purely imaginary.

Note that both the real and imaginary parts of a complex number z are real numbers. Also with this representation of z in terms of real and imaginary parts, the addition and multiplication multiplicative of complex numbers can be carried in the usual way, with the condition that $i^2 = -1$. Thus

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi).(c+di)=(ac-bd)+(ad+bc)i$$

Observe that $a+bi=c+di$ if and only if $a=c$ and $b=d$ *i.e.,* two complex numbers z and z' are equal if and only if $\text{Re}(z)=\text{Re}(z')$ and $\text{Im}(z)=\text{Im}(z')$.

1.2.2 Solved Problems:

1. Problem: Express $\frac{4+2i}{1-2i}+\frac{3+4i}{2+3i}$ in the form $a+bi,\ a,b\in\mathbb{R}$.

Solution: $\frac{4+2i}{1-2i}+\frac{3+4i}{2+3i}=\frac{(4+2i)(1+2i)}{(1-2i)(1+2i)}+\frac{(3+4i)(2-3i)}{(2+3i)(2-3i)}$

$$=\frac{(4-4)+i(2+8)}{1^2-(2i)^2}+\frac{(6+12)+i(8-9)}{2^2-(3i)^2}$$

$$=\frac{10i}{5}+\frac{18-i}{13}=\frac{130i+90-5i}{65}$$

$$=\frac{90+125i}{65}=\frac{18+25i}{13}=\frac{18}{13}+\frac{25}{13}i.$$

2. Problem: Express $(1-i)^3(1+i)$ in the form of $a+bi$.

Solution: $(1-i)^3(1+i)=(1-i)^2(1-i)(1+i)=(1+i^2-2i)(1^2-i^2)$

$$=(1-1-2i)(1+1)=(-2i)(2)=-4i=0+i(-4).$$

3. Problem: Find the real and imaginary parts of the complex number $\frac{a+ib}{a-ib}$.

Solution: $\frac{a+ib}{a-ib}=\frac{(a+ib)(a+ib)}{(a-ib)(a+ib)}=\frac{(a+ib)^2}{a^2+b^2}=\frac{a^2-b^2+2iab}{a^2+b^2}$

$$=\frac{a^2-b^2}{a^2+b^2}+i\frac{2ab}{a^2+b^2}$$

$\therefore$ Real part $=\frac{a^2-b^2}{a^2+b^2}$, imaginary part$=\frac{2ab}{a^2+b^2}$.

4. Problem: Find the multiplicative inverse of $7+24i$

Solution: Since $(x+iy)\left(\dfrac{x-iy}{x^2+y^2}\right)=1,$ it follows that the multiplicative inverse of $x+iy$ is $\dfrac{x-iy}{x^2+y^2}$. Hence the multiplicative inverse of $7+24i$ is $\dfrac{7-24i}{7^2+24^2}=\dfrac{7-24i}{49+576}=\dfrac{7-24i}{625}$.

1.2.3 Conjugate of a complex number:

We have already defined a complex number and art of and defined some fundamental operations on complex numbers. We have also represented a complex number z in the form $a+ib$ and called $a,$ the real part and $b,$ the imaginary part of $z.$

In this section, we shall introduce the concepts of the conjugate of a complex number $z,$ the square root of a complex number.

1.2.4 Definition: For any complex number $z=a+ib,$ we define the conjugate of z as $a+i(-b)$ and denoted by $\bar{z}$ *i.e.*, $\bar{z}=a-ib.$

1.2.5 Note: (i) $\overline{a+ib}=a-ib.$

(ii) For $z\in\mathbb{C},$ $\dfrac{z+\bar{z}}{2}$ is the real part of z and $\dfrac{z-\bar{z}}{2i}$ is the imaginary part of $z.$

This means that if $z=a+ib,$ then $a=\dfrac{z+\bar{z}}{2}$ and $b=\dfrac{z-\bar{z}}{2i}.$

(iii) If $z=(a,0),$ then $\bar{z}=\overline{(a,0)}=\overline{(a+i0)}=\overline{(a,0)}=z.$

Hence z is the real number $\Leftrightarrow \bar{z}=z\Leftrightarrow \operatorname{Im}(z)=0.$

(iv) If $z=(0,b),$ then $\bar{z}=\overline{(0,b)}=\overline{(0+ib)}=0-ib=(0,-b)=-z.$

From this we have $\bar{z}=-z\Leftrightarrow \operatorname{Re}(z)=0.$ In particular $\bar{i}=-i.$

1.2.6 Example:

(i) If $z=2+5i,$ then $\bar{z}=\overline{2+5i}=2-5i.$

(ii) If $z_1=15+8i, z_2=7-20i$ and $z=z_1-z_2,$ then $z=(15+8i)-(7-20i)=8+28i$

and $\bar{z}=8-28i.$

1.2.7 Theorem: If $\alpha,\beta\in C$ then the following results hold.

(i) $\overline{\alpha+\beta}=\bar{\alpha}+\bar{\beta}$ (ii) $\overline{\alpha.\beta}=\bar{\alpha}.\bar{\beta}$

(iii) $\bar{\bar{\alpha}}=\alpha$ (iii) If $\beta\neq(0,0),\overline{(\alpha/\beta)}=\bar{\alpha}/\bar{\beta}$

Proof: (i) Let us take $\alpha = a+ib, \beta = c+id \in \mathbb{C},$ where $a,b,c,d \in \mathbb{R}.$

Then $\alpha+\beta = (a+ib)+(c+id) = (a+c)+i(b+d)$

$\overline{\alpha+\beta} = \overline{(a+c)+i(b+d)} = (a+c)-i(b+d) = (a-ib)+(c-id) = \overline{\alpha}+\overline{\beta}$

(ii) We have $\alpha\beta = (a+ib)(c+id) = (ac-bd)+i(ad+bc)$

$\overline{\alpha\beta} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd)-i(ad+bc) = (a-ib)(c-id) = \overline{\alpha}\,\overline{\beta}$

(iii) $\overline{\overline{\alpha}} = \overline{\overline{(a+ib)}} = \overline{(a-ib)} = a+ib = \alpha$

(iv) Let $\alpha/\beta = \gamma.$ Then $\alpha = \beta\gamma,$ so that

$\overline{\alpha} = \overline{\beta\gamma} = \overline{\beta}\,\overline{\gamma},$ (by (ii))

$\therefore \overline{\gamma} = \overline{\alpha}/\overline{\beta},$ (since $\beta = 0$ and $\overline{\beta} = 0$).

$\therefore \overline{(\alpha/\beta)} = \overline{\alpha}/\overline{\beta}$

1.2.8 Square root of a complex number:

Let $z \in \mathbb{C}.$ If $w \in \mathbb{C}$ is such that $w^2 = z,$ then w is called square root of $z.$ We now find the square root of $z = a+ib.$

Case (i): Suppose $b = 0.$ Then, if $a > 0, w^2 = z = a$ or $w = \pm\sqrt{a},$

if $a < 0, w^2 = z = a$ or $w = \pm i\sqrt{-a}$

Case (ii): Suppose $b \neq 0.$

Let $x+iy$ be such that $(x+iy)^2 = a+ib$

Then $a+ib = x^2-y^2+2ixy.$

On equating the real and imaginary parts on both sides of the above equation, we obtain $a = x^2-y^2, b = 2xy.$

$\therefore \left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + 4x^2y^2 = a^2+b^2,$ and $x^2+y^2 = \sqrt{a^2+b^2}.$

Since $x^2-y^2 = a,$ we have $2x^2 = a+\sqrt{a^2+b^2}$ and $2y^2 = \sqrt{a^2+b^2}-a.$

Hence $x = \pm\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}, y = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}$

Since $2xy = b,$ both x and y have the same sign $-$either both positive or both negative if $b > 0$ and x and y have opposite signs if $b < 0.$

$$\text{Hence } x+iy = \begin{cases} \pm\left[\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right] & if\ b > 0 \\ \pm\left[\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} - i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right] & if\ b < 0 \end{cases}$$

If $x+iy$ is defined as above, then it can be verified that $(x+iy)^2 = a+ib,$ so that $x+iy$ is a square root of $a+ib.$

1.2.9 Note: (i) For a positive real number $a,$ the notation $\sqrt{a}$ is used to denote the square root of $a.$ For a complex number $z,$ the notation $\sqrt{z}$ stands for any one of the square root of $z.$

(ii) If $\sqrt{a+ib} = \pm(x+iy),$ then $\sqrt{a-ib} = \pm(x-iy).$

(iii) $\sqrt{i} = \pm\dfrac{1+i}{\sqrt{2}}, \sqrt{-i} = \pm\dfrac{1-i}{\sqrt{2}}.$

(iv) $\sqrt{a+ib}+\sqrt{a-ib} = \pm\sqrt{2a+2\sqrt{a^2+b^2}}$ or $\pm\sqrt{2\sqrt{a^2+b^2}-2a}.$

1.2.10 Solved Problems:

1. Problem: Find the complex conjugate of $(3+4i)(2-3i).$

Solution: The given complex number $(3+4i)(2-3i) = 6-9i+8i+12 = 18-i.$

It's complex conjugate $=18+i.$

2. Problem: Show that $z_1 = \dfrac{2+11i}{25}, z_2 = \dfrac{-2+i}{(1-2i)^2}$ are conjugate to each other.

Solution: $\dfrac{-2+i}{(1-2i)^2} = \dfrac{-2+i}{1+4i^2-4i} = \dfrac{-2+i}{1+4(-1)-4i} = \dfrac{-2+i}{1-4-4i} = \dfrac{-2+i}{-3-4i} = \dfrac{2-i}{3+4i}$

$$= \frac{(2-i)(3-4i)}{(3+4i)(3-4i)} = \frac{(6-4)+i(-3-8)}{3^2+4^2} = \frac{2-11i}{25}.$$

Since this complex number is the conjugate of $\dfrac{2+11i}{25},$ the given complex numbers z_1, z_2 are conjugate to each other.

3. Problem: Find the square root of the complex number $-5+12i.$

Solution: From 1.2.8 we have $\sqrt{a+ib} = \pm\left[\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right]$ *if* $b>0$

Here $a=-5, b=12.$ Hence $\sqrt{-5+12i} = \pm\left[\sqrt{\dfrac{-5+\sqrt{5^2+12^2}}{2}} + i\sqrt{\dfrac{\sqrt{5^2+12^2}-(-5)}{2}}\right]$

$$= \pm\left[\sqrt{\frac{\sqrt{25+144}-5}{2}} + i\sqrt{\frac{\sqrt{25+144}+5}{2}}\right] = \pm\left[\sqrt{\frac{\sqrt{169}-5}{2}} + i\sqrt{\frac{\sqrt{169}+5}{2}}\right]$$

$$= \pm\left[\sqrt{\frac{13-5}{2}} + i\sqrt{\frac{13+5}{2}}\right] = \pm\left[\sqrt{\frac{8}{2}} + i\sqrt{\frac{18}{2}}\right] = \pm\left[\sqrt{4} + i\sqrt{9}\right] = \pm(2+3i)$$

1.3 Modulus and amplitude of a complex number:

In this section, we shall first define the modulus of a complex number. Then we shall introduce the concept of amplitude of a complex number and its principle amplitude. We shall learn how to express a given complex number in the modulus-amplitude or polar form.

1.3.1 Definition: The modulus or absolute value of a complex number $z = x + iy$ is defined to be the non-negative real number $\sqrt{x^2 + y^2}$. It is denoted by $|z|$ and is called *mod* z. Geometrically the modulus $|z| = |x + iy|$ is the distance from $(0,0)$ to the point (x, y).

1.3.2 Note: (i) $|z| \geq 0 \; \forall \; z \in \mathbb{C}$.

(ii) Let $z = (x, y) \in \mathbb{C}$. Then, $|z| = 0 \Leftrightarrow \sqrt{x^2 + y^2} = 0$

$$\Leftrightarrow x^2 + y^2 = 0 \;\; \Leftrightarrow x = y = 0 \;\; \Leftrightarrow z = 0$$

1.3.3 Example: Let $z = 3 + i$. Then, $|z| = |3 + i| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$.

1.3.4 Theorem: If $\alpha, \beta \in \mathbb{C}$, then.

(i) $|\alpha| = |\overline{\alpha}|$ $\quad$ (ii) $|\alpha|^2 = |\overline{\alpha}|^2 = \alpha\overline{\alpha}$ $\quad$ (iii) $|\alpha\beta| = |\overline{\alpha}||\overline{\beta}|$

Proof: Let $\alpha = x + iy$; $x, y \in \mathbb{R}$. Then, $\overline{\alpha} = \overline{x + iy} = x - iy = x + i(-y)$.

(i) $|\overline{\alpha}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |\alpha|$.

(ii) $\alpha\overline{\alpha} = (x + iy)(x - iy) = x^2 + y^2 = |\alpha|^2$

(iii) $|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta})$, (from above result (ii))

$= \alpha\,\beta.\overline{\alpha}\,\overline{\beta}$, (from 1.2.7$(ii)$)

$= \alpha\,\overline{\alpha}.\;\beta\overline{\beta} = |\alpha|^2\,|\beta|^2$, (from above result (ii))

$\therefore |\alpha\beta| = |\overline{\alpha}||\overline{\beta}|$

1.3.5 Theorem: If $\alpha, \beta \in \mathbb{C}$, then.

(i) $|\text{Re}\,\alpha| \leq |\alpha|; |\text{Im}\,\alpha| \leq |\alpha|$

(ii) $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{\beta})$

(iii) $|\alpha+\beta|^2 + |\alpha-\beta|^2 = 2\left(|\alpha|^2 + |\beta|^2\right)$

(iv) $|\alpha+\beta| \le |\alpha| + |\beta|$

Proof: Let $\alpha = a+ib;\ \beta = c+id$.

(i) Since $|\overline{\alpha}|^2 = a^2+b^2, a^2 \le |\overline{\alpha}|^2$

Hence $|\operatorname{Re}\alpha| = |a| \le |\alpha|$

In a similar way, we can show that $|\operatorname{Im}\alpha| \le |\alpha|$

(ii) $|\alpha+\beta|^2 = (\alpha+\beta)(\overline{\alpha+\beta})$ ($\because$ from theorem 1.3.4(ii))

$= (\alpha+\beta)(\overline{\alpha}+\overline{\beta})$ ($\because$ from theorem 1.2.7(i))

$= \alpha\overline{\alpha} + \alpha\overline{\beta} + \beta\overline{\alpha} + \beta\overline{\beta}$

$= \alpha\overline{\alpha} + \beta\overline{\beta} + \alpha\overline{\beta} + \overline{\alpha\overline{\beta}}$ ($\because$ from theorem 1.2.7(ii), (iii))

$= |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{\beta})$

(iii) $|\alpha+\beta|^2 + |\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{\beta}) + |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{(-\beta)})$

(from above result (ii))

$= |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{\beta}) + |\alpha|^2 + |\beta|^2 - 2\operatorname{Re}(\alpha\overline{\beta})$

$= 2\left(|\alpha|^2 + |\beta|^2\right)$

(iv) We have $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\operatorname{Re}(\alpha\overline{\beta})$ (from above result (ii))

$\le |\alpha|^2 + |\beta|^2 + 2|\alpha\overline{\beta}|$

$= |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta|$ ($\because$ from theorem 1.3.4(i)(iii))

$= \left(|\alpha| + |\beta|\right)^2$

Hence $|\alpha+\beta| \le |\alpha| + |\beta|$

1.3.6 Definition (Amplitude of a complex number, principle amplitude):

For any given non-zero complex number $z=(x,y)$ there exists a $\theta \in \mathrm{R}$ such that

$\cos\theta = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2+y^2}}$; $\sin\theta = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2+y^2}}$. Any real number θ satisfying the above equations is called an amplitude or argument of z. However, for a given complex number $z \neq 0$ there exists a unique θ in the interval $[-\pi,\pi]$ satisfying the above equations. We call such θ the principle amplitude or principle argument of z and denote it by $\operatorname{Arg} z$. For principle amplitude θ, $\tan\theta = \dfrac{y}{x}$ and therefore $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$, if $x>0$.

If n is an integer and θ is an amplitude of z, then $(2n\pi+\theta)$ is also an amplitude of z and is called the general value of the amplitude of z.

1.3.7 Note: The principle amplitude or principle argument of $z=(x,y)$ lies in $\left(0,\dfrac{\pi}{2}\right)$ or $\left(\dfrac{\pi}{2},\pi\right)$ or $\left(-\pi,-\dfrac{\pi}{2}\right)$ or $\left(-\dfrac{\pi}{2},0\right)$ according as the point (x,y) lies in the first quadrant or second quadrant or third quadrant or fourth quadrant and is not on the axes.

1.3.8 Definition: $r(\cos\theta + i\sin\theta), \theta \in (-\pi,\pi]$, is called the polar form or modulus amplitude form of the complex number $z=(x,y)$.

1.3.9 Example: Find the modulus and the principle amplitude of the complex number $-1-\sqrt{3}i$.

Solution: Let $-1-\sqrt{3}i = r(\cos\theta + i\sin\theta)$

Then $r\cos\theta = -1$ and $r\sin\theta = -\sqrt{3}$

$\therefore r = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$

Hence $\cos\theta = -\dfrac{1}{2}$ and $\sin\theta = -\dfrac{\sqrt{3}}{2}$

Since the point $-1-\sqrt{3}i$ lies in the third quadrant, we look for a solution of the above equation that lie in $\left[-\pi,-\dfrac{\pi}{2}\right)$. We find that $\theta = -\pi + \dfrac{\pi}{3} = -\dfrac{2\pi}{3}$ is such a solution.

$\therefore$ For the given complex number, modulus $=2$ and principle amplitude $= -\dfrac{2\pi}{3}$.

1.3.10 Note: (i) We write $(\cos\theta + i\sin\theta)$ in the simplified form as $\text{cis}\,\theta.$ You will learn $\text{cis}\,\theta = e^{i\theta}$ later. Since $i^2 = -1$ and $\sin^2\theta + \cos^2\theta = 1,$ we have $\cos^2\theta - i^2\sin^2\theta = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1.$

From this $\text{cis}\,\theta = \dfrac{1}{\cos\theta - i\sin\theta}$

$$\text{cis}(-\theta) = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{\text{cis}\,\theta}.$$

(ii) We know that $\sin\theta + i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right) = \text{cis}\left(\dfrac{\pi}{2} - \theta\right)$

1.3.11 Some operations on complex numbers in the modulus-amplitude form:

Let $z_1 = (x, y) = r_1(\cos\theta_1 + i\sin\theta_1);\, z_2 = (p, q) = r_2(\cos\theta_2 + i\sin\theta_2).$ We shall now express $z_1 z_2;\, z_1 / z_2,\, z_2 \neq 0$ in the modulus-amplitude form:

$$z_1 = (x, y) = r_1(\cos\theta_1 + i\sin\theta_1) \Rightarrow x = r_1\cos\theta_1,\, y = r_1\sin\theta_1.$$

$$z_2 = (p, q) = r_2(\cos\theta_2 + i\sin\theta_2) \Rightarrow p = r_2\cos\theta_2,\, q = r_2\sin\theta_2.$$

$$z_1 z_2 = (x, y)(p, q) = (xp - yq, xq + yp)$$

$$= (r_1\cos\theta_1.r_2\cos\theta_2 - r_1\sin\theta_1.r_2\sin\theta_2,\, r_1\cos\theta_1.r_2\sin\theta_2 + r_1\sin\theta_1.r_2\cos\theta_2)$$

$$= (r_1 r_2\cos\theta_1\cos\theta_2 - r_1 r_2\sin\theta_1\sin\theta_2,\, r_1 r_2\cos\theta_1\sin\theta_2 + r_1 r_2\sin\theta_1\cos\theta_2)$$

$$= (r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2),\, r_1 r_2(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))$$

$$= (r_1 r_2\cos(\theta_1 + \theta_2),\, r_1 r_2\sin(\theta_1 + \theta_2))$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2),\, \sin(\theta_1 + \theta_2))$$

$$= r_1 r_2\,\text{cis}(\theta_1 + \theta_2).$$

In a similar way, we can prove by mathematical induction that, when n is a positive integer, $z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \text{cis}(\theta_1 + \theta_2 + \ldots + \theta_n).$

$$\frac{z_1}{z_2} = \frac{x + iy}{p + iq} = \frac{(x, y)}{(p, q)} = \left(\frac{xp + yq}{p^2 + q^2}, \frac{yp - xq}{p^2 + q^2}\right)$$

$$= \left(\frac{r_1\cos\theta_1.r_2\cos\theta_2 + r_1\sin\theta_1.r_2\sin\theta_2}{(r_2\cos\theta_2)^2 + (r_2\sin\theta_2)^2}, \frac{r_1\sin\theta_1 r_2\cos\theta_2 - r_1\cos\theta_1 r_2\sin\theta_2}{(r_2\cos\theta_2)^2 + (r_2\sin\theta_2)^2}\right)$$

$$=\left(\frac{r_1r_2\cos(\theta_1-\theta_2)}{r_2^{\ 2}},\frac{r_1r_2\sin(\theta_1-\theta_2)}{r_2^{\ 2}}\right)=\frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2),\sin(\theta_1-\theta_2)\right)=\frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2).$$

1.3.12 Note:

(i) $\text{cis}\,\theta_1\text{cis}\,\theta_2=\text{cis}(\theta_1+\theta_2)$

(ii) $\dfrac{\text{cis}\,\theta_1}{\text{cis}\,\theta_2}=\text{cis}(\theta_1-\theta_2)$

(iii) $\text{mod}(z_1z_2)=\text{mod}\,z_1.\text{mod}\,z_2$

(iv) $\text{Arg}(z_1z_2)=\text{Arg}\,z_1+\text{Arg}\,z_2+n\pi,$ for some $n\in\{-1,0,1\}$.

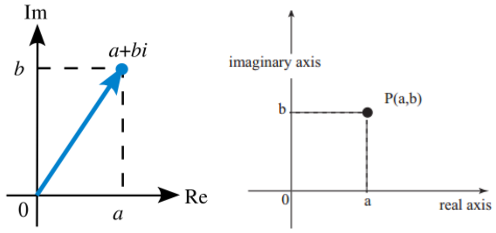
(v) $\text{Arg}(z_1/z_2)=\text{Arg}\,z_1-\text{Arg}\,z_2+n\pi,$ for some $n\in\{-1,0,1\}$.

1.4 Argand plane:

In the earlier discussion a complex number is defined as an ordered pair of real numbers and also defined the fundamental operations of addition, subtraction, multiplication and division algebraically. In this section we give a geometrical representation of a complex number.

1.4.1 Geometrical representation of a complex numbers:

Gauss was one of the mathematicians who first thought that complex numbers can be represented on a two-dimensional plane. Gauss introduced a pair of perpendicular coordinate axes and fixed a point $\mathrm{P}(x,y)$ corresponding to a given complex number (x,y). This means that the first coordinate x and the second coordinate y in the ordered pair denote the coordinates of the point corresponding to $z=x+iy$. We call this plane the complex plane or the z^- plane and the $\mathrm{X,Y}$ axes as real and imaginary axes respectively. In the discussion that follows, a point in a plane and a complex number in are used in the same sense. For example the complex number $a+ib$ denotes the point $\mathrm{P}(a,b)$ as shown in the following figure.



When complex numbers are represented in this way on a complex plane, we call that plane, the Argand plane. In the complex plane any point other than the origin denotes a vector. It is nothing but the directed line segment the origin and the given point. In particular if the point coincides with the origin it results in a null vector. From this it follows that the complex numbers can be represented by vectors in a plane.

1.4.2 Solved Problems:

1. Problem: Write $z = -1 + \sqrt{3}i$ in the polar form.

Solution: If $z = -1 + \sqrt{3}i = x + iy,$ then $x = -1$ and $y = \sqrt{3},$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

Since the given point lies in the second quadrant, we look for a solution of $\tan\theta = -\sqrt{3}$ which lies in $\left[\frac{\pi}{2}, \pi\right]$. We find that $\theta = \frac{2\pi}{3}$ is such a solution.

$$\therefore -1 + \sqrt{3}i = 2\text{cis}\frac{2\pi}{3} \text{ or } 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

2. Problem: Express $z = -2 - 2i$ in the polar form with principle value of amplitude.

Solution: If $z = -2 - 2i = x + iy,$ then $x = -2$ and $y = -2,$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{-2} = 1$$

Since θ satisfies $-\pi \le \theta < \pi,$ the value of θ satisfying the equation $\tan\theta = 1$ is $\theta = -\frac{3\pi}{4}.$

$$\therefore -2 - 2i = 2\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right) \text{ or } 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

Exercise 1(a)

1. If $Z_1 = (3,5)$ and $Z_2 = (2,6)$ find (i) $Z_1 Z_2$ (ii) $\frac{Z_1}{Z_2}$

2. If $Z_1 = (6,3)$ and $Z_2 = (2,-1)$ find (i) $Z_1 Z_2$ (ii) $\frac{Z_1}{Z_2}$

3. If $Z = (\cos\theta, \sin\theta)$ find $Z - \dfrac{1}{Z}$

4. Find the multiplicative inverse of (i) $(\cos\theta, \sin\theta)$ (ii) $(7, 24)$ (iii) $(-2,1)$

5. Express the following complex numbers in the form of $a + ib$

(i) $(2-3i)(3+4i)$ (ii) $\dfrac{a-ib}{a+ib}$ $(iii)\dfrac{4+3i}{(2+3i)(4-3i)}$ (iv) $\dfrac{2+5i}{3-2i}+\dfrac{2-5i}{3+2i}$

6. Find the conjugate of the following complex numbers

(i) $(15+3i)-(4-20i)$ (ii) $(2+5i)(-4+6i)$ $(iii)\dfrac{5i}{7+i}$

7. Find a square root of the following complex numbers

(i) $7+24i$ (ii) $-8-6i$ $(iii)-47+i8\sqrt{3}$ $(iv)-5+12i$

8. If $(a+ib)^2 = x+iy,$ find $x^2+y^2.$

9. If $(x-iy)^{1/3} = a-ib,$ then show that $\dfrac{x}{a}+\dfrac{y}{b} = 4(a^2-b^2).$

10. Express $\left(\dfrac{a+ib}{a-ib}\right)^2 - \left(\dfrac{a-ib}{a+ib}\right)^2$ in the form of $x+iy.$

11. Express the following complex numbers in modulus-amplitude form

(i) $1-i$ (ii) $1+i\sqrt{3}$ (ii) $-1-i\sqrt{3}$ $(iv)\sqrt{3}+i$

12. Express $-\sqrt{7}+i\sqrt{21}$ in polar form

13. Express $-1-i$ in polar form

Key concepts

1. If $z_1 = (a,b)$ and $z_2 = (c,d),$ we define $z_1 + z_2 = (a,b)+(c,d)$ to be the complex number $(a+c, b+d).$

2. The negative of any complex number $z = (a,b)$ denoted by $-z$ is defined as $-z = (-a, -b).$

3. If $z_1 = (a,b)$ and $z_2 = (c,d),$ we define $z_1 - z_2 = (a,b)-(c,d)$ to be the complex number $(a-c, b-d).$

4. If $z_1 = (a,b)$ and $z_2 = (c,d)$, then we define their product by
$z_1.z_2 = (a,b).(c,d) = (ac - bd, ad + bc)$.

5. In $\mathbb{C}$, addition is associative *i.e.*, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, for all $\alpha, \beta, \gamma \in \mathbb{C}$

6. In $\mathbb{C}$, additive identity exists and is unique *i.e.*, there exists unique $\tau \in \mathbb{C}$ such that $\alpha + \tau = \tau + \alpha = \alpha$, for all $\alpha \in \mathbb{C}$. This element τ is called the additive identity.

7. In $\mathbb{C}$, additive inverse exists and is unique *i.e.*, there exists unique $\alpha' \in \mathbb{C}$ such that $\alpha + \alpha' = \alpha' + \alpha = \tau$, for all $\alpha \in \mathbb{C}$. This element α' is called the additive inverse of α.

8. In $\mathbb{C}$, addition is commutative *i.e.*, $\alpha + \beta = \beta + \alpha$, for all $\alpha, \beta \in \mathbb{C}$.

9. The additive identity in $\mathbb{C}$ is denoted by 0. The additive inverse of $\alpha \in \mathbb{C}$ is denoted by $-\alpha$.

10. In $\mathbb{C}$, multiplication is associative *i.e.*, $\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma$, for all $\alpha, \beta, \gamma \in \mathbb{C}$

11. In $\mathbb{C}$, multiplicative identity exists and is unique *i.e.*, there exists unique $\tau \in \mathbb{C}$ such that $\alpha.\tau = \tau.\alpha = \alpha$, for all $\alpha \in \mathbb{C}$. This element τ is called the multiplicative identity.

12. In $\mathbb{C}$, multiplicative inverse exists and is unique *i.e.*, there exists unique $\alpha' \in \mathbb{C}$ such that $\alpha.\alpha' = \alpha'.\alpha = \tau$, for all $\alpha \in \mathbb{C}$. This element α' is called the multiplicative inverse of α.

13. In $\mathbb{C}$, multiplication is commutative *i.e.*, $\alpha.\beta = \beta.\alpha$, for all $\alpha, \beta \in \mathbb{C}$.

14. In $\mathbb{C}$, distributive laws holds *i.e.*, $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$, $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$, for all $\alpha, \beta, \gamma \in \mathbb{C}$.

15. If $\beta = (a,b) \neq (0,0)$, then $\beta^{-1} = \left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2} \right)$.

16. For any $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \beta \neq (0,0)$, the unique $z \in \mathbb{C}$ satisfying $\alpha = \beta z$ denoted α / β. From this we can define the division of $\alpha = (a,b)$ by $\beta = (c,d)$ as $z = \dfrac{\alpha}{\beta} = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2} \right)$.

17. For any complex number $z = a + ib$, we define the conjugate of z as $a + i(-b)$ and denoted by $\bar{z}$ *i.e.*, $\bar{z} = a - ib$.

18. $\overline{a + ib} = a - ib$.

19. If $z = a + ib$, then $a = \dfrac{z + \bar{z}}{2}$ and $b = \dfrac{z - \bar{z}}{2i}$.

20. If z is the real number $\Leftrightarrow \bar{z} = z \Leftrightarrow \mathrm{Im}(z) = 0.$

21. If z is purely imaginary number then $\bar{z} = -z \Leftrightarrow \mathrm{Re}(z) = 0.$ In particular $\bar{i} = -i.$

22.If $\alpha, \beta \in \mathrm{C}$ then

$(i)\ \overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta}$ $\qquad$ $(ii)\ \overline{\alpha.\beta} = \bar{\alpha}.\bar{\beta}$

$(iii)\ \bar{\bar{\alpha}} = \alpha$ $\qquad$ (iii) If $\beta \neq (0,0), \overline{(\alpha/\beta)} = \bar{\alpha}/\bar{\beta}$

23. Let $x+iy$ be such that $(x+iy)^2 = a+ib$

Then $x = \pm\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}, y = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}$

24. If $\sqrt{a+ib} = \pm(x+iy),$ then $\sqrt{a-ib} = \pm(x-iy).$

25. $\sqrt{i} = \pm\dfrac{1+i}{\sqrt{2}}, \sqrt{-i} = \pm\dfrac{1-i}{\sqrt{2}}.$

26. $\sqrt{a+ib} + \sqrt{a-ib} = \pm\sqrt{2a+2\sqrt{a^2+b^2}}$ or $\pm\sqrt{2\sqrt{a^2+b^2}-2a}.$

27. If $\sqrt{a+ib} = \pm(x+iy),$ then $\sqrt{a-ib} = \pm(x-iy).$

28. The modulus or absolute value of a complex number $z = x+iy$ is defined to be the non-negative real number $\sqrt{x^2+y^2}.$ It is denoted by $|z|$ and is called *mod* z.

29. $|z| \geq 0 \ \forall \ z \in \mathbb{C}.$

30. Let $z = (x,y) \in \mathbb{C}.$ Then, $|z| = 0 \ \Leftrightarrow z = 0$

31. If $\alpha, \beta \in \mathbb{C},$ then.

$(i)\ |\alpha| = |\bar{\alpha}|$ $\qquad$ $(ii)\ |\alpha|^2 = |\bar{\alpha}|^2 = \alpha\bar{\alpha}$ $\qquad$ $(iii)\ |\alpha\beta| = |\bar{\alpha}||\bar{\beta}|$

32. If $\alpha, \beta \in \mathbb{C},$ then.

$(i)\ |\mathrm{Re}\,\alpha| \leq |\alpha|; |\mathrm{Im}\,\alpha| \leq |\alpha|$

$(ii)\ |\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\,\mathrm{Re}(\alpha\bar{\beta})$

$(iii)\ |\alpha+\beta|^2 + |\alpha-\beta|^2 = 2\left(|\alpha|^2 + |\beta|^2\right)$

$(iv)\ |\alpha+\beta| \leq |\alpha| + |\beta|$

33. For any given non-zero complex number $z = (x,y)$ there exists a $\theta \in \mathbb{R}$ such that

$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$; $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$. Any real number θ satisfying the above equations is called an amplitude or argument of z. However, for a given complex number $z \neq 0$ there exists a unique θ in the interval $[-\pi, \pi]$ satisfying the above equations. We call such θ the principle amplitude or principle argument of z and denote it by $\operatorname{Arg} z$. For principle amplitude $\theta, \tan\theta = \frac{y}{x}$ and therefore $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, if $x > 0$.

34. If n is an integer and θ is an amplitude of z, then $(2n\pi + \theta)$ is also an amplitude of z and is called the general value of the amplitude of z.

35. $r(\cos\theta + i\sin\theta), \theta \in (-\pi, \pi]$, is called the polar form or modulus amplitude form of the complex number $z = (x, y)$.

36. We write $(\cos\theta + i\sin\theta)$ in the simplified form as $\operatorname{cis}\theta$.

37. $\operatorname{cis}(-\theta) = \frac{1}{\operatorname{cis}\theta}$.

38. $\operatorname{cis}\theta = \frac{1}{\cos\theta - i\sin\theta}$

39. $\sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right) = \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$

40. Let $z_1 = (x, y) = r_1(\cos\theta_1 + i\sin\theta_1)$; $z_2 = (p, q) = r_2(\cos\theta_2 + i\sin\theta_2)$.
$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$.

41. $\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$.

42. $\operatorname{cis}\theta_1 \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2)$

43. $\frac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \operatorname{cis}(\theta_1 - \theta_2)$

44. $\operatorname{mod}(z_1 z_2) = \operatorname{mod} z_1 . \operatorname{mod} z_2$

45. $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + n\pi$, for some $n \in \{-1, 0, 1\}$.

46. $\operatorname{Arg}(z_1 / z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + n\pi$, for some $n \in \{-1, 0, 1\}$.

Answers

Exercise 1(a)

1. (*i*) $(-24, 28)$ (*ii*) $\left(\frac{9}{10}, \frac{-2}{10}\right)$ 2. (*i*) $(15, 0)$ (*ii*) $\left(\frac{9}{5}, \frac{12}{5}\right)$ 3. $(0, 2\sin\theta)$

4. (*i*) $(\cos\theta, -\sin\theta)$ (*ii*) $\left(\frac{7}{625}, \frac{-24}{625}\right)$ (*iii*) $\left(\frac{-2}{5}, \frac{-1}{5}\right)$

5. $(i)\ 18+i(-1)$ $(ii)\ \dfrac{a^2-b^2}{a^2+b^2}+i\left(\dfrac{-2ab}{a^2+b^2}\right)$ $(iii)\ \dfrac{86}{325}+\dfrac{27}{325}i$ $(iv)\ \dfrac{-8}{13}+i(0)$

6. $(i)\ 11-23i$ $(ii)\ -38+8i$ $(iii)\ \dfrac{1-7i}{10}$

7. $(i) \pm(4+3i)$ $(ii)\ \pm(1-3i)$ $(iii)\ \pm\left(1+4\sqrt{3}i\right)$ $(iv) \pm(2+3i)$

8. $\left(a^2+b^2\right)^2$. 10. $0+i\dfrac{8ab\left(a^2-b^2\right)}{\left(a^2+b^2\right)^2}$

11. $(i)\ \sqrt{2}\left[\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right]$ $(ii) 2\left[\cos\left(\dfrac{\pi}{3}\right)+i\sin\left(\dfrac{\pi}{3}\right)\right]$

$(iii) 2\left[\cos\left(-\dfrac{2\pi}{3}\right)+i\sin\left(-\dfrac{2\pi}{3}\right)\right]$ $(iv)\ 2\left[\cos\left(\dfrac{\pi}{6}\right)+i\sin\left(\dfrac{\pi}{6}\right)\right]$

12. $2\sqrt{7}\left[\cos\left(\dfrac{2\pi}{3}\right)+i\sin\left(\dfrac{2\pi}{3}\right)\right]$ 13. $\sqrt{2}\left[\cos\left(-\dfrac{3\pi}{4}\right)+i\sin\left(-\dfrac{3\pi}{4}\right)\right]$

2. QUADRATIC EXPRESSIONS AND EQUATIONS

Introduction:

Algebra is an important part of mathematics, an understanding of its advanced branches. With algebra, we can solve problems that would be difficult or impossible to solve with arithmetic alone. Algebra has many practical applications in science, engineering, business and industry.

Al-Khwarizmi introduced the solutions of equations. his equations were linear or quadratic. His mathematics was done entirely in words with no symbols. The originality of the concepts and the style of the Al-Khwarizmi's Algebra are indeed remarkable. The word algorithm is derived from his name.

With a system of algebraic symbols, mathematicians could think in terms of types of problems rather than individual ones. The principal instrument for solving problems in algebra is the equation. Algebra is also concerned with expressions and inequalities. In algebra, we learn more about the properties of numbers and about the rules that govern operations with numbers.

In mathematics, we cultivate appreciation of abstract concepts, for which algebra forms the basis. It is the study of properties of abstract mathematical systems.

Now, in this chapter, we discuss some basic concepts of quadratic expressions and equations in one variable, extreme values and relation between the coefficients and roots of a equation up to 4^{th} order.

2.1 Solving Quadratic equation and finding nature of roots:

In this section we discuss about the quadratic equations and their roots.

2.1.1 Definition: A polynomial is of the form $ax^2+bx+c,$ where a,b,c are real or complex numbers and $a \neq 0,$ is called a quadratic expression in the variable $x.$ ax^2+bx+c is called the standard form of the quadratic expression. In this expression a is the coefficient $x^2,$ b is the coefficient x and c is the constant term.

2.1.2 Example: $2x^2+5x+7$ and $3ix^2-(i+2)x-5$ are quadratic expressions. $0.x^2+2x+3$ is not a quadratic expression, because the coefficient x^2 is zero. It is a first degree expression.

2.1.3 Definition: Any equation is of the form $ax^2+bx+c=0,$ where a,b,c are real or complex numbers and $a \neq 0,$ is called a quadratic equation in the variable $x.$ The numbers a,b,c are called coefficients of the equation.

2.1.4 Examples:

(i) $x^2+2x+3=0$ is a quadratic equation in $x.$

(ii) $5x^2 - 8x = 2x + 4$ is also a quadratic equation, since it can be rewritten as

$5x^2 - 10x - 4 = 0.$

(iii) $2x^2 + 3x + 2 = 0$ is not a quadratic equation.

(iv) $0.x^2 + 2x + 5 = 0$ is not a quadratic equation, because the coefficient x^2 is zero.

(v) $ax^2 + c = 0$ is a quadratic equation, if $a \neq 0.$

2.1.5 The roots of a quadratic equation: A complex number α is said to be a *root or solution* of the quadratic equation $ax^2 + bx + c = 0$ if $a\alpha^2 + b\alpha + c = 0.$ For example, 2 is to be a root or solution of the quadratic equation $3x^2 - 5x - 2 = 0,$ since $3.2^2 - 5.2 - 2 = 12 - 10 - 2 = 0.$

2.1.6 Note: The zeroes of the quadratic expression $ax^2 + bx + c$ are the same as the roots of the quadratic equation $ax^2 + bx + c = 0.$

2.1.7 Theorem: The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Proof: α is a root of the quadratic equation $ax^2 + bx + c = 0$

$$\Leftrightarrow a\alpha^2 + b\alpha + c = 0$$

$$\Leftrightarrow 4a\left[a\alpha^2 + b\alpha + c\right] = 0 \quad (\because a \neq 0)$$

$$\Leftrightarrow (2a\alpha + b)^2 - b^2 + 4ac = 0$$

$$\Leftrightarrow (2a\alpha + b)^2 = b^2 - 4ac$$

$$\Leftrightarrow 2a\alpha + b = \pm\sqrt{b^2 - 4ac}$$

$$\Leftrightarrow \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\because a \neq 0)$$

2.1.8 Corollary: A quadratic equation $ax^2 + bx + c = 0$ has two roots (not necessarily distinct)

2.1.9 Example: We find the roots of the equation $x^2 - 7x + 12 = 0$

On comparing the given equation with $ax^2 + bx + c = 0,$ we have $a = 1, b = -7$ and $c = 12.$

By Theorem 2.1.7, the roots of $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Therefore, the roots of the equation are $\dfrac{-(-7)\pm\sqrt{(-7)^2-4(1)(12)}}{2(1)}$

$$=\frac{7\pm\sqrt{49-48}}{2}=\frac{7\pm\sqrt{1}}{2}=\frac{7\pm1}{2}=\frac{7+1}{2},\frac{7-1}{2}=\frac{8}{2},\frac{6}{2}=4,3$$

Hence the roots of the equation $x^2-7x+12=0$ are 4 and 3.

2.1.10 Definition: b^2-4ac is called the discriminant of the quadratic expression ax^2+bx+c as well as the quadratic equation $ax^2+bx+c=0$ and it is denoted by the symbol Δ.

2.1.11 Nature of the roots of a quadratic equation: Let α,β be the roots of the quadratic equation $ax^2+bx+c=0,$ where a,b,c are real numbers.

Case (i) $\Delta=0\Leftrightarrow\alpha=\beta=-\dfrac{b}{2a}$ (a repeated root or a double root of $ax^2+bx+c=0$).

Case (ii) $\Delta>0\Leftrightarrow\alpha$ and β are real and distinct.

Case (iii) $\Delta<0\Leftrightarrow\alpha$ and β are non real complex numbers conjugate to each other.

2.1.12 Note: Let a,b and c be rational numbers, α and β be the roots of the quadratic equation $ax^2+bx+c=0.$ Then

(i) α,β are equal rational numbers if $\Delta=0.$

(ii) α,β are distinct rational numbers if Δ is the square of a non zero rational number.

(iii) α,β are conjugate surds if $\Delta>0$ and Δ is not the square of a non zero rational number.

2.1.13 Example: We show that the equation $2x^2-6x+7=0$ has no real root.

On comparing the given equation with $ax^2+bx+c=0,$ we have $a=2,b=-6$ and $c=7.$

So $\Delta=b^2-4ac=(-6)^2-4(2)(7)=36-56=-20.$

Therefore the solutions are given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$=\frac{-(-6)\pm\sqrt{-20}}{2(2)}=\frac{6\pm i\sqrt{20}}{4}=\frac{6\pm i2\sqrt{5}}{4}=\frac{3\pm i\sqrt{5}}{2}$$

Which are non real complex numbers.

Hence the given equation has no real root.

2.1.14 Example: Find all k such that the equation $x^2+2(k+2)x+9k=0$ has equal roots.

On comparing the given equation with $ax^2+bx+c=0,$ we have $a=1, b=2(k+2)$ and $c=9k.$

The condition that the equation $ax^2+bx+c=0,$ has real roots is $\Delta=b^2-4ac=0.$

$$i.e., \left[2(k+2)\right]^2-4(1)(9k)=0$$

$$i.e., (k+2)^2-9k=0$$

$$i.e., k^2-5k+4=0$$ This is a quadratic equation in $k.$

By Theorem 2.1.7, the roots of quadratic equation $ax^2+bx+c=0$ are $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}.$

Therefore the roots are given by $k=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(4)}}{2(1)}=\frac{5\pm\sqrt{25-16}}{2}=\frac{5\pm\sqrt{9}}{2}=\frac{5\pm 3}{2}$$

Hence $k=1; k=4.$

2.1.15 Example: We show that the roots of the equation $x^2-2px+p^2-q^2+2qr-r^2=0$ are rational, given that p,q,r are rational.

On comparing the given equation with $ax^2+bx+c=0,$ we have $a=1, b=-2p$ and $c=p^2-q^2+2qr-r^2.$

So $\Delta=b^2-4ac=(-2p)^2-4(1)(p^2-q^2+2qr-r^2)$

$$=4p^2-4(p^2-q^2+2qr-r^2)$$

$$=4(q^2-2qr+r^2)=4(q-r)^2=\left[2(q-r)\right]^2$$

The coefficients of the given equation are rational numbers, since p,q,r are rational numbers. Since Δ is the square of the rational number $2(q-r),$ the roots of the given equation are rational numbers.

2.1.16 Note: If α and β be the roots of the quadratic equation $ax^2+bx+c=0.$ Then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

We have $\alpha+\beta=\dfrac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=-\dfrac{2b}{2a}=-\dfrac{b}{a}=-\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and $\alpha\beta=\left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)=\dfrac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}=\dfrac{b^2-\left(b^2-4ac\right)}{4a^2}$

$$=\frac{4ac}{4a^2}=\frac{c}{a}=\frac{c}{a}=-\frac{\text{constant term}}{\text{coefficient of } x^2},\ (\text{Note that } a\neq 0)$$

so that $ax^2+bx+c=a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)$

$$=a\left(x^2-(\alpha+\beta)x+\alpha\beta\right)=a(x-\alpha)(x-\beta)$$

2.1.17 Example: We find a quadratic equation whose roots are 3 and $-2.$

Since 3 and -2 are roots, the quadratic equation is $x^2-(3+(-2))x+(3)(-2)=0$

$$i.e., x^2-x-6=0$$

This is a quadratic equation whose roots are 3 and $-2.$

2.1.18 Example: We find a quadratic equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}.$

Let $\alpha=2+\sqrt{3}$ and $\beta=2-\sqrt{3}.$

Now, $\alpha+\beta=2+\sqrt{3}+2-\sqrt{3}=4,\ \alpha.\beta=(2+\sqrt{3})(2-\sqrt{3})=4-2\sqrt{3}+2\sqrt{3}-3=1$

Since a quadratic equation having roots are α and β is of the form $x^2-(\alpha+\beta)x+\alpha\beta=0$

$$i.e., x^2-4x+1=0$$

This is a quadratic equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}.$

2.1.19 Example: We find a quadratic equation whose roots are $-a+ib$ and $-a-ib.$

Let $\alpha=-a+ib$ and $\beta=-a-ib.$

Now, $\alpha+\beta=-a+ib-a-ib=-2a,\ \alpha.\beta=(-a+ib)(-a-ib)=(-a)^2-(-ib)^2=a^2+b^2$

Since a quadratic equation having roots are α and β is of the form $x^2-(\alpha+\beta)x+\alpha\beta=0$

$$i.e., x^2+2ax+a^2+b^2=0$$

This is a quadratic equation whose roots are $-a+ib$ and $-a-ib.$

2.1.20 Solved Problems:

1. Problem: Find the roots of the equation $3x^2+2x-5=0$.

Solution: By Theorem 2.1.7, the roots of the quadratic equation

$$ax^2+bx+c=0 \text{ are } \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Here $a=3, b=2$ and $c=-5$.

Therefore the roots of the quadratic equation are

$$\frac{-2\pm\sqrt{(2)^2-4(3)(-5)}}{2(3)}=\frac{-2\pm\sqrt{4+60}}{2}=\frac{-2\pm\sqrt{64}}{2}=\frac{-2\pm 8}{2}=1,-\frac{5}{3}$$

Hence 1 and $-\frac{5}{3}$ are the roots of the quadratic equation.

2. Problem: Find the roots of the equation $4x^2-4x+17=3x^2-10x-17$.

Solution: The given equation can be rewritten as $x^2+6x+34=0$.

By Theorem 2.1.7, the roots of the quadratic equation

$$ax^2+bx+c=0 \text{ are } \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Here $a=1, b=6$ and $c=34$.

Therefore the roots of the quadratic equation are

$$\frac{-6\pm\sqrt{(6)^2-4(1)(34)}}{2(1)}=\frac{-6\pm\sqrt{36-136}}{2}=\frac{-6\pm\sqrt{-100}}{2}=\frac{-6\pm 10i}{2}=-3+5i,-3-5i$$

Hence $-3+5i$ and $-3-5i$ are the roots of the quadratic equation.

3. Problem: Find the roots of the equation $\sqrt{3}x^2+10x-8\sqrt{3}=0$.

Solution: The given equation is $\sqrt{3}x^2+10x-8\sqrt{3}=0$.

By Theorem 2.1.7, the roots of the quadratic equation

$$ax^2+bx+c=0 \text{ are } \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Here $a=\sqrt{3}, b=10$ and $c=-8\sqrt{3}$.

Therefore the roots of the quadratic equation are

$$\frac{-10 \pm \sqrt{(10)^2 - 4(\sqrt{3})(-8\sqrt{3})}}{2(\sqrt{3})} = \frac{-10 \pm \sqrt{100+96}}{2\sqrt{3}}$$

$$= \frac{-10 \pm \sqrt{196}}{2\sqrt{3}} = \frac{-10 \pm 14}{2\sqrt{3}} = \frac{2}{\sqrt{3}}, -4\sqrt{3}$$

Hence $\frac{2}{\sqrt{3}}$ and $-4\sqrt{3}$ are the roots of the quadratic equation.

4. Problem: Find the nature of the roots of the equation $4x^2 - 20x + 25 = 0.$

Solution: The given equation is $4x^2 - 20x + 25 = 0.$

It is of the form $ax^2 + bx + c = 0$

Here $a = 4, b = -20$ and $c = 25.$

Hence $\Delta = b^2 - 4ac = (-20)^2 - 4(4)(25) = 400 - 400 = 0.$

Since Δ is zero and a, b, c are real, the roots of the given equation are real and equal.

5. Problem: Find the nature of the roots of the equation $3x^2 + 7x + 2 = 0.$

Solution: The given equation is $3x^2 + 7x + 2 = 0.$

It is of the form $ax^2 + bx + c = 0$

Here $a = 3, b = 7$ and $c = 2.$

Hence $\Delta = b^2 - 4ac = (7)^2 - 4(3)(2) = 49 - 24 = 25 = 5^2 > 0.$

Since Δ is a square number, the roots of the given equation are rational and unequal.

6. Problem: For what values of $m,$ the equation $x^2 - 2(1+3m)x + 7(3+2m) = 0$ will have equal roots.

Solution: The given equation is $x^2 - 2(1+3m)x + 7(3+2m) = 0$

It is of the form $ax^2 + bx + c = 0$

Here $a = 1, b = -2(1+3m)$ and $c = 7(3+2m).$

The given equation have equal roots $\Leftrightarrow \Delta = 0.$

Now, $\Delta = b^2 - 4ac = (-2(1+3m))^2 - 4(1)(7(3+2m))$

$$= 4(1+3m)^2 - 4(21+14m)$$

$$= 4\left[(1+9m^2+6m)-(21+14m)\right]$$

$$= 4(9m^2 - 8m - 20) = 4(9m+10)(m-2) = 36(m+\frac{10}{9})(m-2)$$

Since $\Delta = 0 \Leftrightarrow m = -\frac{10}{9}, m = 2$

Therefore the roots of the given equation are equal iff $m = -\frac{10}{9}, m = 2.$

7. Problem: Form a quadratic equation whose roots are $2\sqrt{3}-5$ and $-2\sqrt{3}-5.$

Solution: Let $\alpha = 2\sqrt{3}-5$ and $\beta = -2\sqrt{3}-5.$

Now, $\alpha+\beta = 2\sqrt{3}-5-2\sqrt{3}-5 = -10,$

$\alpha.\beta = (2\sqrt{3}-5)(-2\sqrt{3}-5)$

$$= (2\sqrt{3})(-2\sqrt{3}) - (2\sqrt{3})(5) - (5)(-2\sqrt{3}) - (5)(-5)$$

$$= -12 - 10\sqrt{3} + 10\sqrt{3} + 25 = 13.$$

Since a quadratic equation having roots are α and β is of the form $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$$i.e., x^2 + 10x + 13 = 0$$

This is a quadratic equation whose roots are $2\sqrt{3}-5$ and $-2\sqrt{3}-5.$

2.2Finding out Maximum and Minimum values of a Quadratic expression:

In this section we shall discuss the Maximum and Minimum values of a quadratic expression ax^2+bx+c with real coefficients depends on the coefficient a of x^2.

2.2.1 Maximum and Minimum values:

The extreme values of a quadratic expression with real coefficients depend on the sign of the coefficient of x^2. We state this fact in the following theorem.

2.2.2 Theorem: Suppose $a, b, c \in \mathrm{R}, a \neq 0$ and $f(x) = ax^2+bx+c$

(i) If $a > 0,$ then $f(x)$ has absolute minimum at $x = -\frac{b}{2a}$ and the minimum value is

$\dfrac{4ac-b^2}{4a}$

(ii) If $a<0$, then $f(x)$ has absolute maximum at $x=-\dfrac{b}{2a}$ and the maximum value is

$\dfrac{4ac-b^2}{4a}$

(The proof of this theorem is beyond the scope of this book).

2.2.3 Note: When $a,b,c\in \mathrm{R}$, the minimum or maximum values of $f(x)=ax^2+bx+c$

can be found by using method of calculus. On differentiating $f(x)$, we get

$f'(x)=2ax+b$ and $f''(x)=2a\,\forall\, x\in \mathrm{R}$.

Now $f'(x)=0 \Leftrightarrow 2ax+b=0 \Leftrightarrow x=-\dfrac{b}{2a}$.

(i) If $a>0$, then $f''(x)>0\,\forall\, x\in \mathrm{R}$ and hence from the differentiation rules $f(x)$ has local minimum at $x=-\dfrac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has absolute minimum at $x=-\dfrac{b}{2a}$ and the minimum value of $f(x)$ is $\dfrac{4ac-b^2}{4a}$

(ii) If $a<0$, then $f''(x)<0\,\forall\, x\in \mathrm{R}$ and hence from the differentiation rules $f(x)$ has local maximum at $x=-\dfrac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has absolute maximum at $x=-\dfrac{b}{2a}$ and the maximum value of $f(x)$ is $\dfrac{4ac-b^2}{4a}$

2.2.4 Note: When $a,b,c\in \mathrm{R}$, the quadratic expression ax^2+bx+c has no maximum value when $a>0$ and no minimum value when $a<0$.

2.2.5 Solved Problems:

1. Problem: Find the maximum or minimum value of the expression $3x^2+4x+1$.

Solution: The given expression is $3x^2+4x+1$.

It is of the form ax^2+bx+c.

Here $a=3, b=4$ and $c=1$.

Since $a=3>0,$ the expression $3x^2+4x+1$ has absolute minimum and the minimum value is $\frac{4ac-b^2}{4a}=\frac{4(3)(1)-(4)^2}{4(3)}=\frac{12-16}{12}=-\frac{1}{3}.$

This expression has no maximum.

2. Problem: Find the maximum or minimum value of the expression $4x-x^2-10.$

Solution: The given expression is $4x-x^2-10.$

It is of the form $ax^2+bx+c.$

Here $a=-1, b=4$ and $c=-10.$

Since $a=-1<0,$ the expression $4x-x^2-10$ has absolute maximum and the maximum value is $\frac{4ac-b^2}{4a}=\frac{4(-1)(-10)-(4)^2}{4(-1)}=\frac{40-16}{-4}=-6.$

This expression has no minimum.

3. Problem: Find the value of x at which the following expression x^2+5x+6 have maximum or minimum.

Solution: The given expression is x^2+5x+6

It is of the form $ax^2+bx+c.$

Here $a=1, b=5$ and $c=6.$

Since $a=1>0,$ the expression x^2+5x+6 has absolute minimum at $x=-\frac{b}{2a}=-\frac{5}{2(1)}=-\frac{5}{2}.$

4. Problem: Find the value of x at which the following expression $2x-x^2+7$ have maximum or minimum.

Solution: The given expression is $2x-x^2+7$

It is of the form $ax^2+bx+c.$

Here $a=-1, b=2$ and $c=7.$

Since $a=-1<0,$ the expression x^2+5x+6 has absolute maximum at $x=-\frac{b}{2a}=-\frac{2}{2(-1)}=1.$

Exercise 2(a)

1. Find the value of K, if the equation $x^2+2(K+2)x+9K=0$ has equal roots.

2. Find the nature of the roots of the following equations

   $(i)\ 4x^2-20x+25=0$ $(ii) 3x^2+7x+2=0$ $(iii) 2x^2-8x+3=0$

   $(iv) 9x^2-30x+25=0$ $(v) x^2-12x+32=0$ $(vi) 2x^2-7x+10=0$

3. Obtain the quadratic equations whose roots are given below:

   $(i)\ \frac{m}{n}, \frac{n}{m} (m \neq 0, n \neq 0)$ $(ii)\ -3 \pm 5i$

4. If the following equations have equal roots, find the value of m :

   $(i)\ (m+1)x^2+2(m+3)x+(m+8)=0$

   $(ii)\ (2m+1)x^2+2(m+3)x+(m+5)=0$

   $(iii)\ (3m+1)x^2+2(m+1)x+m=0$

   $(iv)\ x^2-m(2x-8)-15=0$

5. Find the minimum & maximum values of the following quadratic expressions:

   $(i)\ 3x^2+2x+11$ $(ii) 2x-7-5x^2$ $(iii) 4x-5x^2+2$ $(iv) x^2-5x+6$

   $(v) 15+4x-3x^2$ $(vi) x^2-x-7$ $(vii) 2x+5-3x^2$ $(vi) 12x-x^2-32$

2.3 Relation between the roots and coefficients in an equation up to order 4:

In this section we define a polynomial of degree n and derive the relation between roots and coefficients of the polynomial equation up to order 4.

2.3.1 Definition: If n is a non-negative integer and $a_0, a_1, a_2, ...a_n$ are real or complex numbers and $a_0 \neq 0,$ then an expression $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_n$ ...(I) is called a *polynomial* in x of degree n. Here $a_0, a_1, a_2, ...a_n$ are called the coefficients of the polynomial $f(x),$ while a_0 is called the *leading coefficient* of $f(x), a_n$ is called the *constant term or absolute term* of $f(x)$ and a_i is called the *coefficient* of x^{n-i}.

Sometimes a polynomial in x of degree n is represented by $f(x)=a_0+a_1x+a_2x^2+...+a_nx^n$ ...(II) If $a_n \neq 0,$ then a_n is called the *leading coefficient* of $f(x), a_0$ is called the *constant term or absolute term* of $f(x)$ and a_i is called the *coefficient* of x^i.

By definition $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_n=b_0x^n+b_1x^{n-1}+b_2x^{n-2}+...+b_n$ if and only if $a_k=b_k, \forall k=1,2,3,...n.$

We note that a non-zero constant is a polynomial of degree zero. The constant zero is called the zero polynomial and its degree is not defined.

A polynomial with leading coefficient 1 is called a *monic polynomial*.

Any polynomial $p(x)$ of degree $n \geq 1$ can be written as $p(x) = a_0 p_0(x),$ where a_0 is the *leading coefficient* of $p(x)$ and $p_0(x)$ is a *monic polynomial* of degree $n.$

If $f(x)$ is a polynomial of degree $n > 0,$ then the equation $f(x) = 0$ is called a *polynomial equation* of degree $n.$ It is also called as an *algebraic equation* of degree $n.$

If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ is a polynomial and α is a real or complex number, then we write $f(\alpha)$ for $a_0 \alpha^n + a_1 \alpha^{n-1} + a_2 \alpha^{n-2} + ... + a_n.$

A real or complex number α is said to be zero of a polynomial $f(x)$ or a root of the equation $f(x) = 0,$ if $f(\alpha) = 0.$

If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ and $g(x) = b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + ... + b_m$ are polynomials of degree n and m respectively with $n \leq m,$ with out loss of generality then we can define the addition and multiplication of polynomials which satisfy all the arithmetic properties of real or complex numbers except the existence of reciprocals of nonzero polynomial.

The main object of the theory of equations is to find roots of a polynomial equation $f(x) = 0,$ *i.e.,* to solve the equation $f(x) = 0.$

2.3.2 Theorem (Remainder Theorem):

Let $f(x)$ be a polynomial of degree $n > 0.$ Let $a \in \mathbb{C}.$ Then there exists a polynomial $q(x)$ of degree $n-1$ such that $f(x) = (x-a)q(x) + f(a).$

Proof: Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$

Now define $b_0 = a_0,\ b_1 = a_1 + ab_0, b_2 = a_2 + ab_1, ..., b_n = a_{n-1} + ab_{n-2}$ and

$$q(x) = b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + ... + b_{n-2}\, x + b_{n-1}$$

Then, $(x-a)q(x) = xq(x) - aq(x)$

$$= x(b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + ... + b_{n-2}\, x + b_{n-1})$$
$$- a(b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + ... + b_{n-2}\, x + b_{n-1})$$

$$= b_0 x^n + (b_1 - ab_0) x^{n-1} + (b_2 - ab_1) x^{n-2} + ... + (b_{n-1} - ab_{n-2})\, x - ab_{n-1}$$

$$= a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1}\, x - ab_{n-1}$$

$= f(x) - a_n - ab_{n-1}$

Hence, $f(x) = (x-a)q(x) + a_n + ab_{n-1} = (x-a)q(x) + c,$ where $c = a_n + ab_{n-1}.$

By substituting $x = a$ we get that $f(a) = 0.q(a) + c$ so that $c = f(a).$

Therefore $f(x) = (x-a)q(x) + f(a).$

2.3.3 Note: Let $f(x)$ be a polynomial of degree $n > 0.$ Let $a \in \mathrm{C}.$ We say that $x - a$ is a *factor* of $f(x),$ if there exists a polynomial $q(x)$ such that $f(x) = (x-a)q(x).$

2.3.4 Corollary: Let $f(x)$ be a polynomial of degree $n > 0.$ Then $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0.$

2.3.5 Theorem (Fundamental Theorem of Algebra):

Every non constant polynomial equation has at least one root.

(The proof of this theorem is beyond the scope of this book).

2.3.6 Theorem: The set of all roots of a polynomial equation $f(x) = 0$ of degree $n > 0$ is non empty and has at most n elements. Also there exist $\alpha_1, \alpha_2, ..., \alpha_n$ (which may not be distinct) in C such that $f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n),$ where a is a leading coefficient of $f(x).$

Proof: We prove the theorem by induction. The result is true when $n = 1$ (since in this case $f(x) = a(x-\alpha)$ for some α).

Assume the truth of the theorem for $n-1$ where $n \geq 2.$ Now suppose that $f(x)$ is a polynomial of degree n with leading coefficient $a.$ Then by Fundamental Theorem of Algebra $f(x)$ has at least one root say $\alpha_1.$ Then by Corollary 2.3.4, $x - \alpha_1$ is a factor of $f(x)$ and $f(x) = (x-\alpha_1)q(x)$ ...(I)

where $q(x)$ is a polynomial of degree $n-1.$ Since the leading coefficient of $f(x)$ is $a,$ the leading coefficient of $q(x)$ is also $a.$ Now, by the induction hypothesis, there exist $\alpha_1, \alpha_2, ..., \alpha_n$ in C such that $q(x) = a(x-\alpha_2)...(x-\alpha_n)$ ...(II)

From (I) and (II), we have $f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n).$

2.3.7 Note: (i) From Theorem 2.3.6 it follows that a polynomial equation of degree $n > 0$ has at most n distinct roots.

(ii) Let $f(x)$ is a polynomial of degree $n > 0$ with leading coefficient $a.$ If $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ are complex numbers such that

$f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$

$= a(x-\beta_1)(x-\beta_2)...(x-\beta_n),$

then using the principle of mathematical induction, it can be shown that $(\beta_1, \beta_2, ..., \beta_n)$ is a permutation of $(\alpha_1, \alpha_2, ..., \alpha_n)$.

(iii) If $f(x)$ is a polynomial of degree $n > 0$ with leading coefficient a and $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ are such that $f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$ then $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the polynomial equation $f(x) = 0$.

For example if $f(x) = (x-1)^2(x-2)$ then $1,1,2$ are the roots of $f(x) = 0$.

2.3.8 Corollary: Suppose n is a positive integer, $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are complex numbers such that

$$a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + ... + a_n = b_0\alpha^n + b_1\alpha^{n-1} + b_2\alpha^{n-2} + ... + b_n$$

for more than n distinct elements α in $\mathbb{R}$. Then $a_k = b_k$ for $0 \le k \le n$.

2.3.9 Note: If $f(x)$ and $g(x)$ are two polynomials such that $f(\alpha) = g(\alpha)$ for infinitely many numbers α, then $f(x) = g(x)$.

2.3.10 The relation between the roots and the coefficients:

Let us consider the n^{th} degree polynomial equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_{n-1}x + p_n = 0.$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be its roots.

Then we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_{n-1}x + p_n = (x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$$

$$= x^n - (\alpha_1 + \alpha_2 + ... + \alpha_n)x^{n-1}$$
$$+ (\alpha_1\alpha_2 + \alpha_2\alpha_3 + ... + \alpha_{n-1}\alpha_n)x^{n-2}$$
$$+ ... + (-1)^n \alpha_1\alpha_2...\alpha_n.$$

On equating the coefficients of like powers of x in this equation and denoting the sum of products of the roots taken r at a time by s_r, we get

$$-p_1 = s_1 = \sum_{i=1}^{n} \alpha_i \text{ (sum of the roots),}$$

$p_2 = s_2 = \sum_{1 \le i < j \le n}^{n} \alpha_i \alpha_j$ (sum of the product of the roots taken two at a time),

$-p_3 = s_3 = \sum_{1 \le i < j < k \le n}^{n} \alpha_i \alpha_j \alpha_k$ (sum of the product of the roots taken three at a time),

... ...

$(-1)^n p_n = s_n = \alpha_1 \alpha_2 \alpha_3 ... \alpha_n$ (product of the roots).

These equalities give the relation between the roots and the coefficients for any polynomial equation whose leading coefficient is 1.

2.3.11 Note: (i) If the leading coefficient of $f(x)$ is a_0 then on dividing each term of the equation $f(x) = 0$ by $a_0 \ne 0,$ we get

$$x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + ... + \frac{a_{n-1}}{a_0} x + \frac{a_n}{a_0} = 0$$

whose roots coincide with those of $f(x) = 0.$

In this case, the above relations reduce to

$$s_1 = -\frac{a_1}{a_0}, s_2 = \frac{a_2}{a_0}, ..., s_n = (-1)^n \frac{a_n}{a_0}; \; i.e., s_r = (-1)^r \frac{a_r}{a_0} \text{ for } 1 \le r \le n.$$

(ii) For $n = 2,$ we get a quadratic equation $x^2 + p_1 x + p_2 = 0.$

Let α_1 and α_2 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 = -p_1$ and $s_2 = \alpha_1 \alpha_2 = p_2$.

(iii) For $n = 3,$ we get a cubic equation $x^3 + p_1 x^2 + p_2 x + p_3 = 0.$

Let α_1, α_2 and α_3 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 + \alpha_3 = -p_1,$ $s_2 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 = p_2$ and $s_3 = \alpha_1 \alpha_2 \alpha_3 = p_3$.

(iv) For $n = 4,$ we get a bi-quadratic equation $x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = 0.$

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1,$

$s_2 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_4 + \alpha_1 \alpha_3 + \alpha_1 \alpha_4 + \alpha_2 \alpha_4 = p_2,$

$$s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_4 = -p_3$$

and $s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = p_4$.

2.3.12 Solved Problems:

1. Problem: Find the polynomial equation of degree 3 whose roots are $2, 3$ and 6.

Solution: The required polynomial equation is $(x-2)(x-3)(x-6)=0$.

On simplification, this reduces to $x^3 - 11x^2 + 36x - 36 = 0$.

2. Problem: If α, β are the roots of the equation $ax^2 + bx + c = 0$ then find the following

$(i)\,\alpha^2+\beta^2 \quad (ii)\,\alpha^3+\beta^3 \ (iii)\,\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2} \ (iv)\,\alpha^4\beta^7+\alpha^7\beta^4 \ (v)\,\dfrac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}}$

Solution: Given that α, β are the roots of the equation $ax^2 + bx + c = 0$

On dividing the equation by a, we get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ...(I)

On comparing equation (I) with $x^2 + p_1 x + p_2 = 0$, we have $p_1 = \frac{b}{a}, p_2 = \frac{c}{a}$

Since α, β are the roots of the equation (I). Then $\alpha + \beta = -p_1 = -\frac{b}{a}, \alpha\beta = p_2 = \frac{c}{a}$

Now, $(i)\,\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2.\frac{c}{a} = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$(ii)\,\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$$= \left(-\frac{b}{a}\right)^3 - 3.\frac{c}{a}\left(-\frac{b}{a}\right) = \frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{b^3 + 3abc}{a^3}$$

$$(iii)\,\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2.\beta^2} = \frac{\frac{b^2-2ac}{a^2}}{\left(\frac{c}{a}\right)^2} = \left(\frac{b^2-2ac}{a^2}\right)\left(\frac{a^2}{c^2}\right) = \frac{b^2-2ac}{c^2}$$

$$(iv)\,\alpha^4\beta^7 + \alpha^7\beta^4 = \alpha^4\beta^4(\alpha^3+\beta^3) = \left(\frac{c}{a}\right)^4\left(\frac{b^3+3abc}{a^3}\right)$$

$$= \left(\frac{c^4}{a^4}\right)\left(\frac{b^3+3abc}{a^3}\right) = \frac{b^3c^4 + 3abc^5}{a^7}$$

$(v)\dfrac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}} = \dfrac{\alpha^2+\beta^2}{\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}} = \dfrac{\alpha^2+\beta^2}{\dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2}}$

$$= (\alpha^2+\beta^2)\frac{\alpha^2\beta^2}{(\alpha^2+\beta^2)} = \alpha^2\beta^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}.$$

3. Problem: If α, β, γ are the roots of the equation $3x^3 - 10x^2 + 7x + 10 = 0$ then find $\alpha\beta + \beta\gamma + \gamma\alpha$

Solution: Given that α, β, γ are the roots of the equation $3x^3 - 10x^2 + 7x + 10 = 0$

On dividing the equation by 3, we get $x^3 - \frac{10}{3}x^2 + \frac{7}{3}x + \frac{10}{3} = 0$ ...(I)

On comparing equation (I) with $x^3 + p_1x^2 + p_2x + p_3 = 0.$ we have $p_1 = -\frac{10}{3}, p_2 = \frac{7}{3}, p_3 = \frac{10}{3}$

Since α, β, γ are the roots of the equation (I). Then $\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = p_2 = \frac{7}{3}.$

4. Problem: Find the relation between the roots and the coefficients of the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0.$

Solution: Given equation is $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ ...(I)

On comparing equation (I) with $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0.$ we have $p_1 = -2, p_2 = 4, p_3 = 6$ and $p_4 = -21.$

Let α, β, γ and δ be its roots.

Then $s_1 = \sum \alpha = -p_1 = 2,$

$s_2 = \sum \alpha\beta = p_2 = 4,$

$s_3 = \sum \alpha\beta\gamma = -p_3 = -6$

and $s_4 = \alpha\beta\gamma\delta = p_4 = -21.$

5. Problem: If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ then find the values of $a, b, c, d.$

Solution: Given equation is $x^4 + ax^3 + bx^2 + cx + d = 0$ ...(I)

The equation having roots $1, 2, 3$ and 4 is $(x-1)(x-2)(x-3)(x-4) = 0$

$$i.e., x^4 - 10x^3 + 35x^2 - 50x + 24 = 0 \qquad ...(II)$$

On comparing equation (I) with (II) we obtain $a = -10, b = 35, c = -50$ and $d = 24.$

6. Problem: If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0,$ then find $(i) \sum \alpha\beta \quad (ii) \sum \frac{1}{\alpha} \quad (iii) \sum \alpha^2 \; (iv) \sum \alpha^3$

Solution: Given that α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0 \qquad ...(I)$

Since α, β, γ are the roots of the equation (I). Then

$$\sum \alpha = \alpha + \beta + \gamma = -p, \sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q, \; \alpha\beta\gamma = -r.$$

Now, $(i) \sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q,$

$$(ii) \;\; \sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = -\frac{q}{r}.$$

$$(iii) \;\; \sum \alpha^2 = \alpha^2 + \beta^2 + \gamma^2$$

We have $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (-p)^2 - 2(q) = p^2 - 2q.$$

$$(iv) \;\; \sum \alpha^3 = \alpha^3 + \beta^3 + \gamma^3$$

We have $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$

$$= (\alpha + \beta + \gamma)((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$= (\alpha + \beta + \gamma)((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha))$$

$$\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)) + 3\alpha\beta\gamma$$

$$\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = (-p)((-p)^2 - 3(q)) + 3(-r) = 3pq - p^3 - 3r.$$

Exercise 2(b)

1. Find the polynomial equation of lowest degree whose roots are given below:
   $(i) 1, 2, 3 \quad (ii) -1, -2, -3 \quad (iii) 2, -2, 3 \quad (iv) 2, 2, -4 \quad (v) 2 \pm 3i, -4$

2. If α, β are the roots of the equation $x^2 + x + 1 = 0$ then find the following

$(i)\,\alpha^2+\beta^2 \quad (ii)\,\alpha^3+\beta^3 \; (iii)\,\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2} \; (iv)\,\alpha^4\beta^7+\alpha^7\beta^4 \; (v)\,\dfrac{\alpha^2+\beta^2}{\alpha^{-2}+\beta^{-2}}$

3. If α,β,γ are the roots of equations $4x^3-6x^2+7x+3=0$ then find $\alpha\beta+\beta\gamma+\gamma\alpha$

4. If the product of the roots of the equation $4x^3+16x^2-9x-a=0,$ is 9 then find a.

5. If $-1, 2, \alpha$ are the roots of the equation $2x^3+x^2-7x-6=0,$ then find α.

6. If $1,-2,3$ are the roots of the equation $x^3-2x^2+ax+6=0,$ then find a.

7. If α,β,γ are the roots of the equation $x^3+px^2+qx+r=0,$ then find $(i)\sum\dfrac{1}{\alpha^2\beta^2}$

$(ii)\sum\dfrac{1}{\alpha}$

8. Solve the equation $x^3-3x^2-16x+48=0,$ given that the sum of two of its roots is zero.

9. Find the condition that $x^3-px^2-qx-r=0$ may have the sum of two of its roots is zero.

10. Find the relation between the roots and the coefficients of the equation

$3x^3-10x^2+7x+10=0$

11. From a polynomial equation of the lowest degree, whose roots are

(i) −2, −2, 2, 2 and (ii) 1, 3, 5, 7

Key concepts

1. A polynomial is of the form $ax^2+bx+c,$ where a,b,c are real or complex numbers and $a\neq 0,$ is called a quadratic expression in the variable x. ax^2+bx+c is called the standard form of the quadratic expression. In this expression a is the coefficient x^2, b is the coefficient x and c is the constant term.
2. Any equation is of the form $ax^2+bx+c=0,$ where a,b,c are real or complex numbers and $a\neq 0,$ is called a quadratic equation in the variable x. The numbers a,b,c are called coefficients of the equation.
3. A complex number α is said to be a *root or solution* of the quadratic equation $ax^2+bx+c=0$ if $a\alpha^2+b\alpha+c=0.$
4. The zeroes of the quadratic expression ax^2+bx+c are the same as the roots of the quadratic equation $ax^2+bx+c=0.$

5. The roots of the quadratic equation $ax^2+bx+c=0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

6. A quadratic equation $ax^2+bx+c=0$ has two roots (not necessarily distinct)

7. b^2-4ac is called the discriminant of the quadratic expression ax^2+bx+c as well as the quadratic equation $ax^2+bx+c=0$ and it is denoted by the symbol Δ.

8. Let α,β be the roots of the quadratic equation $ax^2+bx+c=0$, where a,b,c are real numbers (i) $\Delta=0 \Leftrightarrow \alpha=\beta=-\frac{b}{2a}$ (a repeated root of $ax^2+bx+c=0$). (ii) $\Delta>0 \Leftrightarrow \alpha$ and β are real and distinct.

   (iii) $\Delta<0 \Leftrightarrow \alpha$ and β are non real complex numbers conjugate to each other.

9. Let a,b and c be rational numbers, α and β be the roots of the quadratic equation $ax^2+bx+c=0$. Then (i) α,β are equal rational numbers if $\Delta=0$. (ii) α,β are distinct rational numbers if Δ is the square of a non zero rational number.

   (iii) α,β are conjugate surds if $\Delta>0$ and Δ is not the square of a non zero rational number.

10. If α and β be the roots of the quadratic equation $ax^2+bx+c=0$. Then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

We have $\alpha+\beta=$ $-\frac{b}{a}=-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$ and $\alpha\beta$ $=\frac{c}{a}=-\frac{\text{constant term}}{\text{coefficient of } x^2}$, ($a\neq 0$)

11. If α and β be the roots of the quadratic equation $ax^2+bx+c=0$. Then

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

12. Suppose $a,b,c\in \mathrm{R}, a\neq 0$ and $f(x)=ax^2+bx+c$ (i) If $a>0$, then $f(x)$ has absolute minimum at $x=-\frac{b}{2a}$ and the minimum value is $\frac{4ac-b^2}{4a}$ (ii) If $a<0$, then $f(x)$ has absolute maximum at $x=-\frac{b}{2a}$ and the maximum value is $\frac{4ac-b^2}{4a}$

13. When $a,b,c\in \mathrm{R}$, $f(x)=ax^2+bx+c$

(i) If $a>0$, then $f''(x)>0\ \forall\, x\in \mathrm{R}$ and hence from the differentiation rules $f(x)$ has local minimum at $x = -\frac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has

absolute minimum at $x = -\frac{b}{2a}$ and the minimum value of $f(x)$ is $\frac{4ac-b^2}{4a}$

(ii) If $a < 0$, then $f''(x) < 0 \, \forall \, x \in \mathrm{R}$ and hence from the differentiation rules $f(x)$ has local

maximum at $x = -\frac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has

absolute maximum at $x = -\frac{b}{2a}$ and the maximum value of $f(x)$ is $\frac{4ac-b^2}{4a}$

14. When $a, b, c \in \mathrm{R}$, the quadratic expression $ax^2 + bx + c$ has no maximum value when $a > 0$ and no minimum value when $a < 0$.

15. If n is a non-negative integer and $a_0, a_1, a_2, ...a_n$ are real or complex numbers and $a_0 \neq 0$, then an expression $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ ...(I) is called a *polynomial* in x of degree n. Here $a_0, a_1, a_2, ...a_n$ are called the coefficients of the polynomial $f(x)$, while a_0 is called the *leading coefficient* of $f(x), a_n$ is called the *constant term or absolute term* of $f(x)$ and a_i is called the *coefficient* of x^{n-i}.

16. If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ is a polynomial and α is a real or complex number, then we write $f(\alpha)$ for $a_0 \alpha^n + a_1 \alpha^{n-1} + a_2 \alpha^{n-2} + ... + a_n$.

17. A real or complex number α is said to be zero of a polynomial $f(x)$ or a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

18. **(Remainder Theorem):** Let $f(x)$ be a polynomial of degree $n > 0$. Let $a \in \mathrm{C}$. Then there exists a polynomial $q(x)$ of degree $n-1$ such that $f(x) = (x-a)q(x) + f(a)$.

19.Let $f(x)$ be a polynomial of degree $n > 0$. Let $a \in \mathrm{C}$. We say that $x - a$ is a *factor* of $f(x)$, if there exists a polynomial $q(x)$ such that $f(x) = (x-a)q(x)$.

20. Let $f(x)$ be a polynomial of degree $n > 0$. Then $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

21. Every non constant polynomial equation has at least one root.

22.The set of all roots of a polynomial equation $f(x)=0$ of degree $n>0$ is non empty and has at most n elements. Also there exist $\alpha_1,\alpha_2,...,\alpha_n$ (which may not be distinct) in $\mathbb{C}$ such that $f(x)=a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n),$ where a is a leading coefficient of $f(x).$

23. A polynomial equation of degree $n>0$ has at most n distinct roots. If $f(x)$ is a polynomial of degree $n>0$ with leading coefficient a and $\alpha_1,\alpha_2,...,\alpha_n \in \mathbb{C}$ are such that $f(x)=a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$ then $\alpha_1,\alpha_2,...,\alpha_n$ are the roots of the polynomial equation $f(x)=0.$

24.Suppose n is a positive integer, $a_1,a_2,...,a_n$ and $b_1,b_2,...,b_n$ are complex numbers such that $a_0\alpha^n+a_1\alpha^{n-1}+a_2\alpha^{n-2}+...+a_n=b_0\alpha^n+b_1\alpha^{n-1}+b_2\alpha^{n-2}+...+b_n$ for more than n distinct elements α in $\mathbb{R}$. Then $a_k=b_k$ for $0\le k\le n.$

25.. If $f(x)$ and $g(x)$ are two polynomials such that $f(\alpha)=g(\alpha)$ for infinitely many numbers α, then $f(x)=g(x).$

26. .Let us take $x^n+p_1x^{n-1}+p_2x^{n-2}+...+p_{n-1}x+p_n=0$ as n^{th} degree polynomial equation.

Let $\alpha_1,\alpha_2,...,\alpha_n$ be its roots. Then we have

$$-p_1=s_1=\sum_{i=1}^{n}\alpha_i \text{ (sum of the roots),}$$

$$p_2=s_2=\sum_{1\le i<j\le n}^{n}\alpha_i\alpha_j \text{ (sum of the product of the roots taken two at a time),}$$

$$-p_3=s_3=\sum_{1\le i<j<k\le n}^{n}\alpha_i\alpha_j\alpha_k \text{ (sum of the product of the roots taken three at a time),}$$

... ...

$$(-1)^n p_n=s_n=\alpha_1\alpha_2\alpha_3...\alpha_n \text{ (product of the roots).}$$

These equalities give the relation between the roots and the coefficients for any polynomial equation whose leading coefficient is 1.

27. (i) If the leading coefficient of $f(x)$ is a_0 then on dividing each term of the equation $f(x)=0$ by $a_0 \neq 0,$ we get $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$ whose roots coincide with those of $f(x)=0.$ In this case, the above relations reduce to

$$s_1 = -\frac{a_1}{a_0}, s_2 = \frac{a_2}{a_0}, ..., s_n = (-1)^n \frac{a_n}{a_0};\ i.e., s_r = (-1)^r \frac{a_r}{a_0} \text{ for } 1 \le r \le n.$$

(ii) For $n=2,$ we get a quadratic equation $x^2 + p_1x + p_2 = 0.$ Let α_1 and α_2 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 = -p_1$ and $s_2 = \alpha_1\alpha_2 = p_2$.

(iii) For $n=3,$ we get a cubic equation $x^3 + p_1x^2 + p_2x + p_3 = 0.$ Let α_1, α_2 and α_3 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 + \alpha_3 = -p_1,\ s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = p_2$ and $s_3 = \alpha_1\alpha_2\alpha_3 = p_3$.

(iv) For $n=4,$ we get a bi-quadratic equation $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0.$

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be its roots.

Then $s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1,\ s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 = p_2,$

$$s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_4 = -p_3 \text{ and } s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = p_4.$$

Answers
Exercise 2(a)

3. $\text{K} = 1, \text{K} = 4$

4. (i) real & equal (ii) rational & distinct (iii) real & distinct
(iv) rational & equal (v) rational & distinct (vi) conjugate complex numbers

3. (i) $mnx^2 + \left(n^2 - m^2\right)x - mn = 0$ $(ii) x^2 + 6x + 34 = 0$

4. (i) $\frac{1}{3}$ (ii) $\frac{-5 \pm \sqrt{41}}{2}$ $(iii) -\frac{1}{2}, 1$ $(iv) 3, 5$

5. (i) minimum value $\frac{32}{3}$ (ii) maximum value $-\frac{34}{3}$ (iii) maximum value $\frac{14}{5}$

(iv) minimum value $-\frac{1}{4}$ (v) maximum value $\frac{49}{3}$ (vi) minimum value $-\frac{29}{4}$

(vii) maximum value $\frac{16}{3}$ (vi) maximum value 4

Exercise 2(b)

3. $(i)\, x^3-6x^2+11x-6=0$ $(ii)\, x^3+6x^2+11x+6=0$ $(iii)\, x^3-3x^2-4x+12=0$

   $(iv)\, x^3-12x+16=0$ $(v)\, x^3-3x+52=0$

4. $(i)-1$ $(ii)2$ $(iii)-1$ $(iv)\,2$ $(v)1$ 3. $\frac{7}{4}$ 4. 36 5. $-\frac{3}{2}$ 6. -5

7. $(i)\,\frac{p^2-2q}{r^2}$ $(ii)-\frac{q}{r}$ 8. $3, 4, -4$

9. $pq=r$ 10. $\sum\alpha=\frac{10}{3}, \sum\alpha\beta=7, \alpha\beta\gamma=-\frac{10}{3}$

11. $(i)\, x^4-8x^2+16=0$ $(ii)\, x^4-16x^3+86x^2-176x+105=0$

3. BINOMIAL THEOREM

Introduction:

Binomial means two terms connected by either $+$ or $-$. We have come across many expansions of Squares, Cubes etc. of a binomial in earlier classes. For example

$$(x+y)^1 = x^1 + y^1 = x+y$$
$$(x-y)^1 = x^1 - y^1 = x-y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Each of these is a expansion of a power of the sum or the difference of two terms. These are called *binomial expansions*. The quotients $1,1$ in the expansion of $(x+y)^1$, $1,2,1$ in the expansion of $(x+y)^2$, $1,3,3,1$ in the expansion of $(x+y)^3$, $1,4,6,4,1$ in the expansion of $(x+y)^4$ etc. are called *binomial coefficients.* From the above examples we observe that the coefficients in these expansions are as follows.

Index	*Coefficients*
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	....

Fig 3.1

From the above diagram we observe the following pattern obtaining a row from the previous row from the second row onwards.

(i) Each row begins and ends with 1 (one).

(ii) The n^{th} row has $(n+1)$ terms for any $n \in Z^+$. gins and ends with 1 (one).

(iii) The other numbers (except the first and last) in a row are obtained by adding the two numbers in the previous row on either side of it.

The diagram in is called *Pascal triangle* which is named after its inventor, a French mathematician Blaise Pascal (1623-1662). But this was mentioned in a different form under a different name Meru-Prastara by the renowned Indian scientist Pingala in his book *Chanda Shastra* as early as 200 B.C.

The expansion of $(x+y)^n$ using multiplication, as shown in the beginning, become difficult as n increases. In this chapter, we derive the expansion of $(x+y)^n$ when n is a positive integer. This result is known as the *Binomial theorem*. The coefficients of the terms $x^i y^j$ are called *Binomial coefficients.* We study the properties of these binomial coefficients, give methods to find the middle term(s) finding out the coefficients of x^p and independent terms.

3.1Binomial Theorem for positive index:

In the expansion of $(x+y)^n$, mentioned in the introduction, we observe the following points.

(i) As we proceed from left to right, the index of x decreases and the index of y increases by 1 at a time.

(ii) The coefficients in the expansion of

$(x+y)^1$ are $1_{C_0}, 1_{C_1}$

$(x+y)^2$ are $2_{C_0}, 2_{C_1}, 2_{C_2}$

$(x+y)^3$ are $3_{C_0}, 3_{C_1}, 3_{C_2}, 3_{C_3}$

$(x+y)^4$ are $4_{C_0}, 4_{C_1}, 4_{C_2}, 4_{C_3}, 4_{C_4}$

From these observations we can easily guess the general formula for the expansion of $(x+y)^n$ for any positive integer n is shown in the following without proof.

3.1.1 Theorem (Binomial Theorem):

Let n be a positive integer and x, a be real numbers, then

$$(x+a)^n = n_{C_0} x^n.a^0 + n_{C_1} x^{n-1}.a^1 + n_{C_2} x^{n-2}.a^2 + ... + n_{C_r} x^{n-r}.a^r + ... + n_{C_n} x^0.a^n$$

3.1.2 Note:

Let n be a positive integer and x, a be real numbers, then

(i) $(x+a)^n = \sum_{r=0}^{n} n_{C_r} x^{n-r}.a^r$

(ii) The expansion of $\left(x+a\right)^n$ has $(n+1)$ terms.

In the following we define the general term in the expansion of $\left(x+a\right)^n$.

3.1.3 Definition:

In the expansion of $\left(x+a\right)^n$ the $(r+1)^{th}$ term is called the *general term* and it is given by $T_{r+1} = n_{C_r} x^{n-r}.a^r$ for $0 \le r \le n$.

In the expansion of $\left(x+a\right)^n, a$ is either a positive or a negative real number and hence there is no need to give the binomial expansion of $\left(x-a\right)^n$ separately. But still it will be useful to the reader to have the expansion of $\left(x-a\right)^n$ clearly.

3.1.4 Note:

On replacing a by $-a$ in the expansion of $\left(x+a\right)^n$ given in the theorem 3.1.1 we get

$$\left(x-a\right)^n = n_{C_0} x^n.(-a)^0 + n_{C_1} x^{n-1}.(-a)^1 + n_{C_2} x^{n-2}.(-a)^2 + \ldots + n_{C_r} x^{n-r}.(-a)^r + \ldots + n_{C_n} x^0.(-a)^n$$
$$= n_{C_0} x^n.a^0 - n_{C_1} x^{n-1}.a^1 + n_{C_2} x^{n-2}.a^2 - \ldots + (-1)^r n_{C_r} x^{n-r}.a^r + \ldots + (-a)^n n_{C_n} x^0.a^n$$

In this expansion the general term T_{r+1} is given by $T_{r+1} = (-1)^r n_{C_r} x^{n-r}.a^r$ for $0 \le r \le n.$

3.1.5 Definition: If n is even then the expansion of $\left(x+a\right)^n$ has $\left(n+1\right)$ number (odd number) of terms. Hence there is only one middle term, which is the $\left(\frac{n}{2}+1\right)^{th}$ term and $T_{\frac{n}{2}+1} = n_{C_{\frac{n}{2}}} x^{\frac{n}{2}}.a^{\frac{n}{2}}$.

If n is odd then the expansion of $\left(x+a\right)^n$ has $\left(n+1\right)$ number (even number) of terms. Hence there is two middle terms. They are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms. These terms are given by $T_{\frac{n+1}{2}} = n_{C_{\frac{n-1}{2}}} x^{\frac{n+1}{2}}.a^{\frac{n-1}{2}}$ and $T_{\frac{n+3}{2}} = n_{C_{\frac{n+1}{2}}} x^{\frac{n-1}{2}}.a^{\frac{n+1}{2}}$

3.2Problems on expansions, middle terms. Finding out coefficient of x^p and independent terms:

3.2.1 Solved Problems:

1. Problem: Write and simplify the first three terms of the expansion $\left(\frac{2x}{3}+\frac{7y}{4}\right)^5$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$

$i.e\ T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n=5, a=\frac{2}{3}, b=\frac{7}{4}$

Now $i.e\ T_{r+1} = 5_{c_r}\left(\frac{2x}{3}\right)^{5-r}\left(\frac{7y}{4}\right)^r$ ....(II)

since $n=5$ the number of terms are $6,$ hence first three terms are $1,2,3$

Put $r=0$ in equation (II) we get.

$$T_{0+1} = 5_{c_0}\left(\frac{2x}{3}\right)^{5-0}\left(\frac{7y}{4}\right)^0 \Rightarrow T_1 = 1.\left(\frac{2x}{3}\right)^5.1 \Rightarrow T_1 = \left(\frac{2x}{3}\right)^5$$

Put $r=1$ in equation (II) we get.

$$T_{1+1} = 5_{c_1}\left(\frac{2x}{3}\right)^{5-1}\left(\frac{7y}{4}\right)^1 \Rightarrow T_2 = 5\left(\frac{2x}{3}\right)^4\left(\frac{7y}{4}\right) \Rightarrow T_2 = \frac{80x^4}{81}.\frac{7y}{4} \Rightarrow T_2 = \frac{140x^4y}{81}$$

Put $r=2$ in equation (II) we get.

$$T_{2+1} = 5_{c_2}\left(\frac{2x}{3}\right)^{5-2}\left(\frac{7y}{4}\right)^2 \Rightarrow T_3 = 5_{c_2}\left(\frac{2x}{3}\right)^3\left(\frac{7y}{4}\right)^2 \Rightarrow T_3 = 10\left(\frac{8x^3}{27}\right)\left(\frac{49y^2}{16}\right)$$

$$\Rightarrow T_3 = \frac{245x^3y^2}{27}$$

2. Problem: Write down the last three terms of the expansion $(3x-4y)^{10}$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$

$i.e\ T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n=10, a=3, b=-4$

Now $i.e\ T_{r+1} = 10_{c_r}(3x)^{10-r}(-4y)^r$ ....(II)

since $n=10$ the number of terms are $11,$ hence last three terms are $9,10,11$

Put $r=8$ in equation (II) we get

$T_{8+1} = 10_{c_8}(3x)^{10-8}(-4y)^8 \Rightarrow T_9 = 45(3x)^2(-4y)^8 \Rightarrow T_9 = 405.4^8 x^2 y^8$
$\Rightarrow T_9 = 26542080x^2y^8$

Put $r = 9$ in equation (II) we get

$T_{9+1} = 10_{c_9}(3x)^{10-9}(-4y)^9 \Rightarrow T_{10} = 10(3x)^1(-4y)^9 \Rightarrow T_{10} = -30.4^9 xy^9$

Put $r = 10$ in equation (II) we get

$T_{10+1} = 10_{c_{10}}(3x)^{10-10}(-4y)^{10} \Rightarrow T_{11} = (4y)^{10} \Rightarrow T_{11} = 4^{10} y^{10}$

3. Problem: Write the expansion of $(2a+3b)^6$

Solution: We have from binomial theorem
$(x+a)^n = n_{C_0} x^n.a^0 + n_{C_1} x^{n-1}.a^1 + n_{C_2} x^{n-2}.a^2 + ... + n_{C_r} x^{n-r}.a^r + ... + n_{C_n} x^0.a^n$

Take $n = 6, x = 2a, a = 3b$

Now
$$(2a+3b)^6 = 6_{C_0}(2a)^6(3b)^0 + 6_{C_1}(2a)^{6-1}(3b)^1 + 6_{C_2}(2a)^{6-2}(3b)^2$$
$$+ 6_{C_3}(2a)^{6-3}(3b)^3 + 6_{C_4}(2a)^{6-4}(3b)^4 + 6_{C_5}(2a)^{6-5}(3b)^5 + 6_{C_6}(2a)^0(3b)^6$$
$$= 1.2^6.a^6.1 + 6.2^5.a^5.3.b + 15.2^4.a^4.3^2.b^2$$
$$+ 20.2^3.a^3.3^3.b^3 + 15.2^2.a^2.3^4.b^4 + 6.2.a.3^5.b^5 + 1.1.3^6.b^6$$

$$= 64a^6 + 576a^5b + 2160a^4b^2$$
$$+ 4320a^3b^3 + 4860a^2b^4 + 2916ab^5 + 729b^6$$

4. Problem: Find the 5^{th} term in the expansion of $(3x-4y)^7$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$
i.e $T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n = 7, a = 3, b = -4$

Now *i.e* $T_{r+1} = 7_{c_r}(3x)^{7-r}(-4y)^r$ ....(II)

since $n = 7$ the number of terms are 8, so that the 5^{th} term is T_5.

No $T_5 = T_{4+1} = 7_{c_4}(3x)^{7-4}(-4y)^4 = 7_{c_4}(3x)^3(-4y)^4 = (-1)^4 7_{c_4}3^3.x^3.4^4.y^4 = 7_{c_4}3^3.4^4.x^3.y^4$

5. Problem: Find the 4^{th} term from the end in the expansion of $(2a+5b)^8$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$

i.e $T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n=8, x=2, y=5$

Now *i.e* $T_{r+1} = 8_{c_r}(2a)^{8-r}(5b)^r$ ....(II)

since $n=8$ the number of terms are 9, so that the 4^{th} term from the right is 6^{th} term

Now $T_6 = T_{5+1} = 8_{c_5}(2a)^{8-5}(5b)^5 = 8_{c_5}(2a)^3(5b)^5 = 8_{c_5}2^3.a^3.5^5.b^5 = 8_{c_5}2^3.5^5.a^3.b^5$

6. Problem: Find the middle term (s) in the expansion of $(3a-5b)^6$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$

i.e $T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n=6, x=3, y=-5$

Now *i.e* $T_{r+1} = 6_{c_r}(3a)^{6-r}(-5b)^r$ ....(II)

since $n=6$ the number of terms are 7 (odd number), hence there is only one middle term and it is T_4.

Now $T_4 = T_{3+1} = 6_{c_3}(3a)^{6-3}(-5b)^3 = -6_{c_3}3^3.a^3.5^3.b^3 = -6_{c_3}15^3.a^3.b^3$

7. Problem: Find the middle term (s) in the expansion of $(2x+3y)^7$

Solution: We have T_{r+1} is the general term in the expansion of $(ax+by)^n$

i.e $T_{r+1} = n_{c_r}(ax)^{n-r}(by)^r$ ....(I)

Take $n=7, a=2, b=3$

Now *i.e* $T_{r+1} = 7_{c_r}(2x)^{7-r}(3y)^r$ ....(II)

since $n=7$ the number of terms are 8 (even number), hence there are two middle terms. They are T_4 and T_5.

Now $T_4 = T_{3+1} = 7_{c_3}(2x)^{7-3}(3y)^3 = 7_{c_3}(2x)^4(3y)^3 = 7_{c_3}2^4.x^4.3^3.y^3 = 7_{c_3}2^4.3^3.x^4.y^3$

$$T_5 = T_{4+1} = 7_{c_4}(2x)^{7-4}(3y)^4 = 7_{c_4}(2x)^3(3y)^4 = 7_{c_4}2^3.x^3.3^4.y^4 = 7_{c_4}2^3.3^4.x^3.y^4$$

8. Problem: Find the coefficient of x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$

Solution: We have T_{r+1} is the general term in the expansion of $(X+Y)^n$

i.e $T_{r+1} = n_{c_r}(X)^{n-r}(Y)^r$ ....(I)

Take $n = 20, X = 2x^2, Y = -\frac{1}{x}$

Then $T_{r+1} = 20_{c_r}\left(2x^2\right)^{20-r}\left(-\frac{1}{x}\right)^r = 20_{c_r}2^{20-r}\left(x^2\right)^{20-r}.\frac{(-1)^r}{x^r} = (-1)^r\, 20_{c_r}2^{20-r}x^{40-3r}$ ...(II)

To find the coefficient of x^{10} in the expansion, we should consider $40-3r=10 \Rightarrow r=10.$

Hence the coefficient of x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$ is $= (-1)^{10}\, 20_{c_{10}}2^{20-10}$

$$= 20_{c_{10}}2^{10}$$

9. Problem: Find the term independent of x (*i.e* the constant term) in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$

Solution: We have T_{r+1} is the general term in the expansion of $(X+Y)^n$

i.e $T_{r+1} = n_{c_r}(X)^{n-r}(Y)^r$ ....(I)

Take $n = 10, X = \sqrt{\frac{x}{3}}, Y = \frac{3}{2x^2}$.

Then $T_{r+1} = 10_{c_r}\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{3}{2x^2}\right)^r = 10_{c_r}\left(\left(\frac{x}{3}\right)^{\frac{1}{2}}\right)^{10-r}\left(\frac{3}{2x^2}\right)^r$

$$= 10_{c_r}\left(\frac{x^{\frac{10-r}{2}}}{3^{\frac{10-r}{2}}}\right)\frac{3^r}{2^r x^{2r}} = 10_{c_r}x^{\frac{10-r}{2}-2r}\frac{1}{2^r.3^{\frac{10-r}{2}-r}} = 10_{c_r}x^{\frac{10-5r}{2}}\frac{1}{2^r.3^{\frac{10-3r}{2}}} \quad \text{.......(II)}$$

To find the term independent of x in the expansion, we should consider

$$\frac{10-5r}{2} = 0 \Rightarrow r = 2.$$

Therefore, the term independent of x in the given expansion is

$$T_3 = T_{2+1} = 10_{c_2} x^0 \frac{1}{2^2 . 3^{\frac{10-3.2}{2}}}$$

$$\therefore T_3 = \frac{5}{4}.$$

10. Problem: If the coefficient of x^{10} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-10} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, find the relation between a and b where a and b are real numbers.

Solution: We have T_{r+1} is the general term in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

$$i.e\ T_{r+1} = 11_{c_r} \left(ax^2\right)^{11-r} \left(\frac{1}{bx}\right)^r = 11_{c_r} a^{11-r} x^{22-2r} \frac{1}{b^r x^r} = 11_{c_r} \frac{a^{11-r}}{b^r} x^{22-3r}$$

To find the coefficient of x^{10} in the expansion, we should consider $22 - 3r = 10 \Rightarrow r = 4.$

Hence the coefficient of x^{10} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is $= 11_{c_4} \frac{a^7}{b^4}$.......(I)

We have T_{r+1} is the general term in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

$$i.e\ T_{r+1} = 11_{c_r} (ax)^{11-r} \left(\frac{-1}{bx^2}\right)^r = (-1)^r 11_{c_r} a^{11-r} x^{11-r} \frac{1}{b^r x^{2r}} = (-1)^r 11_{c_r} \frac{a^{11-r}}{b^r} x^{11-3r}$$

To find the coefficient of x^{-10} in the expansion, we should consider
$11 - 3r = -10 \Rightarrow r = 7.$

Hence the coefficient of x^{-10} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is $= -11_{c_7} \frac{a^4}{b^7}$.......(II)

Hence from equations (I) and (II), we get $11_{c_4} \frac{a^7}{b^4} = -11_{c_7} \frac{a^4}{b^7}$

$$i.e\ \frac{a^7}{b^4} = -\frac{a^4}{b^7} \left(\because 11_{c_4} = 11_{c_7}\right)$$

$$i.e\ a^3 = -\frac{1}{b^3}$$

$i.e \quad a^3b^3 = -1$

$i.e \quad ab = -1 \; (\because a, b \text{ are real.})$

Exercise 3(a)

I Expand the following using binomial theorem

1.$(4x+5y)^7$ 2.$\left(\frac{2x}{3}+\frac{7y}{4}\right)^5$ 3.$\left(\frac{2p}{5}-\frac{3q}{7}\right)^6$

II Write down and simplify the following

1. 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$ 2. 7^{th} term in $(3x-4y)^{10}$ 3. 10^{th} term in $\left(\frac{3p}{4}-5q\right)^{14}$

4. 7^{th} term in $\left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14}$ 5. 3^{rd} term from the end $\left(x^{\frac{-2}{3}}-\frac{3}{x^2}\right)^8$

III Find the number of terms in the following expansions. Also find the middle term (s) in each expansion.

1. $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ 2. $(3p+4q)^{14}$ 3. $(2x+3y)^7$

IV Find the coefficient of

1. x^{-6} in $\left(3x-\frac{4}{x}\right)^{10}$ 2. x^{11} in $\left(2x^2+\frac{3}{x^3}\right)^{13}$ 3. x^{-7} in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$

4. x^2 in $\left(7x^3-\frac{2}{x^2}\right)^9$ 5. x^9 in $\left(2x^2-\frac{1}{x}\right)^{20}$ 6. x^{10} in $\left(ax^2+\frac{1}{bx}\right)^{11}$

V Find the term independent of x in the expansion of

1. $\left(\frac{\sqrt{x}}{3}-\frac{4}{x^2}\right)^{10}$ 2. $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$ 3. $\left(4x^3+\frac{7}{x^2}\right)^{14}$ 4. $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^9$

VI Find the middle term (s) in each expansion.

1. $\left(\frac{3x}{7}-2y\right)^{10}$ 2. $\left(4a+\frac{3b}{2}\right)^{11}$ 3. $(4x^2+5x^3)^{17}$ 4. $\left(\frac{3}{a^3}+5a^4\right)^{20}$

Key concepts

1. **Binomial Theorem:** Let n be a positive integer and x, a be real numbers, then

$$\left(x+a\right)^n = n_{C_0} x^n.a^0 + n_{C_1} x^{n-1}.a^1 + n_{C_2} x^{n-2}.a^2 + ... + n_{C_r} x^{n-r}.a^r + ... + n_{C_n} x^0.a^n$$

2. Let n be a positive integer and x, a be real numbers, then

(i) $\left(x+a\right)^n = \sum_{r=0}^{n} n_{C_r} x^{n-r}.a^r$ (ii) The expansion of $\left(x+a\right)^n$ has $(n+1)$ terms.

3. In the expansion of $\left(x+a\right)^n$ the $(r+1)^{th}$ term is called the *general term* and it is given by $T_{r+1} = n_{C_r} x^{n-r}.a^r$ for $0 \le r \le n.$

4. $\left(x-a\right)^n = n_{C_0} x^n.a^0 - n_{C_1} x^{n-1}.a^1 + n_{C_2} x^{n-2}.a^2 - ... + (-1)^r n_{C_r} x^{n-r}.a^r + ... + (-a)^n n_{C_n} x^0.a^n$

In this expansion the general term T_{r+1} is given by $T_{r+1} = (-1)^r n_{C_r} x^{n-r}.a^r$ for $0 \le r \le n.$

5. If n is even then the expansion of $\left(x+a\right)^n$ has $\left(n+1\right)$ number (odd number) of terms.

Hence there is only one middle term, which is the $\left(\frac{n}{2}+1\right)^{th}$ term and $T_{\frac{n}{2}+1} = n_{C_{\frac{n}{2}}} x^{\frac{n}{2}}.a^{\frac{n}{2}}$

If n is odd then the expansion of $\left(x+a\right)^n$ has $\left(n+1\right)$ number (even number) of terms.

Hence there is two middle terms. They are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms. These terms are given by $T_{\frac{n+1}{2}} = n_{C_{\frac{n-1}{2}}} x^{\frac{n+1}{2}}.a^{\frac{n-1}{2}}$ and $T_{\frac{n+3}{2}} = n_{C_{\frac{n+1}{2}}} x^{\frac{n-1}{2}}.a^{\frac{n+1}{2}}$

Answers

Exercise 3(a)

I 1. $\sum_{r=0}^{7} 7_{c_r} \left(4x\right)^{7-r} \left(5y\right)^r$ 2. $\sum_{r=0}^{5} 5_{c_r} \left(\frac{2}{3}x\right)^{5-r} \left(\frac{7}{4}y\right)^r$ 3. $\sum_{r=0}^{6} \left(-1\right)^r 6_{c_r} \left(\frac{2p}{5}\right)^{6-r} \left(\frac{3q}{7}\right)^r$

II 1. $189x^4y^5$ 2. $280\left(12\right)^5 x^4 y^6$ 3. $\frac{-2002\left(3\right)^5\left(5\right)^9}{\left(4\right)^5} p^5 q^9$ 4. $14_{c_6}.\frac{4^5}{x^{12}}$ 5. $\frac{28 \times 3^6}{x^{40/3}}$

III 1. 10; $9_{c_4}\left(\frac{3a}{4}\right)^5\left(\frac{b}{2}\right)^4, 9_{c_5}\left(\frac{3a}{4}\right)^4\left(\frac{b}{2}\right)^5$ 2. 15; $14_{c_7}\left(3p\right)^7\left(4q\right)^7$

3. 8; $7_{c_3}(2x)^4(3y)^3, 7_{c_4}(2x)^3(3y)^4$

IV 1. 405×4^8 2. $286\times2^{10}\times3^3$ 3. $\frac{-4375}{324}$ 4. $-126\times7^4\times2^5$ 5. 0 6. $11_{c_4}\frac{a^7}{b^4}$.

V 1. $T_3=\frac{80}{729}$ 2. $T_{11}=25_{c_{10}}\times3^{15}\times5^{10}$ 3. 0 4. $T_7=\frac{3^7\times5^7\times7}{2^7}$

VI 1. $T_6=-10_{c_5}.\left(\frac{6}{7}\right)^5.x^5.y^5$ 2. $T_6=77\times2^8\times3^6\times a^6b^5$ & $T_7=77\times2^5\times3^7\times a^5b^6$

3. $T_9=17_{c_8}\times4^9\times5^8\times x^{42}$ & $T_{10}=17_{c_9}\times4^8\times5^9\times x^{43}$ 4. $T_{11}=20_{c_{10}}15^{10}a^{10}$

4. PARTIAL FRACTIONS

Introduction:

In chapter four we have defined a polynomial in x of degree n and learnt the methods of finding roots of polynomial equations. We frequently come across quotient of polynomials, which we may call polynomial fractions or rational fractions or simply fractions. We often require expansion of these fractions in power series. Therefore, we have to express the fractions like $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x) \neq 0$ are polynomials as the sum of certain terms known as partial fractions. In this chapter we learnt the partial fraction decomposition of a fraction. It is useful in many situations like finding particular integrals of differential equations, evaluation of integrals expanding infinite series in some cases, summing the infinite series etc.

4.1Rational Fractions:

In the next section, we describe some concepts that are required for our discussion and we define the term rational fraction, reducible and irreducible polynomial. Throughout this chapter we consider polynomials with real coefficients. Henceforth a polynomial means a polynomial with real coefficients.

4.1.1 Definition (Rational Fraction): If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then $\frac{f(x)}{g(x)}$ is called a rational fraction or polynomial fraction or simply a fraction.

Examples: $\frac{5x+1}{(x-1)(x+2)}$ and $\frac{x^3+5x+7}{x^2-2}$ are rational fractions.

4.1.2 Definition (Proper and Improper Fractions): A rational fraction $\frac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than the degree of $g(x)$. Otherwise it is called an improper fraction.

Examples: (i) $\frac{5x+1}{(x-1)(x+2)}$ is a proper fraction.

(ii) $\frac{x^3}{x^2-3x+2}$ is an improper fraction.

(iii) $\frac{x^3}{(2x-1)(x+2)(x-3)}$ is an improper fraction.

4.1.3 Definition (Irreducible Polynomial): A polynomial $f(x)$ is said to be irreducible if it cannot be expressed as a product of two polynomials $g(x)$ and $h(x)$ such that the degree of each polynomial is less than the degree of $f(x)$. If $f(x)$ is not irreducible then we say that $f(x)$ is reducible.

Examples:

(i) $2x+1$ is an irreducible polynomial.

(ii) x^2+x+2 is an irreducible polynomial.

(iii) $x^3-6x^2+11x-6$ is a reducible polynomial, since

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$$

(iv) $x^2-4x+13$ is an irreducible polynomial, since it cannot be expressed as product of linear polynomials whose coefficients are real. Though $x^2-4x+13=[x-(2+3i)][x-(2-3i)]$, it is not irreducible as the coefficients of the fractions on the right hand side are not real.

4.1.4 Note :

(i) Every linear polynomial is an irreducible polynomial.

For example $2x+1$ is irreducible

(ii) If $a\neq 0, ax^2+bx+c$ is irreducible iff $b^2-4ac<0$. For example x^2-x+1 is irreducible polynomial since $b^2-4ac=-3<0$.

4.1.5 Division Algorithm for polynomials:

We state the division algorithm without proof.

If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=q(x)g(x)+r(x)$ where either $r(x)=0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

4.1.6 Definition (Proper Fraction): If a proper fraction is expressed as the sum of two or more proper fractions, where in the denominators are powers of irreducible polynomials, then each proper fraction in the sum is called a partial fraction of the given fraction.

4.2Non repeated linear factors, repeated linear factors and irreducible non repeated linear factors:

In earlier classes we learnt how to add two proper fractions to get another proper fraction. For example $\frac{3}{2x+5}+\frac{5}{x+6}=\frac{13x+43}{2x^2+17x+30}$. Here $\frac{3}{2x+5},\frac{5}{x+6}$ are called partial fraction of $\frac{13x+43}{2x^2+17x+30}$.

Now we learn using some rules to express a proper fraction as a sum of two or more proper fractions. This process is known as 'resolving into partial fractions'. Resolution of a proper fraction $\frac{f(x)}{g(x)}$ into a sum of partial fractions depends upon the factorization of $g(x)$ into linear and/o r irreducible (quadratic) factors. We assume that such a decomposition is possible and is unique. We list out here under some useful rules for resolution of $\frac{f(x)}{g(x)}$ into a sum of partial fractions without presenting proofs.

4.2.1 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains non-repeated linear factors:

We find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains non-repeated linear factors, we use the following rule.

4.2.2 Rule I:

Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated factor $(ax+b)$ of $g(x)$ there will be partial fraction of the form $\frac{\mathrm{A}}{ax+b}$ where A is a non-zero real number, to be determined.

4.2.3 Solved Problems:

1. Problem: Resolve $\frac{5x+1}{(x-1)(x+2)}$ into partial fractions.

Solution: Let $\frac{5x+1}{(x-1)(x+2)}=\frac{\mathrm{A}}{x-1}+\frac{\mathrm{B}}{x+2}$..........(I)

where A,B are constants to be determined.

From equation (I) we get $\frac{5x+1}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$

$\therefore A(x+2)+B(x-1) = 5x+1$..........(II)

Put $x=1$ in equation (II) we get

$A(1+2)+B(1-1) = 5.1+1 \Rightarrow A(3)+B(0) = 5+1 \Rightarrow 3A = 6 \Rightarrow A = 2$

Put $x=-2$ in equation (II) we get

$A(-2+2)+B(-2-1) = 5(-2)+1 \Rightarrow A(0)+B(-3) = -10+1 \Rightarrow -3B = -9 \Rightarrow B = 3$

Now substitute the values of $A \& B$ in equation (I) we get

$$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$$

2. Problem: Resolve $\frac{2x+3}{(x+2)(2x+1)}$ into partial fractions.

Solution: Let $\frac{2x+3}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1}$..........(I)

where A, B are constants to be determined.

$$\therefore \frac{2x+3}{(x+2)(2x+1)} = \frac{A(2x+1)+B(x+2)}{(x+2)(2x+1)}$$

$\therefore A(2x+1)+B(x+2) = 2x+3$..........(II)

Put $x=-2$ in equation (II) we get

$A(2(-2)+1)+B(-2+2) = 2(-2)+3 \Rightarrow A(-4+1)+B(0) = -4+3 \Rightarrow -3A = -1 \Rightarrow A = \frac{1}{3}$

Put $x = -\frac{1}{2}$ in equation (II) we get

$A(2(-\frac{1}{2})+1)+B(-\frac{1}{2}+2) = 2(-\frac{1}{2})+3 \Rightarrow A(-1+1)+B(\frac{3}{2}) = 2 \Rightarrow \frac{3}{2}B = 2 \Rightarrow B = \frac{4}{3}$

Now substitute the values of $A \& B$ in equation (I) we get

$$\frac{2x+3}{(x+2)(2x+1)} = \frac{1/3}{x+2} + \frac{4/3}{2x+1}$$

4.2.4 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains repeated and/or non-repeated linear factors:

We find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains repeated linear factors, we use the following rule.

4.2.5 Rule II:

Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated factor $(ax+b)^n, a \neq 0,$ where n is a positive integer, of $g(x)$ there will be partial fractions of the form $\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots\frac{A_n}{(ax+b)^n}$ where $A_1, A_2, A_3, \ldots A_n$ are to be determined constants. Note that $A_n \neq 0$ and Rule I is a particular case of Rule II for $n=1$.

4.2.6 Solved Problems:

1. Problem: Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

Solution: Let $\frac{x^2+5x+7}{(x-3)^3}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}$..........(I)

where A, B, C are constants to be determined.

From equation (I) we get $\frac{x^2+5x+7}{(x-3)^3}=\frac{A(x-3)^2+B(x-3)+C}{(x-3)^3}$

$\therefore A(x-3)^2+B(x-3)+C=x^2+5x+7$..........(II)

Put $x=3$ in equation (II) we get

$A(3-3)^2+B(3-3)+C=3^2+5.3+7 \Rightarrow A(0)^2+B(0)+C=9+15+7 \Rightarrow C=31$

Now equating on both sides the coefficient of x^2, x and constant we get

$A=1, B-6A=5, 9A-3B+C=7$

Solving these equations, we get $A=1, B=11, C=31$

Now substitute the values of A,B&C in equation (I) we get

$$\frac{x^2+5x+7}{(x-3)^3}=\frac{1}{x-3}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}$$

Another Method

Let $\frac{x^2+5x+7}{(x-3)^3}$..........(I)

Put $x-3=y \Rightarrow x=y+3$ in equation (I) we get

$$\frac{(y+3)^2+5(y+3)+7}{(y)^3}=\frac{(y^2+6y+9)+(5y+15)+7}{y^3}=\frac{y^2+11y+31}{y^3}=\frac{y^2}{y^3}+\frac{11y}{y^3}+\frac{31}{y^3}$$

$$=\frac{1}{y}+\frac{11}{y^2}+\frac{31}{y^3}=\frac{1}{x-3}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}$$

2. Problem: Resolve $\frac{x-1}{(x-2)^2(x+1)}$ into partial fractions.

Solution: Let $\frac{x-1}{(x-2)^2(x+1)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}$..........(I)

$$\Rightarrow \frac{x-1}{(x-2)^2(x+1)}=\frac{A(x-2)(x+1)+B(x+1)+C(x-2)^2}{(x-2)^2(x+1)}$$

$$\Rightarrow \frac{x-1}{(x-2)^2(x+1)}=\frac{A(x-2)(x+1)+B(x+1)+C(x-2)^2}{(x-2)^2(x+1)}$$

$\Rightarrow A(x-2)(x+1)+B(x+1)+C(x-2)^2=x-1$..........(II)

Put $x=2$ in equation (II) we get

$$A(2-2)(2+1)+B(2+1)+C(2-2)^2=2-1$$

$$\Rightarrow A(0)(3)+B(3)+C(0)^2=1 \Rightarrow 3B=1 \Rightarrow B=\frac{1}{3}$$

Put $x=-1$ in equation (II) we get

$$A(-1-2)(-1+1)+B(-1+1)+C(-1-2)^2=-1-1$$

$$\Rightarrow A(-3)(0)+B(0)+C(-3)^2=-2 \Rightarrow 9C=-2 \Rightarrow C=-\frac{2}{9}$$

equation (II) $\Rightarrow A(x^2-x-2)+B(x+1)+C(x^2-4x+4)=x-1$

$$\Rightarrow (A+C)x^2+(-A+B-4C)x+(-2A+B+4C)=x-1$$

Now equating on both sides the coefficient of x^2 we get

$$A+C=0 \Rightarrow A=-C=\frac{2}{9}$$

Now substitute the values of A,B&C in equation (I) we get

$$\frac{x-1}{(x-2)^2(x+1)}=\frac{2}{9(x-2)}+\frac{1}{3(x-2)^2}+\frac{-2}{9(x+1)}$$

3. Problem: Resolve $\frac{x+4}{(x^2-4)(x+1)}$ into partial fractions.

Solution: Let $\frac{x+4}{(x^2-4)(x+1)}=\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+1}$..........(I)

$$\Rightarrow \frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A(x+2)(x+1)+B(x-2)(x+1)+C(x+2)(x-2)}{(x-2)(x+2)(x+1)}$$

$$\Rightarrow A(x+2)(x+1)+B(x-2)(x+1)+C(x+2)(x-2)=x+4..........(II)$$

Put $x=2$ in equation (II) we get

$$A(2+2)(2+1)+B(2-2)(2+1)+C(2+2)(2-2)=2+4$$

$$\Rightarrow A(4)(3)+B(0)(3)+C(4)(0)=6 \Rightarrow 12A=6 \Rightarrow A=\frac{1}{2}$$

Put $x=-2$ in equation (II) we get

$$A(-2+2)(-2+1)+B(-2-2)(-2+1)+C(-2+2)(-2-2)=-2+4$$

$$\Rightarrow A(0)(-1)+B(-4)(-1)+C(0)(-4)=2 \Rightarrow 4B=2 \Rightarrow B=\frac{1}{2}$$

Put $x=-1$ in equation (II) we get

$$A(-1+2)(-1+1)+B(-1-2)(-1+1)+C(-1+2)(-1-2)=-1+4$$

$$\Rightarrow A(1)(0)+B(-3)(0)+C(1)(-3)=3 \Rightarrow -3C=3 \Rightarrow C=-1$$

Now substitute the values of A,B&C in equation (I) we get

$$\frac{x+4}{(x^2-4)(x+1)}=\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{1}{2(x-2)}+\frac{1}{2(x+2)}+\frac{-1}{x+1}$$

4.2.7 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains irreducible factors:

We find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains non-repeated irreducible factors, we use the following rule.

4.2.8 Rule III:

Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated quadratic factor $ax^2+bx+c, a \neq 0$ where n is a positive integer, of $g(x)$ there will be partial fraction of the form $\frac{\mathrm{A}x+\mathrm{B}}{ax^2+bx+c}$ where A, B are real numbers to be determined.

4.2.9 Solved Problems:

1. Problem: Resolve $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$ into partial fractions.

Solution: Let $\frac{2x^2+3x+4}{(x-1)(x^2+2)} = \frac{\mathrm{A}}{x-1} + \frac{\mathrm{B}x+\mathrm{C}}{x^2+2}$..........(I)

$$\Rightarrow \frac{2x^2+3x+4}{(x-1)(x^2+2)} = \frac{\mathrm{A}(x^2+2)+(\mathrm{B}x+\mathrm{C})(x-1)}{(x-1)(x^2+2)}$$

$$\Rightarrow \mathrm{A}(x^2+2)+(\mathrm{B}x+\mathrm{C})(x-1) = 2x^2+3x+4 \text{..........(II)}$$

Put $x=1$ in equation (II) we get

$$\mathrm{A}(1^2+2)+(\mathrm{B}.1+\mathrm{C})(1-1) = 2.1^2+3.1+4$$

$$\Rightarrow \mathrm{A}(3)+(\mathrm{B}+\mathrm{C})(0) = 2+3+4 \Rightarrow 3\mathrm{A}=9 \Rightarrow \mathrm{A}=3$$

equation (II) $\Rightarrow \mathrm{A}(x^2+2)+\mathrm{B}x(x-1)+\mathrm{C}(x-1) = 2x^2+3x+4$

$$\Rightarrow \mathrm{A}(x^2+2)+\mathrm{B}(x^2-x)+\mathrm{C}(x-1) = 2x^2+3x+4$$

$$\Rightarrow (\mathrm{A}+\mathrm{B})x^2+(-\mathrm{B}+\mathrm{C})x+(2\mathrm{A}-\mathrm{C}) = 2x^2+3x+4$$

Now equating on both sides the coefficient of x^2 we get

$$\mathrm{A}+\mathrm{B}=2 \Rightarrow \mathrm{B}=2-\mathrm{A} \Rightarrow \mathrm{B}=2-3 \Rightarrow \mathrm{B}=-1$$

Now equating on both sides the coefficient of x we get

$$-\mathrm{B}+\mathrm{C}=3 \Rightarrow \mathrm{C}=3+\mathrm{B} \Rightarrow \mathrm{C}=3-1 \Rightarrow \mathrm{C}=2$$

Now substitute the values of A,B&C in equation (I) we get

$$\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{3}{x-1}+\frac{-x+2}{x^2+2}$$

2. Problem: Resolve $\frac{1}{(x^2+4)(x^2+9)}$ into partial fractions.

Solution: Let $\frac{1}{(x^2+4)(x^2+9)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+9}$..........(I)

$$\Rightarrow \frac{1}{(x^2+4)(x^2+9)}=\frac{(Ax+B)(x^2+9)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+9)}$$

$$\Rightarrow (Ax+B)(x^2+9)+(Cx+D)(x^2+4)=1\text{..........(II)}$$

equation (II) $\Rightarrow Ax(x^2+9)+B(x^2+9)+Cx(x^2+4)+D(x^2+4)=1$

$\Rightarrow A(x^3+9x)+B(x^2+9)+C(x^3+4x)+D(x^2+9)=1$

$\Rightarrow (A+C)x^3+(B+D)x^2+(9A+4C)x+(9B+4D)=1$

Now equating on both sides the coefficients of x^3, x^2, x and the constant value we get

$A+C=0, B+D=0, 9A+4C=0, 9B+4D=1$

Now by solving $A+C=0, 9A+4C=0$ $\Rightarrow$ we get $A=0, C=0$

Now by solving $B+D=0, 9B+4D=1$ $\Rightarrow$ we get $B=\frac{1}{5}, D=-\frac{1}{5}$

Now substitute the values of A,B,C&D in equation (I) we get

$$\frac{1}{(x^2+4)(x^2+9)}=\frac{1}{5(x^2+4)}-\frac{1}{5(x^2+9)}$$

Exercise 4(a)

I Resolve the following functions into partial fractions

1. $\frac{x}{(x+1)(2x+1)}$ 2. $\frac{2x+1}{(x-1)(2x+3)}$ 3. $\frac{x-4}{(x-2)(x-3)}$ 4. $\frac{x-1}{(x-2)(x+3)}$ 5. $\frac{1}{(x-3)(x+1)}$

6. $\frac{x}{(x-3)(x+2)}$ 7. $\frac{1}{(x-1)(x-3)}$ 8. $\frac{1}{(x+1)(x-8)}$ 9. $\frac{x-4}{x^2-5x+6}$ 10. $\frac{13x+43}{(x+6)(2x+5)}$

11. $\frac{5x+6}{(x-1)(x+2)}$ 12. $\frac{x+4}{(x^2-4)(x+1)}$ 13. $\frac{2x+3}{(x-1)^3}$ 14. $\frac{1}{(x-1)^2(x-2)}$ 15. $\frac{1}{(1-2x)^2(1-3x)}$

16. $\frac{x^2-x+1}{(x-1)^2(x+1)}$ 17. $\frac{1}{(x^2+16)(x^2+9)}$ 18. $\frac{x^2-3}{(x+2)(x^2+1)}$ 19. $\frac{3x^2+2x}{(x-3)(x^2+2)}$

20. $\frac{3x}{(x-1)(x-2)^2}$ 21. $\frac{1}{(x^2+16)(x^2+25)}$

Key concepts

1. If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then $\frac{f(x)}{g(x)}$ is called a rational fraction or polynomial fraction or simply a fraction.

2. A rational fraction $\frac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than the degree of $g(x)$. Otherwise it is called an improper fraction.

3. A polynomial $f(x)$ is said to be irreducible if it cannot be expressed as a product of two polynomials $g(x)$ and $h(x)$ such that the degree of each polynomial is less than the degree of $f(x)$. If $f(x)$ is not irreducible then we say that $f(x)$ is reducible.

4. Every linear polynomial is an irreducible polynomial.

5. If $a \neq 0, ax^2+bx+c$ is irreducible iff $b^2-4ac<0$.

6. **Division Algorithm for polynomials:** If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=q(x)g(x)+r(x)$ where either $r(x)=0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

7. If a proper fraction is expressed as the sum of two or more proper fractions, where in the denominators are powers of irreducible polynomials, then each proper fraction in the sum is called a partial fraction of the given fraction.

Rule I: Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated factor $(ax+b)$ of $g(x)$ there will be partial fraction of the form $\frac{\text{A}}{ax+b}$ where A is a non-zero real number, to be determined.

Rule II: Let $\dfrac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated factor $(ax+b)^n, a \neq 0,$ where n is a positive integer, of $g(x)$ there will be partial fractions of the form $\dfrac{A_1}{(ax+b)}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}+\ldots\dfrac{A_n}{(ax+b)^n}$ where $A_1, A_2, A_3, \ldots A_n$ are to be determined constants. Note that $A_n \neq 0$ and Rule I is a particular case of Rule II for $n=1$.

Rule III: Let $\dfrac{f(x)}{g(x)}$ be a proper fraction. To each non-repeated quadratic factor $ax^2+bx+c, a \neq 0$ where n is a positive integer, of $g(x)$ there will be partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$ where A, B are real numbers to be determined.

Answers

Exercise 4(a)

1. $\dfrac{1}{x+1}-\dfrac{1}{2x+1}$ 2. $\dfrac{3/5}{x-1}+\dfrac{4/5}{2x+3}$ 3. $\dfrac{2}{x-2}-\dfrac{1}{x-3}$ 4. $\dfrac{1/5}{x-2}+\dfrac{4/5}{x+3}$ 5. $\dfrac{1/4}{x-3}-\dfrac{1/4}{x+1}$

6. $\dfrac{3/5}{x-3}+\dfrac{2/5}{x+2}$ 7. $\dfrac{-1/2}{x-1}+\dfrac{1/2}{x-3}$ 8. $\dfrac{1/9}{x-8}-\dfrac{1/9}{x+1}$ 9. $\dfrac{2}{x-2}-\dfrac{1}{x-3}$ 10. $\dfrac{5}{x+6}+\dfrac{3}{2x+5}$

11. $\dfrac{11/3}{x-1}+\dfrac{4/3}{x+2}$ 12. $\dfrac{1/2}{x-2}+\dfrac{1/2}{x+2}-\dfrac{1}{x+1}$ 13. $\dfrac{2}{(x-1)^2}+\dfrac{5}{(x-1)^3}$ 14. $\dfrac{-1}{x-1}-\dfrac{1}{(x-1)^2}+\dfrac{1}{x-2}$

15. $\dfrac{-6}{1-2x}-\dfrac{2}{(1-2x)^2}+\dfrac{9}{1-3x}$ 16. $\dfrac{-2}{x-1}-\dfrac{1}{(x-1)^2}+\dfrac{3}{x-2}$ 17. $\dfrac{1}{7}\left[\dfrac{1}{x^2+9}-\dfrac{1}{x^2+16}\right]$

18. $\dfrac{1}{5}\left[\dfrac{1}{x+2}+\dfrac{4x-8}{x^2+1}\right]$ 19. $\left[\dfrac{3}{x-3}+\dfrac{2}{x^2+2}\right]$ 20. $\dfrac{3}{x-1}-\dfrac{3}{x-2}+\dfrac{6}{(x-2)^2}$

21. $\dfrac{1}{9}\left[\dfrac{1}{x^2+16}-\dfrac{1}{x^2+25}\right]$

5. MEASURES OF DISPERSION

Introduction:

In the earlier classes you have studied some methods of representing data graphically and pictorially. This representation reveals certain salient features or characteristics of the data. We have learnt how to construct a frequency table for a given data and find various measures that provide a single representative value of the data, called a measure of central tendency. Recall that arithmetic mean, median, mode, geometric mean, and harmonic mean are the measures of central tendency. A measure of central tendency gives us a rough idea where data points are centered. But this single representative value cannot adequately describe the variation of a set of data. For example the mean of the set of values 2, 10, 87 is 33. The mean of the distribution 30, 32, 37 is 33. So, to make better representation of the data is spread or scattered or dispersed or how much they are bunched around a measure of central tendency.

A measure of dispersion or variation describes the spread or scattering the individual values around the central value. To illustrate the concept of dispersion, we shall consider the following example.

Consider the following data of runs scored by two batsmen A,B in the last ten test matches.

A	28	70	30	2	42	64	80	93	5	116
B	44	38	60	45	52	54	49	54	76	58

Note that the arithmetic mean $\sum \frac{x_i}{\mathrm{N}}$ of the scores of both the players is 53 and the median score $\left(\frac{n+1}{2}\right)$ if n is odd and $\frac{1}{2}\left(\frac{n}{2}+\frac{n+2}{2}\right)$ if n is even is also 53. Based on these values of measures of central tendency, can we say that the performance of these two batsmen is the same? The answer is clearly 'no' because the variability of the scores of A is from 2 to 116 whereas the variability of the runs scored by B is between 38 and 76. This measure 'variability' is another factor required to be studied in Statistics. Like a measure of central tendency, we have to know a measure to describe the variability. This measure is called 'measure of dispersion'.

Measuring dispersion of a data is significant because it determines the reliability of an average by pointing out as to how far an average is representative of the entire data. In this chapter, we shall learn the following measures of dispersion and their methods of calculation for ungrouped and grouped data.

(i) Range (ii) Mean deviation (iii) Standard deviation

such a measure computed for a distribution is called 'Statistic'.

5.1 Range:

For an ungrouped data, range is defined as the difference between the maximum (greatest) value and the minimum (smallest) value of the series of observations.

For a grouped data (*i.e.*, data given in the form of a frequency table), range is approximated as the difference between the upper limit of the largest class and the lower limit of the smallest class.

1. **Example:** In the example of runs scored by the two batsman A and B, we can make some inference on the variability in the scores on the basis of minimum and maximum runs in each series. This difference is called the range of the data.
   In the case of batsman A, the range is $116-2=114$ and for batsman B, the range is $76-38=38$. Clearly, the range of A is greater than the range of B. Therefore, the scores are more scattered or dispersed in the case of A, where as for B, they are less dispersed and close to each other.
2. **Example:** Let us consider the daily sales in (in Rs.) of two firms A and B, for 5 days, as given in the following table:

Firm A	Firm B
5,050	4,900
5,025	3,100
4,950	2,200
4,835	1,800
5,140	13,000
$\overline{X}_A = 5,000$	$\overline{X}_B = 5,000$

   The average sales of both firms is the same but the distribution pattern of the sales is not similar. There is a greater amount of variation in the daily sales of the firm B than that of the firm A.
   Range of sales of firm A=$5140-4835=305$.
   Range of sales of firm B=$13000-1800=11,200$.

The range of a data is very easy to calculate and it gives us some idea about the validity of the data. However, the range is a crude measure of dispersion, since it uses only two extreme values. Also, it does not tell us about the dispersion of the data from a measure of central tendency. Hence we need to know a more realistic measure of dispersion known as (i) mean deviation and (ii) standard deviation.

5.2 Mean Deviation:

To find the dispersion values of x from a central value a, we define the deviation about a. They are $(x-a)'$s. To find the mean deviation we have to sum up all such deviations. Since a measure of central tendency lies between the maximum and minimum values of the distribution, some of the deviations will be negative and some positive. Also, some of the deviations may vanish. If, in particular $a = \bar{x},$ the sum of the deviations from the mean $(\bar{x})$ is zero. In this case,

$$\text{Mean of the deviations from the arithmetic mean } = \frac{\text{Sum of the deviations}}{\text{Number of observations}} = \frac{0}{n} = 0.$$

Hence finding such mean deviation does not serve any purpose.

Recall that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Hence, to find the measure of dispersion from a fixed number $a,$ we may take the mean of the absolute values of the deviations from the arithmetic mean $(\bar{x}).$ Such mean is called the mean deviation from the arithmetic mean and expressed as

$$\text{Mean deviation from the mean } = \frac{\text{Sum of the absolute values of deviations from } \bar{x}}{\text{Number of observations}}.$$

Remark: Mean deviation may be obtained from any measure of central tendency. For instance, it can be obtained from median also. Mean deviation from mean and median are commonly used.

We shall now learn how to compute the mean deviation from mean and median.

5.2.1(a) Mean Deviation from the mean for an ungrouped data:

Suppose we have a discrete data with n observations $x_1, x_2, x_3, ..., x_n.$ Then we adopt the following procedure for computing the mean deviation from the mean of the given data.

Step 1: Calculate the arithmetic mean $(\bar{x})$ of n observations $x_1, x_2, x_3, ..., x_n.$ Let it be $a.$

Step 2: Find the deviations of each x_i from $a,$ *i.e.,* $x_1 - a, x_2 - a, x_3 - a, ..., x_n - a.$

Step 3: Find the absolute value *i.e.,* $|x_1 - a|, |x_2 - a|, |x_3 - a|, ..., |x_n - a|$ of these deviations by

ignoring the negative sign, if any, in the deviations computed in step 2.

Step 4: Find the arithmetic mean of the absolute values of the deviations

$$\textit{i.e.,} \text{ M.D from the mean} = \frac{\sum_{i=1}^{n} |x_i - a|}{n}.$$

3.Example: Find the mean deviation from the mean of the following discrete data:

$6, 7, 10, 12, 13, 4, 12, 16.$

Solution: The arithmetic mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+12+16}{8} = 10.$$

The absolute values of the deviations: $|x_i - \bar{x}|$ are $4, 3, 0, 2, 3, 6, 2, 6.$

$$\therefore \text{ The mean deviation from the mean } = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

$$= \frac{4+3+0+2+3+6+2+6}{8} = \frac{26}{8} = 3.25.$$

5.2.1(b) Mean Deviation from the median for an ungrouped data:

Suppose we have a discrete data with n observations $x_1, x_2, x_3, ..., x_n$. Then we adopt the following procedure for computing the mean deviation from the median of the given data.

Step 1: Calculate the median of n observations $x_1, x_2, x_3, ..., x_n$. Let it be b.

Step 2: Find the deviations of each x_i from b, *i.e.*, $x_1 - b, x_2 - b, x_3 - b, ..., x_n - b.$

Step 3: Find the absolute value *i.e.*, $|x_1 - b|, |x_2 - b|, |x_3 - b|, ..., |x_n - b|$ of these deviations by ignoring the negative sign, if any, in the deviations computed in step 2.

Step 4: Find the arithmetic mean of the absolute values of the deviations

$$i.e., \text{ M.D from the median} = \frac{\sum_{i=1}^{n} |x_i - b|}{n}.$$

4.Example: Find the mean deviation from the median of the following discrete data:

$6, 7, 10, 12, 13, 4, 12, 16.$

Solution: Expressing the data points in ascending order of magnitude, we get

$4, 6, 7, 10, 12, 12, 13, 16.$

Then the median of these 8 observations is $b = \frac{10+12}{2} = 11.$

The absolute values of the deviations: $|x_i - \bar{x}|$ are $7, 5, 4, 1, 1, 1, 2, 5.$

$$\therefore \text{ The mean deviation from the median } = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

$$= \frac{7+5+4+1+1+1+2+5}{8} = \frac{26}{8} = 3.25.$$

Note:

(i) In the problem we considered the above, the obtained mean deviation from the mean and the mean deviation from the median are equal. But in general they need not be equal.

(ii) The value of the mean deviation about the median of an ungrouped data is the least when compared to the mean deviation computed about any other measure of central tendency. This is also called the minimal property of the median.

5.2.2 Mean Deviation for a grouped data:

You have learnt in elementary statistics that a data can be arranged or grouped as a frequency distribution in two ways.

(i) Discrete frequency distribution and

(ii) Continuous frequency distribution.

We shall now discuss the method of finding the mean deviation for both the types of distribution.

(a) Discrete frequency distribution:

Suppose the data consists of n distinct points $x_1, x_2, x_3, ..., x_n$ occurring with frequencies $f_1, f_2, f_3, ..., f_n$ respectively. Then, we can represent this data in the following manner.

x_i	x_1	x_2	x_3	...	x_n
f_i	f_1	f_2	f_3	...	f_n

This form is called the discrete frequency distribution.

5.2.2(i) Mean Deviation about the mean: Recall that the arithmetic mean $(\bar{x})$ of a discrete frequency distribution with data points is obtained using the formula:

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i x_i$$ where $\mathrm{N} = \sum_{i=1}^{n} f_i$ gives the total frequencies in the considered distribution.

Now the mean deviation about the mean *i.e.*, $\mathrm{M.D.}(\bar{x})$ is obtained by finding the absolute values of the deviations of the data points from the mean *i.e.*, $|x_i - \bar{x}|$ and using the formula: M.D from the mean $= \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i |x_i - \bar{x}|$

5.Example: Find the mean deviation about the mean for the following data:

x_i	2	5	7	8	10	35
f_i	6	8	10	6	8	2

Solution: We shall now construct the following table to enable us to compute the required statistic.

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
2	6	12	6	36
5	8	40	3	24
7	10	70	1	10
8	6	48	0	0
10	8	80	2	16
35	2	70	27	54
	$\mathrm{N} = \sum_{i=1}^{n} f_i = 40$	$\sum_{i=1}^{n} f_i x_i = 320$		140

A.M. $\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{320}{40} = 8.$

$$\mathrm{M.D.(mean)} = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i |x_i - \bar{x}| = \frac{1}{40} \times 140 = 3.5.$$

5.2.2(ii) Mean Deviation about the median: To find the mean deviation about the median, we have to find the median of the given discrete frequency distribution. After arranging the observations in either ascending or in descending order, we shall then find the sum of the frequencies: $\sum_{i=1}^{n} f_i = \text{N}$ and compute the cumulative frequencies. Then we shall identify the observation whose cumulative frequency is equal to or just greater than $\frac{\text{N}}{2}$. This is the median of the data.

To obtain the mean deviation about the median, we find the absolute values of the deviations from the median and substitute them in the formula:

$$\text{M.D. from the median } = \frac{1}{\text{N}} \sum_{i=1}^{n} f_i \left| x_i - \text{median} \right|$$

6.Example: Find the mean deviation about the median for the following data:

x_i	6	9	3	12	15	13	21	22
f_i	4	5	3	2	5	4	4	3

Solution: Here $\sum_{i=1}^{n} f_i = \text{N=30}.$ Keeping the observations in the ascending order, we get the following distribution:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

Median of these observations is equal to 13.

Now we compute the absolute values of the deviations *i.e.*, $\left| x_i - \text{median} \right|$, from the median and compute $f_i \left| x_i - \text{median} \right|$, as shown in the following table:

$\left\| x_i - \text{median} \right\|$	10	7	4	1	0	2	8	9
f_i	3	4	5	2	4	5	4	3
$f_i \left\| x_i - \text{median} \right\|$	30	28	20	2	0	10	32	27

Now, $\sum_{i=1}^{8} f_i \left| x_i - \text{median} \right| = 149.$

Hence mean deviation from the median $= \frac{1}{\text{N}} \sum_{i=1}^{n} f_i \left| x_i - \text{median} \right| = \frac{1}{30} \times 149 = 4.97.$

5.2.3(i) Finding Mean Deviation from the mean for a continuous frequency distribution: Recall that a continuous frequency distribution is a series in which the data is classified into different class- intervals (without gaps) along with their respective frequencies (f_i). As an illustration, consider the following grouped data in the form of continuous distribution which relate to the sales of 100 companies:

Sales (in Rs. thousand)	$40-50$	$50-60$	$60-70$	$70-80$	$80-90$	$90-100$
Number of companies	5	15	25	30	20	5

Recall that while computing the arithmetic mean of a continuous frequency distribution, we assumed that the entire frequency (f_i) of the i^{th} class interval is centred at the midpoint x_i of that class interval. In the discussion that follows, we adopt in much the same procedure and write the midpoint (x_i) of each class interval. With these x_i, we proceed to find the mean deviation, as has been done in the case of discrete frequency distribution.

7.Example: Mean deviation from the mean: We shall construct the following table from the given tabulated data:

Sales (in Rs.1000's)	No.of companies (f_i).	Mid point of class interval (x_i).	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
$40-50$	5	45	225	36	130
$50-60$	15	55	825	24	240
$60-70$	25	65	1625	10	150
$70-80$	30	75	2250	0	120
$80-90$	20	85	1700	16	280
$90-100$	5	95	475	54	120
	$N = \sum_{i=1}^{n} f_i = 100$		$\sum_{i=1}^{n} f_i x_i = 710$		$\sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert = 1040$

Here $N = \sum_{i=1}^{n} f_i = 100$ and Mean $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{N} = \dfrac{7100}{100} = 71.$

Hence mean deviation from the mean $= \dfrac{1}{N} \sum_{i=1}^{n} f_i \lvert x_i - \bar{x} \rvert = \dfrac{1040}{100} = 10.4.$

Alternative simple method (Step-deviation method):

Some times when the midpoints of the class intervals x_i as well as their associated frequencies are numerically large, then we find a computational tediousness in the above procedure. To avoid this tediousness, we take an assumed mean a which lies in the middle or just close to it in the data and take the deviations of midpoints x_i from this assumed mean. This amounts to shifting of origin from zero to the assumed mean on the number line.

Some times, if there is a common factor of all the deviations, we divide them by this common factor (h) to further simplify the deviations. These are known as step deviations. Taking step deviations amounts to change of scale on the number line. .

With the assumed mean a and common factor h if we define a new variable $d_i = \frac{x_i - a}{h}$, then the arithmetic mean $\bar{x} = a + \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)h.$

(Remark: When $d_i = \frac{x_i - a}{h}$, then $x_i = a + hd_i$. Multiplying throughout by f_i, taking summation on both sides from 1 to n and dividing throughout by N we get $\bar{x} = a + \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)h.$

We shall now illustrate this simplified procedure with the following example:

8.Example: Find the mean deviation about the mean for the following data:

Marks obtained	0–10	10–20	20–30	30–40	40–50
No. of students	5	8	15	16	6

Solution: Taking the assumed mean $a = 25$ and $h = 10$, we form the following table:

Class Interval	Frequency (f_i)	Mid point of class interval (x_i).	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
0–10	5	5	–2	–10	22	110
10–20	8	15	–1	–8	12	96
20–30	15	25	0	0	2	30
30–40	16	35	1	16	8	128
40–50	6	45	2	12	18	108
	$N = \sum_{i=1}^{n} f_i$ $= 50$			$\sum_{i=1}^{n} f_i d_i$ $= 10$		$\sum_{i=1}^{n} f_i \lvert x_i - \bar{x}\rvert$ $= 472$

Now, Mean $\bar{x} = a + \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right) h = 25 + \left(\frac{10}{50}\right) 10 = 27.$

Hence mean deviation from the mean $= \frac{1}{N}\sum_{i=1}^{n} f_i \left|x_i - \bar{x}\right| = \frac{472}{50} = 9.44$ marks.

5.2.3(ii) Finding Mean Deviation from the median for a continuous frequency distribution: The process of finding mean deviation from the median for a continuous frequency distribution is just similar to the procedure adopted for finding the mean deviation about the mean. The only difference lies in the replacement of mean by median while taking the deviations.

To find the median for a continuous frequency distribution, we identify the class interval in which $(N/2)^{th}$ observation lies. This class is known as *median class*. We then find the median, using the formula:

$$\text{Median} = L + \left[\left(\frac{N}{2} - p.c.f\right)\Big/ f\right] i,$$

Where L is the lower limit of the median class, $p.c.f$ is the preceding cumulative frequency to the median class, f is the frequency of the median class and i is the width of the median class.

After finding the median, the absolute values of the deviations of the midpoint x_i of each class from the median *i.e.*, $|x_i - \text{median}|$ are found.

$$\text{M.D. from the median} = \frac{1}{N}\sum_{i=1}^{n} f_i \left|x_i - \text{median}\right|$$

We shall now illustrate this simplified procedure with the following example:

9.Example: Find the mean deviation about the median for the following data:

Age (years)	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50	50 – 55	55 – 60
No.of workers	120	125	175	160	150	140	100	30

Solution: We form the following table for the given data:

Class Interval	Frequency (f_i)	Cumulative frequency	Mid point of class interval (x_i).	$\lvert x_i - \text{Median}\rvert$	$f_i \lvert x_i - \text{Median}\rvert$
$20-25$	120	120	22.5	15	1800
$25-30$	125	245	27.5	10	1250
$30-35$	175	420	32.5	5	875
$35-40$	160	580	37.5	0	0
$40-45$	150	730	42.5	5	750
$45-50$	140	870	47.5	10	1400
$50-55$	100	970	52.5	15	1500
$55-60$	30	1000	57.5	20	600
	$\text{N} = \sum_{i=1}^{n} f_i$ $=1000$				$\sum_{i=1}^{n} f_i \lvert x_i - \text{Med}\rvert$ $=8175$

Here $(\text{N}/2)^{th}$ observation $= \dfrac{1000}{2} = 500^{th}$ lies in the class interval $35-40$. This is the median class. Now, Median $= \text{L} + \left[\left(\dfrac{\text{N}}{2} - p.c.f\right)\Big/ f\right] i = 35 + \left[\dfrac{(500-420)}{160}\right] \times 5$

$$= 35 + \left[\frac{400}{160}\right] = 35 + 2.5 = 37.5$$

Hence mean deviation from the median $= \dfrac{1}{\text{N}} \sum_{i=1}^{n} f_i \lvert x_i - \text{Median}\rvert = \dfrac{8175}{1000} = 8.175.$

5.3 Variance and Standard Deviation of Ungrouped / grouped data:

In the earlier section, while finding the mean deviation about the mean or median, we have taken the absolute values of the deviations to give meaning to that statistic, otherwise the deviations may cancel among themselves. To overcome this difficulty that arise due to the signs of deviations, we consider the squares of the deviations to make them non-negative. Thus if $x_1, x_2, x_3, ..., x_n$ are n observations and $\bar{x}$ is their mean, then

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + ... + (x_n - \bar{x})^2 \geq 0.$$

We have the following cases:

Case (i): If $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0$, then each $(x_i - \bar{x}) = 0$ which implies all observations are equal to the mean $\bar{x}$ and hence there is no dispersion.

Case (ii): If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is small, then it indicates that each observation x_i is very close to the mean $\bar{x}$ and hence the degree of dispersion is low.

Case (iii): If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is large, then it indicates a higher degree of dispersion of the observation from the mean $\bar{x}$.

On the other hand, if we take the mean of the squared deviations from the mean, *i.e.*, $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$, then it is found that this number leads to a proper measure of dispersion. This number is called *variance* and is denoted by σ^2 (read as sigma square) and is given by $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. Then σ, the *standard deviation* is given by the positive square root of the variance.

$$\therefore \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

5.3.1(i) Calculation of variance and standard deviation for an ungrouped data:

10.Example: Find the variance and standard deviation of the following data:

$$5, 12, 3, 18, 6, 8, 2, 10.$$

Solution: The mean of the given data is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{5+12+3+18+6+8+2+10}{8} = \frac{64}{8} = 8.$$

To find the variance, we construct the following table:

x_i	5	12	3	18	6	8	2	10
$(x_i - \bar{x})$	–3	4	-5	10	–2	0	–6	2
$(x_i - \bar{x})^2$	9	16	25	100	4	0	36	4

Here $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 194$

$\therefore$ Variance $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{8}\times 194 = 24.25.$

Hence standard deviation $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{24.25} = 4.95\,(\text{approx}).$

5.3.1(ii) Calculation of variance and standard deviation for a discrete frequency distribution:

11.Example:

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

We shall construct the following table for computing the required statistic:

x_i	f_i	$f_i x_i$	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	-10	100	300
8	5	40	-6	36	180
11	9	99	-3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	$\text{N} = 30$	$\sum f_i x_i = 420$			$\sum f_i(x_i - \bar{x})^2 = 1374$

Here $\text{N} = 30, \sum_{i=1}^{n} f_i x_i = 420.$

$$\bar{x} = \frac{1}{\text{N}}\sum_{i=1}^{n} f_i x_i = \frac{420}{30} = 14.$$

$$\sum_{i=1}^{n} f(x_i - \bar{x})^2 = 1374$$

$\therefore$ Variance $\sigma^2 = \frac{1}{\text{N}}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \frac{1}{30}\times 1374 = 45.8.$

Hence standard deviation $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2} = \sqrt{45.8} = 6.77$ (approx).

5.3.2 Standard Deviation of a continuous frequency distribution:

Recall that in the case of finding a mean deviation for a continuous frequency distribution, we have transformed it as a discrete distribution by representing each class by its midpoint. Then the standard deviation is calculated by adopting the same procedure that was carried out for a discrete frequency distribution.

If there are n classes in a given distribution, each class represented by its midpoint x_i with frequency f_i, the standard deviation is obtained using the formula.

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}, \text{ where } N = \sum_{i=1}^{n} f_i \text{ and } \bar{x} \text{ is the mean of the distribution.}$$

Aliter: For the purpose of simplifying the computation in finding the standard deviation, we adopt the following alternative formula.

We have, variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i \left(x_i^2 + \bar{x}^2 - 2x_i \bar{x}\right)$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 + \sum_{i=1}^{n} f_i \bar{x}^2 - 2\sum_{i=1}^{n} f_i x_i \bar{x}\right]$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 \sum_{i=1}^{n} f_i - 2\bar{x}\sum_{i=1}^{n} f_i x_i\right]$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 N - 2\bar{x} N\bar{x}\right] \quad \left(\because \bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i \Rightarrow \sum_{i=1}^{n} f_i x_i = N\bar{x}\right)$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 N - 2N\bar{x}^2\right]$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 - 2\bar{x}^2$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right)^2$$

Then, standard deviation $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right)^2}$.

Some times the midpoint x_i of the different class intervals in a continuous distribution are so large that the calculation of mean and variance becomes tedious and time consuming. In such cases, we apply the step deviation method, as detailed here under, to avoid the complexity in computation.

Let h be the width of the class intervals and A be the assumed mean. Assume that the scale is reduced to $1/h$ times in the step deviation method.

Define $y_i = \frac{x_i - A}{h}, i = 1, 2, 3, ..., n.$

Then, $x_i = A + hy_i$ ...(I)

Now $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \frac{1}{N}\sum_{i=1}^{n} f_i (A + hy_i)$ by(I)

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i A + \sum_{i=1}^{n} f_i h y_i\right] = \frac{1}{N}\left[A\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} f_i y_i\right]$$

$$= A + \frac{1}{N}\left[h\sum_{i=1}^{n} f_i y_i\right] = A + h\bar{y} \text{...(II)}$$

Also $\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i (A + hy_i - A - h\bar{y})^2, \text{ using (I) and (II)}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i h^2 (y_i - \bar{y})^2 = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i (y_i - \bar{y})^2\right] = h^2\sigma_y^2 \text{ or } \sigma_x = h\sigma_y \text{...(III)}$$

But we have shown that standard deviation $\sigma_x = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$ or

$$\sigma_y = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$$

Hence from equation (III), $\sigma_x = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$.

We shall exemplify this simple method with the following example.

12.Example: Calculate the variance and standard deviation of the following data:

Class Interval	$30-40$	$30-40$	$30-40$	$30-40$	$30-40$	$30-40$	$30-40$
Frequency	3	7	12	15	8	3	2

Solution: Here $h = 10$.

If we take the assumed mean $A = 65$, then $y_i = \frac{x_i - A}{h} = \frac{x_i - 65}{10}$.

We shall now construct the following table with the given data:

Class Interval	Frequency (f_i)	Mid point of class interval (x_i).	$y_i = \frac{x_i - 65}{10}$	y_i^2	$f_i y_i$	$f_i y_i^2$
$30-40$	3	35	-3	9	-9	27
$40-50$	7	45	-2	4	-14	28
$50-60$	12	55	-1	1	-12	12
$60-70$	15	65	0	0	0	0
$70-80$	8	75	1	1	8	8
$80-90$	3	85	2	4	6	12
$90-100$	2	95	3	9	6	18
	$N = 50$				$\sum f_i y_i = -15$	$\sum f_i y_i^2 = 105$

Mean $\bar{x} = A + \left(\frac{1}{N}\sum_{i=1}^{n} f_i y_i\right) \times h = 65 - \left(\frac{15}{50} \times 10\right) = 62.$

Variance $\sigma_x^2 = \frac{h^2}{N^2}\left[N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2\right]$

$$= \frac{100}{2500}\left[50(105)-(-15)^2\right] = \frac{1}{25}\left[5250-225\right] = 201.$$

Hence standard deviation $\sigma_x = \sqrt{201} = 14.18$ (approx).

5.4 Coefficient of Variation and analysis of frequency distributions with equal means but different variances:

The two measures of dispersion we have studied in this unit namely, mean deviation and standard deviation, have the same units in which the data is given. Whenever we want to compare the variability of two series of data having the same mean (or may differ widely) in their mean or measured in different units, we do not merely calculate the measure of dispersion. Instead, we require a measure which is independent of units. The measure of variability which is pure a pure number and is independent of units. The measure of variability which is a pure number and is independent of units is called the coefficient of variation, denoted by C.V.

The coefficient of variation of a distribution is defined as $\text{C.V} = \frac{\sigma}{\bar{x}} \times 100, \bar{x} \neq 0.$

Where σ is the standard deviation and $\bar{x}$ is the mean of the distribution or data.

The coefficient of variation is a relative measure of variation. For comparing the variability of two series (distributions), we calculate the coefficient of variation, for each series. The series having greater C.V. is said to have more variability than the other. The series having less C.V. is deemed to be more consistent (or homogeneous) than the other.

5.4.1. Analysis / Comparison of two frequency distributions with equal means:

Suppose we have two distributions D_1 and D_2 with the same mean *i.e.,* $\bar{x}_1 = \bar{x}_2 = \bar{x}(say),$ but different standard deviations σ_1 and σ_2 respectively. Then C.V. of $D_1 = \frac{\sigma_1}{\bar{x}} \times 100$ and C.V. of $D_2 = \frac{\sigma_2}{\bar{x}} \times 100$. Then it follows that the two C.V.'s can be compared on the basis of the values of σ_1 and σ_2 only. In this case, the series with lower value of standard deviation is said to be more consistent than the other and the series with greater standard deviation is called more dispersed than the other.

13.Example: Students of two sections A and B of a class show the following performance in a test (conducted for 100marks)

	Section A	Section B
Number of students	50	60
Average marks in the test	45	45
variance of distribution of marks	64	81

Which section of students have greater variability in performance?

Solution: Since the variance of distribution of marks of section A is 64, its standard deviation earlier section, finding the mean deviation $\sigma_1 = 8$. Similarly, since the variance of distribution of marks of section B is 81, its standard deviation earlier section, finding the mean deviation $\sigma_2 = 9$. Since the average marks of both sections of students is the same *i.e.*, 45, the section with greater standard deviation will have more variability. Hence section B has greater variability in the performance.

5.4.2. Comparison of two frequency distributions with unequal means:

We shall illustrate this case by considering an example.

14.Example: List of two models of refrigerators A and B, obtained in a survey, are given below:

Life in years	Model A	Model B
$0-2$	5	2
$2-4$	16	7
$4-6$	13	12
$6-8$	7	19
$8-10$	5	9

Which refrigerator model would you suggest to purchase?

Solution: To find the mean and variance of the lives of model A and model B of refrigerators, we shall construct the following tables for model A and model B

Class Interval	Mid point (x_i)	(f_i)	$f_i x_i$	$f_i x_i^2$
$0-2$	1	5	5	5
$2-4$	3	16	48	144
$4-6$	5	13	65	325
$6-8$	7	7	49	343
$8-10$	9	5	45	405
		$N = 46$	212	1221

Class Interval	Mid point (x_i)	(f_i)	$f_i x_i$	$f_i x_i^2$
$0-2$	1	2	2	2
$2-4$	3	7	21	63
$4-6$	5	12	60	300
$6-8$	7	19	133	931
$8-10$	9	9	81	729
		$N=49$	297	2025

For Model A: $\bar{x}_A = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{212}{46} = 4.6.$

$$\sigma_A^2 = \frac{\sum f_i x_i^2}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)^2 = \frac{1221}{46} - \left(\frac{212}{46}\right)^2 = 5.38$$

$\Rightarrow \sigma_A = \sqrt{5.38} = 2.319.$

C.V. of model A $= \dfrac{\sigma_A}{\bar{x}_A} \times 100 = \dfrac{2.319}{4.6} \times 100 = 50.41$

For Model B: $\bar{x}_B = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{297}{49} = 6.06.$

$$\sigma_B^2 = \frac{\sum f_i x_i^2}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)^2 = \frac{2025}{49} - \left(\frac{297}{49}\right)^2 = 4.61$$

$\Rightarrow \sigma_B = \sqrt{4.61} = 2.147.$

C.V. of model B $= \dfrac{\sigma_B}{\bar{x}_B} \times 100 = \dfrac{2.147}{6.06} \times 100 = 35.43$

Since C.V. of model B $<$ C.V. of model A, we can say that model B is more consistant than the model A, with regard to the life in years. Hence we suggest model B for purchase.

5.5 Solved Problems:

1. Problem: Find the mean deviation from the mean of the following data, using the step deviation method.

Marks obtained	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
N0. of students	6	5	8	15	7	6	3

Solution: Taking the assumed mean $a = 35$ and $h = 10$, we form the following table:

Class Interval	Frequency (f_i)	Mid point of class interval (x_i).	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
0 – 10	6	5	−3	−18	28.4	170.4
10 – 20	5	15	−2	−10	18.4	92
20 – 30	8	25	−1	−8	8.4	67.2
30 – 40	15	35	0	0	1.6	24.0
40 – 50	7	45	1	7	11.6	81.2
50 – 60	6	55	2	12	21.6	129.6
60 – 70	3	65	3	9	31.6	94.8
	$N = \sum_{i=1}^{n} f_i = 50$			$\sum_{i=1}^{n} f_i d_i = -8$		$\sum_{i=1}^{n} f_i \lvert x_i - \bar{x}\rvert = 659.2$

Now, Mean $\bar{x} = a + \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)h = 35 + \left(\frac{-8}{50}\right)10 = 33.4$ marks

Hence mean deviation from the mean $= \frac{1}{N}\sum_{i=1}^{n} f_i \lvert x_i - \bar{x}\rvert = \frac{659.2}{50} = 13.18$ (nearly).

2. Problem: The following table gives the daily wages of workers in a factory. Compute the standard deviation and the coefficient of variation of the wages of the workers.

Wages (Rs.)	125 – 175	175 – 225	225 – 275	275 – 325	325 – 375	375 – 425	425 – 475	475 – 525	525 – 575
N0. of workers	2	22	19	14	3	4	6	1	1

Solution: We shall solve this problem using the step deviation method, since the midpoints of the class intervals are numerically large. Here $a = 300$ and $h = 50.$ Then

$$y_i = \frac{x_i - a}{h} = \frac{x_i - 300}{50}.$$

Frequency (f_i)	Mid point of class interval (x_i).	$y_i = \frac{x_i - 65}{10}$	y_i^2	$f_i y_i$	$f_i y_i^2$
2	150	–3	9	– 6	18
22	200	–2	4	–44	88
19	250	–1	1	–19	19
14	300	0	0	0	0
3	350	1	1	3	3
4	400	2	4	8	16
6	450	3	9	18	54
1	500	4	16	4	16
1	550	5	25	5	25
$N = 72$				$\sum f_i y_i = -31$	$\sum f_i y_i^2 = 239$

$$\text{Mean } \bar{x} = A + \left(\frac{1}{N}\sum_{i=1}^{n} f_i y_i\right) \times h = 300 - \left(\frac{31}{72} \times 50\right) = 278.47.$$

$$\text{Variance } \sigma_x^2 = \frac{h^2}{N^2}\left[N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2\right]$$

$$= \frac{2500}{72 \times 72}\left[72(239) - (31)^2\right]$$

$$\text{Hence standard deviation } \sigma_x = \sqrt{\frac{2500}{72 \times 72}\left[72(239) - (31)^2\right]} = 88.52 \text{ (approx).}$$

$$\text{C.V. model B} = \frac{\sigma_x}{\bar{x}} \times 100 = \frac{88.52}{278.47} \times 100 = 31.79$$

3. Problem: An analysis of monthly wages paid to the workers of two firms A and B belonging to the same industry gives the following data:

	Firm A	Firm B
Number of workers	500	600
Average daily wages (Rs.)	186	175
Variance of distribution of wages	81	100

(i) Which firm A or B has greater variability in individual wages?

(ii) Which firm A or B has larger wage bill?

Solution: (i) Since variance of distribution of wages in Firm A is 81, $\sigma_1^2 = 81$ and hence $\sigma_1 = 9$. Since variance of distribution of wages in Firm B is 100, $\sigma_2^2 = 100$ and hence $\sigma_2 = 10$.

C.V. of distributions of wages of Firm A $= \frac{\sigma_1}{\overline{x_1}} \times 100 = \frac{9}{186} \times 100 = 4.84$

C.V. of distributions of wages of Firm B $= \frac{\sigma_2}{\overline{x_2}} \times 100 = \frac{10}{175} \times 100 = 5.71$

Since C.V. of Firm B $>$ C.V. of Firm A, we can say that Firm B has greater variability in individual wages.

(ii) Firm A has number of workers *i.e.*, wage earners $(n_1) = 500$.

Its average daily wage, say $\overline{x_1} = \text{Rs.}186$.

Since Average daily wages (Rs.) $= \frac{\text{Total wages paid}}{\text{Number of workers}}$, it follows that total wages paid to the workers $= n_1.\overline{x_1} = 500 \times 186 = \text{Rs.}93,000$.

Firm B has number of workers *i.e.*, wage earners $(n_2) = 600$.

Its average daily wage, say $\overline{x_2} = \text{Rs.}175$.

Since Average daily wages (Rs.) $= \frac{\text{Total wages paid}}{\text{Number of workers}}$, it follows that total wages paid to the workers $= n_2.\overline{x_2} = 600 \times 175 = \text{Rs.}1,05,000$.

Hence we see that Firm B has larger wage bill.

4. Problem: The scores of two cricketers A and B in 10 innings are given below. Find who is a better run and who is a more consistent player.

A	28	70	30	2	42	64	80	93	5	116
B	44	38	60	45	52	54	49	54	76	58

Solution: For cricketer A: $\bar{x} = \frac{540}{10} = 54;$

For cricketer B: $\bar{y} = \frac{380}{10} = 38.$

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	y_i	$(y_i - \bar{y})$	$(y_i - \bar{y})^2$
40	-14	196	28	-10	100
25	-29	841	70	32	1024
19	-35	1225	31	-7	49
80	26	676	0	-38	1444
38	-16	256	14	-24	576
8	-46	2116	111	73	5329
67	13	169	66	28	784
121	67	4489	31	-7	49
66	12	144	25	-13	169
76	22	484	4	-34	1156
$\sum x_i = 540$		$\sum (x_i - \bar{x})^2 = 10596$	$\sum y_i = 380$		$\sum (y_i - \bar{y})^2 = 10680$

Standard deviation of scores of A

$$= \sigma_x = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2} = \sqrt{\frac{10596}{10}} = \sqrt{1059.6} = 32.55\,\text{(approx)}.$$

Standard deviation of scores of B

$$= \sigma_y = \sqrt{\frac{1}{n}\sum (y_i - \bar{y})^2} = \sqrt{\frac{1068}{10}} = \sqrt{1068} = 32.68\,\text{(approx)}.$$

C.V. of $A = \frac{\sigma_x}{\bar{x}} \times 100 = \frac{32.55}{54} \times 100 = 60.28$

C.V. of $B = \frac{\sigma_y}{\bar{y}} \times 100 = \frac{32.68}{38} \times 100 = 86$

Since $\bar{x} > \bar{y}$, cricketer A is better run getter (scorer). and we can say that Firm B has greater variability in individual wages.

Since C.V. of B $>$ C.V. of A, we can say that cricketer A has is also a more consistent player.

Exercise 5(a)

1. Find the mean deviation about the mean for the following data:

   (i) 38, 70, 48, 42, 55, 63, 46, 54, 44

   (ii) 3, 6, 10, 4, 10, 9

2. Find the mean deviation from the median for the following data:

   (i) 13, 17, 16, 11, 13, 10, 16, 11, 18, 12, 17

   (ii) 4, 6, 9, 3, 10, 2

3. Find the mean deviation about the mean of the following frequency distributions:

(i)

x_i	10	11	12	13
f_i	3	12	18	12

(ii)

x_i	10	30	50	70	90
f_i	4	24	28	16	8

4. Find the mean deviation about the median of the following frequency distributions:

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

5. Find the variance and standard deviation of the following data:

   (i) 6, 7, 10, 12, 13, 4, 8, 12

   (ii) 350, 361, 370, 373, 376, 379, 385, 387, 394, 395.

6. Find the variance and standard deviation of the following frequency distribution:

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

7. Find the mean deviation about the mean for the following continuous distribution:

Hight (in cms)	95-105	105-115	115-125	125-135	135-145	145-155

Number of boys	9	13	26	30	12	10

8. Find the mean deviation about the median for the following continuous distributions:
(i)

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
Number of boys	6	8	14	16	4	2

(ii)

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
frequency	5	8	7	12	28	20	10	10

9. Find the variance and standard deviation of the following continuous distribution:

Age(years)	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Number of workers	3	61	32	153	140	51	2

10. The coefficient of variation of two distributions are 60 and 70 and their standard deviations are 21 and 16 respectively. Find their arithmetic means.
11. From the prices of shares A and B are given below, for 10 days of trading, find out which share is more stable?

A	35	54	52	53	56	58	52	50	51	49
B	108	107	105	105	106	107	104	103	104	101

Key concepts

- Measures of Dispersion: Range, mean deviation, variance, standard deviation are some measures of dispersion.
- Range is defined as the difference of maximum value and the minimum value of the data.
- Mean Deviation from the mean for an ungrouped data:

  (i) M.D from the mean $= \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$.

  (ii) M.D from the median $= \frac{1}{n}\sum_{i=1}^{n}|x_i - \text{Median}|$.
- Mean Deviation from the mean for a grouped data:

  (i) M.D from the mean $= \frac{1}{\text{N}}\sum_{i=1}^{n} f_i|x_i - \bar{x}|$

(ii) M.D from the median $= \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - \text{median}|$

- Variance and Standard Deviation of Ungrouped data:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2,\ \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

- Variance and Standard Deviation for a discrete frequency distribution:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2,\ \sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}.$$

- The Standard Deviation of a continuous frequency distribution:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right)^2}.$$

- Coefficient of variation C.V $= \frac{\sigma}{\bar{x}} \times 100,\ \bar{x} \neq 0.$

Answers

Exercise 5(a)

1. (i) 8.4 (ii) 2.67 2. (i) 2.45 (ii) 3.29 3. (i) 0.71 (ii) 16

4. 3.23 5. (i) 9.25 (ii) 183.2 6. Variance = 43.4; S.D = 6.59

7. 11.29 8. (i) 10.35 (ii) 14.29 9. Variance = 141.07; S.D = 11.87

10. $\bar{x}_1 = 35$; $\bar{x}_2 = 22.85$ 11. B

6. CIRCLES

Introduction:

Geometry has probably originated in ancient Egypt and flourished in Greece, India and China. In the sixth century B.C., the systematic development of geometry has begun.

Great mathematicians such as Thales, Menachmus and Archimedes worked on the circle and a tangent to it during the fifth century B.C. Thirty or forty years after the work of Aristotle, Euclid (a teacher of mathematics of Alexandria in Egypt) collected all the known works and arranged them in his famous book called “The Elements”

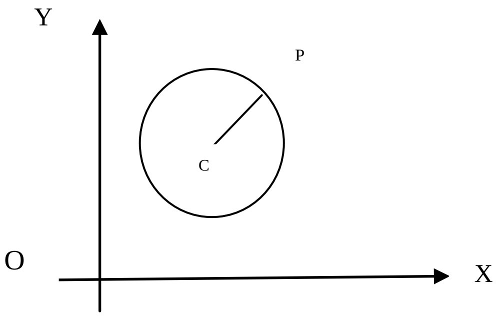
Rene Descartes introduced a very important branch of mathematics known as coordinate geometry and algebra. In honour of Descartes the subject is named as Cartesian Geometry.

The shape of a wheel of a bicycle, a wheel of bullock cart, bangle and some coins are of circular shape. In this chapter, we deal with the circle and obtain its equation. We derive the position of a point in the plane of a circle. We derive the equation of a tangent and the position of a straight line in the plane of a circle. We further derive the condition that the line to be a tangent.

6.1 Equation of a circle, standard form, centre and radius:

6.1.1 Definition:

A circle is the set of points in a plane such that they are equidistant from a fixed point lying in the plane.



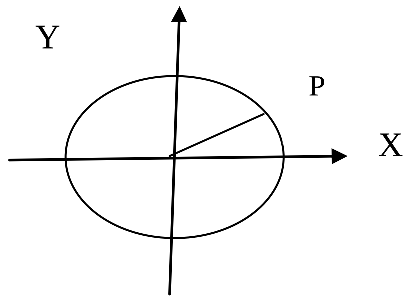
The fixed point is called the centre and the distance from the centre to a point on the circle.

Further, twice of the radius of the circle is called its diameter. In the above figure C is the centre of the circle and CP is its radius.

6.1.2 Standard form: Now, we proceed to find the equation of circle in standard form and its other forms.

6.1.3 Theorem: The equation of a circle with centre $\mathrm{O}(0,0)$ and radius r is $x^2+y^2=r^2$.

Proof: A point $\mathrm{P}(x,y)$ is on the circle if and only if the distance between P and O is r.

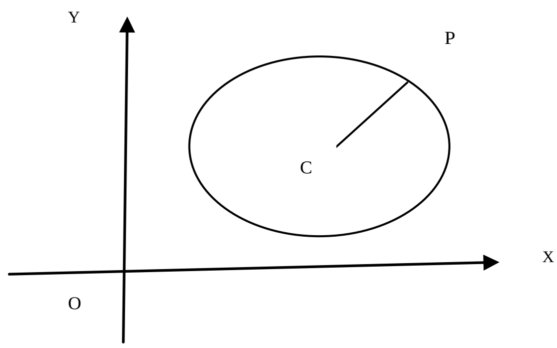


$\therefore \mathrm{PO}=r$

i.e., $x^2+y^2=r^2$...(I)

Which is the required equation of circle. The equation is called standard form of circle.

6.1.4 Theorem: The equation of a circle with centre $C(h,k)$ and radius r is $(x-h)^2+(y-k)^2=r^2$.



Proof: A point $P(x, y)$ is on the circle if and only if the distance between $P(x, y)$ and $C(h,k)$ is r.

$\therefore PC = r$

i.e., $\sqrt{(x-h)^2+(y-k)^2} = r$

i.e., $(x-h)^2+(y-k)^2 = r^2$ ..(I)

Which is the required equation of circle.

In the following, we obtain a necessary and sufficient condition for a second degree equation in x and y to represent a circle. This facilitates us to decide by just looking at the coefficients whether the equation represents a circle.

6.1.5 Theorem: The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$...(I) where the coefficients a, h, b, g, f and c are real numbers, represents a circle if and only if $(i)\ a=b\neq 0$ $(ii)\ h=0$ and $(iii)\ g^2+f^2-ac\geq 0$.

Proof: Suppose that the equation (I) represents a circle.

We shall prove $(i)\ a=b\neq 0$ $(ii)\ h=0$ and $(iii)\ g^2+f^2-ac\geq 0$.

Let (α,β) be the centre and r be the radius of the circle (I). Then by Theorem 6.1.4, the equation of a circle is $(x-\alpha)^2+(y-\beta)^2=r^2$

$$i.e.,\ x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2-r^2=0 \quad \text{...(II)}$$

The equations (I) and (II) represent the same circle. Comparing the coefficients in (I) and (II) we get $h=0$ and

$$\frac{a}{1}=\frac{b}{1}=\frac{2g}{-2\alpha}=\frac{2f}{-2\beta}=\frac{c}{\alpha^2+\beta^2-r^2} \quad \text{...(III)}$$

$a=b$ follows from equation (III).

If $a=0$ then $b=0$ $(\because a=b)$. In such case the given general equation will not be of second degree $(\because a=0, b=0, h=0)$.

$\therefore a\neq 0, b\neq 0.$

Further, from equation (III), we have

$$\frac{a}{1} = \frac{c}{\alpha^2 + \beta^2 - r^2}$$

$$\alpha^2 + \beta^2 - r^2 = \frac{c}{a} \qquad ..(IV)$$

and also we have $\alpha = \frac{-g}{a}$ and $\beta = \frac{-f}{a}$.

substituting these in equation (IV), we get $\frac{g^2}{a^2} + \frac{f^2}{a^2} - r^2 = \frac{c}{a}$

$$\frac{g^2 + f^2 - ac}{a^2} = r^2 \geq 0$$

$$\therefore \frac{g^2 + f^2 - ac}{a^2} \geq 0$$

i.e., $g^2 + f^2 - ac \geq 0$ $(\because a^2 \geq 0)$

Conversely, suppose that $(i)\, a = b \neq 0$ $(ii)\, h = 0$ and $(iii)\, g^2 + f^2 - ac \geq 0$, we shall prove that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle.

Since $a = b$ and $h = 0$ the general equation (I) of second degree becomes $ax^2 + ay^2 + 2gx + 2fy + c = 0$

$$\therefore x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0 \quad (\because a \neq 0)$$

$$\therefore \left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}$$

$$\therefore \left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \left(\sqrt{\frac{g^2 + f^2 - ac}{a^2}}\right)^2 \qquad ..(V)$$

Since $g^2 + f^2 - ac \geq 0$, the equation (V) represents a circle whose centre is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius is $\frac{\sqrt{g^2 + f^2 - ac}}{a}$.

6.1.6 Note: $(i)\, x^2 + y^2 + 2gx + 2fy + c = 0$ is considered as general equation of circle.

(ii) The centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f)$.

(iii) The radius of the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{g^2+f^2-c}$.

(iv) If $g^2+f^2-c=0$ then $x^2+y^2+2gx+2fy+c=0$ represents a point circle. In this case the centre itself is the point circle. The equation of a point circle having the centre at the origin is $x^2+y^2=0$.

(v) The equation of a circle through $(0,0)$ will be in the form $x^2+y^2+2gx+2fy=0$ since $(0,0)$ is a point circle.

(vi) The equation of a circle having the centre on the $x-$axis will be in the form of $x^2+y^2+2gx+c=0$ $(\because y-$ coordinate of the centre is zero$)$.

(vii) The equation of a circle having the centre on the $y-$axis will be in the form of $x^2+y^2+2fy+c=0$ $(\because x-$ coordinate of the centre is zero$)$.

$(viii)$ Two or more circles are said to be concentric if their centres are same.

(ix) The equation of a circle concentric with the circle $x^2+y^2+2gx+2fy+c=0$ will be in the form of $x^2+y^2+2gx+2fy+c'=0$ where c' is any constant.

(x) If the radius of a circle is 1 then it is called a unit circle.

6.1.7 Solved Problems:

1. Problem: Find the equation of the circle having centre $C(1,4)$ and radius $r=5$.

Solution: Given centre of the circle is $C(1,4)$ and radius of the circle is $r=5$.

The equation of a circle having centre $C(a,b)$ and radius r is $(x-a)^2+(y-b)^2=r^2$

$$\Rightarrow (x-1)^2+(y-4)^2=5^2 \Rightarrow x^2-2x+1+y^2-8y+16=25 \Rightarrow x^2+y^2-2x-8y-8=0$$

The required equation of circle is $x^2+y^2-2x-8y-8=0$.

2. Problem: Find the centre and radius of the circle $x^2+y^2+2x-4y-4=0$.

Solution: The given equation of the circle is $x^2+y^2+2x-4y-4=0$..........(I)

It is of the form $x^2+y^2+2gx+2fy+c=0$.........(II)

Compare equations (I) & (II) we get $2g=2,\ 2f=-4, c=-4$

$\Rightarrow g=1,\ f=-2, c=-4$

Centre of the circle is $(-g,-f)=(-1,2)$

Radius of the circle is $\sqrt{g^2+f^2-c}=\sqrt{1^2+(-2)^2-(-4)}=\sqrt{1+4+4}=\sqrt{9}=3$

3. Problem: Find the centre and radius of the circle $2x^2+2y^2-3x+2y-1=0.$

Solution: The given equation of the circle is $2x^2+2y^2-3x+2y-1=0.$

$$\Rightarrow x^2+y^2-\frac{3}{2}x+y-\frac{1}{2}=0..........(I)$$

It is of the form $x^2+y^2+2gx+2fy+c=0.........(II)$

Compare equations (I) & (II) we get $2g=-\frac{3}{2},\ 2f=1, c=-\frac{1}{2}$

$\Rightarrow g=-\frac{3}{4}, f=\frac{1}{2}, c=-\frac{1}{2}$

Centre of the circle is $(-g,-f)=\left(\frac{3}{4},-\frac{1}{2}\right)$

Radius of the circle is $r=\sqrt{g^2+f^2-c}=\sqrt{\left(-\frac{3}{4}\right)^2+\left(\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)}$

$$=\sqrt{\frac{9}{16}+\frac{1}{4}+\frac{1}{2}}=\sqrt{\frac{21}{16}}=\frac{\sqrt{21}}{4}$$

4. Problem: Find the equation of the circle passing through the point $(5,6)$ and having centre at the point $(-1,2)$.

Solution: Let the given point be $P(5,6)$ and the centre of the circle is $C(-1,2)$.

The equation of the circle having centre at $C(h,k)$ and radius r is
$(x-h)^2+(y-k)^2=r^2 \Rightarrow (x+1)^2+(y-2)^2=r^2$

Since it passing through the point $P(5,6)$ $\Rightarrow (5+1)^2+(6-2)^2=r^2 \Rightarrow (6)^2+(4)^2=r^2$
$\Rightarrow r^2=52$

Hence the required equation of circle is $(x+1)^2+(y-2)^2=52$

$\Rightarrow x^2+2x+1+y^2-4y+4=52 \Rightarrow x^2+y^2+2x-4y-47=0$

5. Problem: If $(2,3)$ is the centre of the circle represented by the equation $x^2+y^2+ax+by-12=0$. Find the values of a and b. Also find the centre and radius of the circle.

Solution: The given equation of the circle is $x^2+y^2+ax+by-12=0$ ...(I)

It is of the form $x^2+y^2+2gx+2fy+c=0$ ...(II)

Compare equations (I) & (II) we get $2g=a,\ 2f=b, c=-12$

$\Rightarrow g=\frac{a}{2}, f=\frac{b}{2}, c=-12$ ..(III).

The given centre of the circle is $(-g,-f)=(2,3) \Rightarrow -g=2, -f=3$

$\Rightarrow g=-2, f=-3$ ...(IV)

From equations (III) & (IV) we get $\frac{a}{2}=-2, \frac{b}{2}=-3 \Rightarrow a=-4, b=-6$

Radius of the circle is $r=\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+(-3)^2-(-12)}=\sqrt{4+9+12}=\sqrt{25}=5$

6. Problem: Find the equation of the circle passing through the point $(2,3)$ and concentric with the circle $x^2+y^2+8x+12y+15=0$.

Solution: The given equation of the circle is $x^2+y^2+8x+12y+15=0$ ...(I)

Let the required equation of the concentric circle be $x^2+y^2+8x+12y+c=0$ ...(II)

If it passes through $(2,3)$, we have $2^2+3^2+8.2+12.3+c=0 \Rightarrow 4+9+16+36+c=0$

$\Rightarrow c=-65$

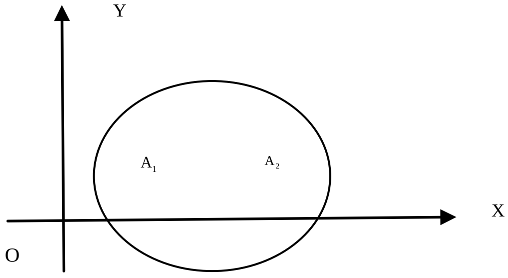
Hence the required equation of circle is $x^2+y^2+8x+12y-65=0$.

6.1.8 Theorem: (i) If $g^2-c>0$ then the intercept made on the $x-$axis by the circle $x^2+y^2+2gx+2fy+c=0$ is $2\sqrt{g^2-c}$.

(ii) If $f^2-c>0$ then the intercept made on the $y-$ axis by the circle $x^2+y^2+2gx+2fy+c=0$ is $2\sqrt{f^2-c}$.

Proof: (i) The points of intersection of the given circle $x^2+y^2+2gx+2fy+c=0$...(I) and $y=0$...(II) ($i.e., x-$ axis equation) are the common points of (I) and (II).

Put $y=0$ in (I) to get the abscissae of the points of intersection. The abscissae of the common points are the roots of $x^2+2gx+c=0$...(III)



The discriminant of this equation is $4(g^2-c)$. Since $g^2-c>0$, the equation (III) has two real and distinct roots, say x_1 and x_2. Suppose the points of intersection are $A_1(x_1,0)$ and $A_2(x_2,0)$. We have to prove that $A_1A_2=2\sqrt{g^2-c}$.

Since x_1 and x_2 are roots of the equation (III), we have

$x_1+x_2=-2g,\ x_1x_2=c.$

Consider $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=(-2g)^2-4c=4(g^2-c)$.

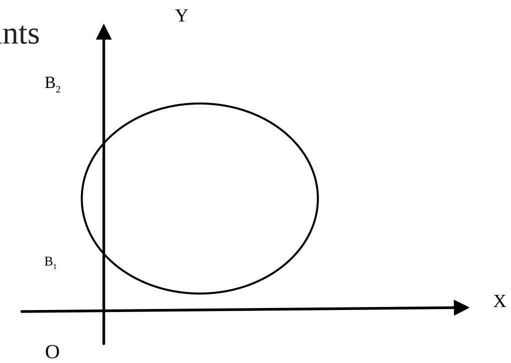
Taking the square root, we get $|x_1-x_2|=2\sqrt{g^2-c}$

i.e., $A_1A_2=2\sqrt{g^2-c}$.

Thus the intercept made by (I) on $x-$axis is $2\sqrt{g^2-c}$.

(ii) The points of intersection of the given circle $x^2+y^2+2gx+2fy+c=0$...(I) and $x=0$...(IV) ($i.e., x-$ axis equation) are the common points of (I) and (IV).

Put $y=0$ in (I) to get the ordinates of the points of intersection. The ordinates of the common points are the roots of $y^2+2fy+c=0$...(V)



The discriminant of this equation is $4(f^2-c)$. Since $f^2-c>0$, the equation (V) has two real and distinct roots, say y_1 and y_2. Suppose the points of intersection are $B_1(0,y_1)$ and $B_2(0,y_2)$. We have to prove that $B_1B_2=2\sqrt{f^2-c}$.

Since y_1 and y_2 are roots of the equation (V), we have

$y_1 + y_2 = -2f,\ y_1 y_2 = c.$

Consider $(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1 y_2 = (-2f)^2 - 4c = 4(f^2 - c).$

Taking the square root, we get $|y_1 - y_2| = 2\sqrt{f^2 - c}$

i.e., $B_1B_2 = 2\sqrt{f^2 - c}.$

Thus the intercept made by (I) on $y-$axis is $2\sqrt{f^2 - c}.$

6.1.9 Note: (i) $g^2 - c = 0 \Rightarrow A_1A_2 = 0 \Rightarrow A_1, A_2$ are coincident *i.e.*, the $x-$axis touches the circle in two coincident points. Thus the $x-$axis touches the circle at the point of coincidence.

(ii) $f^2 - c = 0 \Rightarrow B_1B_2 = 0 \Rightarrow B_1, B_2$ are coincident *i.e.*, the $y-$axis touches the circle in two coincident points. Thus the y^- axis touches the circle at the point of coincidence.

(iii) If $g^2 - c < 0$ then the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet the $x-$axis.

(iv) If $f^2 - c < 0$ then the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet the $y-$axis.

6.1.10 Examples: Let us find the equation of the circle which touches the $x-$axis at a distance of 3 from the origin and making intercept of length 6 on the $y-$axis.

Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$...(I)

This meets the $x-$axis at $(3, 0)$.

Since $(3, 0)$ is a point on (I)

$\therefore 3^2 + 0^2 + 2g.3 + 2f.0 + c = 0$

i.e., $6g + c = -9$ ...(II)

By Note 6.1.9(i), we have $g^2 - c = 0$ ...(III)

Adding (II) and (III) we get $g^2 + 6g + 9 = 0$

i.e., $(g + 3)^2 = 0$

i.e., $g = -3$ ...(IV)

From (III) and (IV), we get $c = 9$ ...(V)

Given that the intercept on $y-$axis made by (I) is 6.

Therefore by Theorem 6.1.9(*ii*), we have $2\sqrt{f^2 - c} = 6$

$$i.e.,\ 2\sqrt{f^2 - 9} = 6$$

$$i.e.,\ \sqrt{f^2 - 9} = 3$$

$$i.e.,\ f^2 - 9 = 9$$

$$i.e.,\ f^2 = 18$$

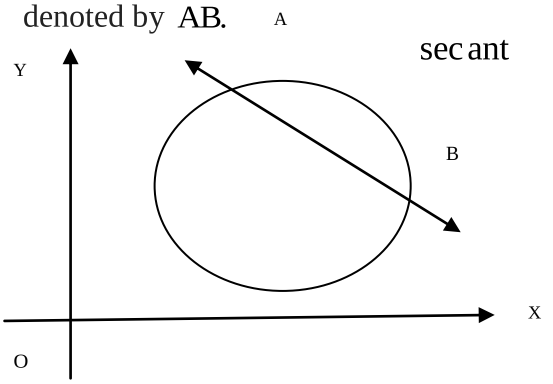
Hence $\quad f = \pm 3\sqrt{2}.$

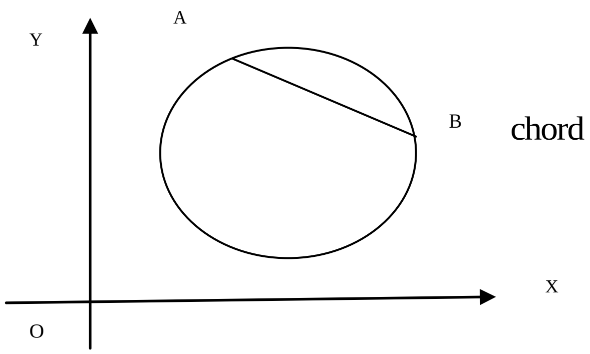
Since $g = -3,\ f = \pm 3\sqrt{2}$ and $c = 9,$ we have two circles satisfying the hypothesis, these circles are $x^2 + y^2 - 6x + 6\sqrt{2}y + 9 = 0$ and $x^2 + y^2 - 6x - 6\sqrt{2}y + 9 = 0.$

6.1.11 Definition: If A and B are two distinct points on a circle then

(i) the line $\overleftrightarrow{AB}$ through A and B is called a secant

(ii) The segment $\overline{AB}$, the join of A and B is called a chord and the length of the chord is denoted by AB.

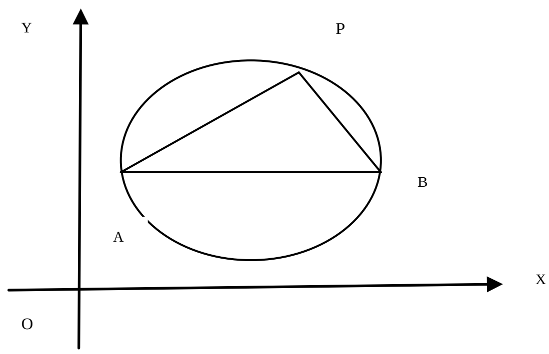




6.1.12 Equation of a circle with a given line segment as diameter:

In this section, we derive the equation of a circle whose diameter extremities are given.

6.1.13 Theorem: The equation of the circle whose diameter extremities are (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$



Proof: Let $A = (x_1, y_1), B = (x_2, y_2)$ and

C be the centre of the circle.

Let $P = (x, y)$ be any point on it other than A and B. Join A and B, and also P and B.

We know that $\angle APB = 90^0$.

i.e., the lines AP and BP are perpendicular.

$\therefore$ (slope of AP) $\times$ (slope of BP) $= -1$.

$$i.e., \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

$$i.e., (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \ldots (I)$$

Also clearly A and B satisfy (I). Therefore any point $P(x,y)$ on the circle satisfies equation (I). Conversely if a point $P(x,y)$ satisfies (I) then $\angle APB = 90^0$ and hence P lies on the circle.

Thus (I) is the equation of the required circle.

6.1.14 Solved Problems:

1. Problem: Find the equation of the circle whose extremities of the diameter are $(1,2)\&(4,5)$.

Solution: Let the extremities of the diameter be $A(1,2), B(4,5)$

The equation of the circle whose extremities of the diameter are $A(x_1,y_1), B(x_2,y_2)$

is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow (x-1)(x-4)+(y-2)(y-5)=0 \Rightarrow x^2-5x+4+y^2-7y+10=0$

$\Rightarrow x^2+y^2-5x-7y+14=0$

Hence the required equation of circle is $x^2+y^2-5x-7y+14=0$

2. Problem: Find the other end of the diameter of the circle $x^2+y^2-8x-8y+27=0$ if one end of it is $(2,3)$.

Solution: Let the extremities of the diameter be $A(2,3), B(a,b)$

The centre of the circle $x^2+y^2-8x-8y+27=0$ is $C(4,4)$.

The centre of the circle whose extremities of the diameter are $A(x_1,y_1), B(x_2,y_2)$

is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\Rightarrow \left(\frac{2+a}{2}, \frac{3+b}{2}\right) = (4,4) \Rightarrow (2+a, 3+b) = (8,8)$

$\Rightarrow a = 6, b = 5$

$\therefore$ The other end of the diameter is $(6,5)$.

6.1.15 Equation of circle through three non-collinear points:

We derive a formula to find the equation of a circle through three given points in the next section.

6.1.16 Theorem: The equation of a circle passing through three non-collinear points $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix} = 0$$

where $c_i = -(x_i^2 + y_i^2) \ (i = 1, 2, 3)$

Proof: Let the equation of a circle passing through three non-collinear points $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ be $x^2 + y^2 + 2gx + 2fy + c = 0 \ldots(\mathrm{I})$

Since the points $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ are lying on (I), we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \ldots(\mathrm{II})$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \ldots(\mathrm{III})$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad \ldots(\mathrm{IV})$$

Let $2g = a, 2f = b$ and $c_i = -(x_i^2 + y_i^2) \ (i = 1, 2, 3)$ ...(V)

The equations $(\mathrm{II}), (\mathrm{III})$ and (IV) can be written as

$$ax_1 + by_1 + c = c_1 \qquad \ldots(\mathrm{VI})$$

$$ax_2 + by_2 + c = c_2 \qquad \ldots(\mathrm{VII})$$

$$ax_3 + by_3 + c = c_3 \qquad \ldots(\mathrm{VIII})$$

Let $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Then $\Delta \neq 0$ since the points P, Q and R are non-collinear.

Consider $\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} ax_1+by_1+c & y_1 & 1 \\ ax_2+by_2+c & y_2 & 1 \\ ax_3+by_3+c & y_3 & 1 \end{vmatrix} = \begin{vmatrix} ax_1 & y_1 & 1 \\ ax_2 & y_2 & 1 \\ ax_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} by_1 & y_1 & 1 \\ by_2 & y_2 & 1 \\ by_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} c & y_1 & 1 \\ c & y_2 & 1 \\ c & y_3 & 1 \end{vmatrix}$

$= a.\Delta + 0 + 0$ ($\because$ two column elements are proportional)

$$\therefore 2g = a = \frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{\Delta} \qquad ...(\mathrm{IX})$$

Similarly $2f = b = \dfrac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{\Delta}$ ...(X) and $c = \dfrac{\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}}{\Delta}$ ...(XI)

Substituting the values of g, f and c in (I), we get the equation of the circle passing through three non-collinear points $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix} = 0$$

6.1.17 Note: (i) The centre of the circle passing through three non-collinear points

$\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ is $\left(\dfrac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{-2 \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \dfrac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{-2 \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}} \right)$

(from equations (IX) and (X) of Theorem 6.1.16)

(ii) We can also find the equation of the circle passing through three non-collinear points in the following way. First we suppose that the equation of the circle passing through the given three points $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ in general form. Substitute the coordinates of $\mathrm{P}(x_1, y_1), \mathrm{Q}(x_2, y_2)$ and $\mathrm{R}(x_3, y_3)$ in this equation. We get the equations involving three unknowns g, f and $c.$ Solve them for g, f and $c.$ Substitute these values in the supposed equation, we get the required circle.

6.1.18 Solved problems:

1. Problem: Find the equation of the circle passing through the points $(0,0),(2,0)\&(0,2).$

Solution: Let the given points be $\mathrm{A}(0,0),\mathrm{B}(2,0)\&\mathrm{C}(0,2).$

The required equation of circle is $x^2+y^2+2gx+2fy+c=0$ ...(I)

Since $\mathrm{A}(0,0)$ passes through equation (I) we get $0^2+0^2+2g.0+2f.0+c=0$
$\Rightarrow 0+0+0+0+c=0\Rightarrow c=0.$ ...(II)

Since $\mathrm{B}(2,0)$ passes through equation (I) we get $2^2+0^2+2g.2+2f.0+c=0$

$\Rightarrow 4+0+4g+0+c=0\Rightarrow 4+0+4g+0+0=0(\because from(\mathrm{II}))\Rightarrow 4+4g=0$

$\Rightarrow g=-1$ ....(III)

Since $\mathrm{C}(0,2)$ passes through equation (I) we get $0^2+2^2+2g.0+2f.2+c=0$

$\Rightarrow 0+4+0+4f+c=0\Rightarrow 4+0+4g+0+0=0(\because from(\mathrm{II}))\Rightarrow 4+4f=0$

$\Rightarrow f=-1$ ....(IV)

Now substitute the values of g, f, c in equation (I) we get
$x^2+y^2+2(-1)x+2(-1)y+0=0$

$\Rightarrow x^2+y^2-2x-2y=0$

Hence the required equation of circle is $x^2+y^2-2x-2y=0$

2. Problem: Find the equation of the circle passing through the points $(1,1),(3,2)\&(2,-1).$

Solution: Let the given points be $\mathrm{A}(1,1),\mathrm{B}(3,2)\&\mathrm{C}(2,-1).$

The required equation of circle is $x^2+y^2+2gx+2fy+c=0$ ...(I)

Since $\mathrm{A}(1,1)$ passes through equation (I) we get $1^2+1^2+2g.1+2f.1+c=0$
$\Rightarrow 2g+2f+c=-2$ ...(II)

Since $\mathrm{B}(3,2)$ passes through equation (I) we get $3^2+2^2+2g.3+2f.2+c=0$

$\Rightarrow 6g+4f+c=-13$ ...(III)

Since $C(2,-1)$ passes through equation (I) we get $2^2+(-1)^2+2g.2+2f(-1)+c=0$

$\Rightarrow 4g-2f+c=-5$ ...(IV)

By solving equations (II) & (III) we get $4g+2f=-11$ ...(V)

By solving equations (III) & (IV) we get $2g+6f=-8$ ...(VI)

By solving equations (V) & (VI) we get $g=-\frac{5}{2},\ f=-\frac{1}{2}$ ...(VII)

By solving equations (II) & (VII) we get $c=4$ ...(VIII)

Now substitute the values of g, f, c in equation (I) we get

$x^2+y^2+2(-\frac{5}{2})x+2(-\frac{1}{2})y+4=0$

$\Rightarrow x^2+y^2-5x-y+4=0$

Hence the required equation of circle is $x^2+y^2-5x-y+4=0$

6.2 Position of a point in the plane of a circle- Definition of a tangent:

In earlier classes, we have learnt that the tangent at any point of a circle is a straight line which meets the circle at that point only. The point is called a point of contact. This tangent is perpendicular to the radius drawn from the centre to the point of contact. In this section we give another definition of a tangent to the circle using the limit concept. Using this definition we find an equation of tangent at any point in section 6.3. We also learn the position of a point with respect to a circle and power of a point. Further, we define the length of a tangent from appoint and obtain a formula for it.

6.2.1 Notation: Now we introduce certain notations that will be used in rest of this section and subsequently.

(i) The expression $x^2+y^2+2gx+2fy+c$ is denoted by S

i.e., $S \equiv x^2+y^2+2gx+2fy+c.$

(ii) The expression $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$ is denoted by S_1

i.e., $S_1 \equiv xx_1+yy_1+g(x+x_1)+f(y+y_1)+c.$

Similarly $S_2 \equiv xx_2+yy_2+g(x+x_2)+f(y+y_2)+c$

$S_{12} \equiv x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c$

$S_{11} \equiv x_1^{\ 2} + y_1^{\ 2} + 2gx_1 + 2fy_1 + c.$

6.2.2 Position of a point with respect to a circle: The circle in a plane divides the circle into three parts namely

(i) the interior of the circle

(ii) the circumference which is the circular curve

(iii) the exterior of the circle.

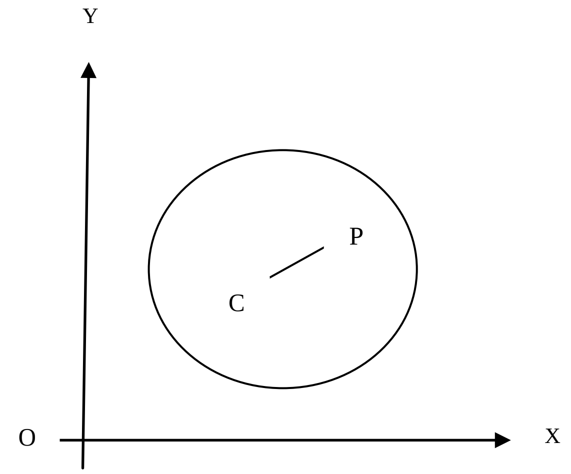
6.2.3 Theorem: Let $S = 0$ be a circle in a plane and $P(x_1, y_1)$ be any point in the same plane. Then

(i) P lies in the interior of the circle $\Leftrightarrow S_{11} < 0,$

(ii) P lies on the circle $\Leftrightarrow S_{11} = 0,$

(iii) P lies in the exterior of the circle $\Leftrightarrow S_{11} > 0.$

Proof: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) be the equation of the given circle and $P(x_1, y_1)$ be any point in the plane. Then $C(-g, -f)$ is the centre and r is the radius of the circle.



(i) P lies in the interior of the circle

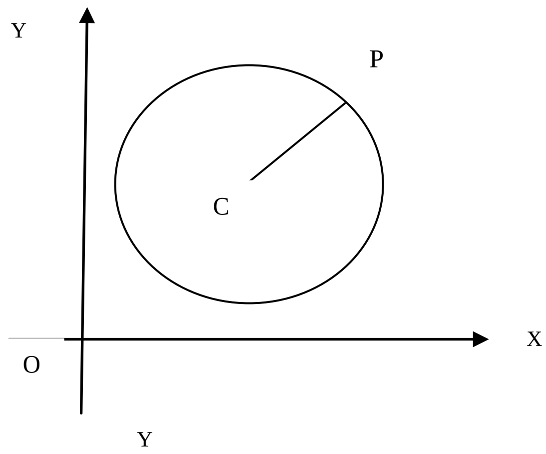
$\Leftrightarrow CP < r$

$\Leftrightarrow CP^2 < r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 < g^2 + f^2 - c$

$\Leftrightarrow x_1^{\ 2} + y_1^{\ 2} + 2gx_1 + 2fy_1 + c < 0.$

$\Leftrightarrow S_{11} < 0.$



(ii) P lies on the circle

$\Leftrightarrow CP = r$

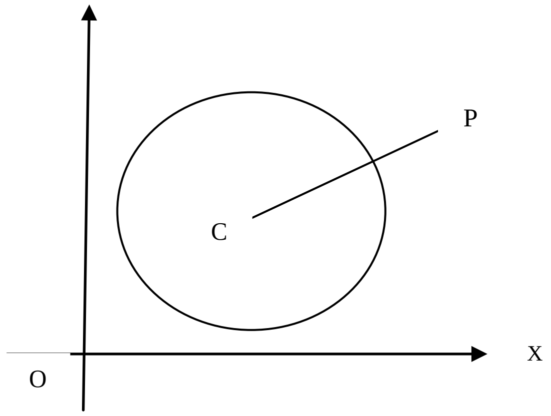
$\Leftrightarrow CP^2 = r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 = g^2 + f^2 - c$

$\Leftrightarrow x_1^{\ 2} + y_1^{\ 2} + 2gx_1 + 2fy_1 + c = 0$

$\Leftrightarrow S_{11} = 0.$

(iii) P lies in the exterior of the circle



$\Leftrightarrow CP > r$

$\Leftrightarrow CP^2 > r^2$

$\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 > g^2 + f^2 - c$

$\Leftrightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0.$

$\Leftrightarrow S_{11} > 0.$

6.2.4 Example: Let $S \equiv x^2 + y^2 + 6x + 8y - 96 = 0$...(I) be the equation of the circle and $P(1,2)$ be a point in the plane. Here $(x_1, y_1) = (1,2)$. $S_{11} = 1^2 + 2^2 + 6(1) + 8(2) - 96 = -69.$ Since $S_{11} < 0$, by Theorem 6.2.3, the point $P(1,2)$ is in the interior of the circle. Note that the centre of the circle is $(-3,-4)$ and radius is 11. The distance from the centre to the point $(1,2)$ is $\sqrt{52}$ which is less than the radius 11. Hence, the point $P(1,2)$ is inside the circle.

6.2.5 Definition: Let P be any point on a given circle and Q be a neighbouring point of P lying on the circle. Join P and Q. Then $\overline{PQ}$ is a secant. The limiting position of the line $\overline{PQ}$ when $Q \to P$ along the circle, is called the *tangent* at P.

6.2.6 Definition: If P is an external point to the circle $S = 0$ and PT is the tangent from P to the circle $S = 0$ Then $\overline{PT}$ is called the *length of the tangent* from P to the circle.

6.2.7 Definition: Suppose $S = 0$ is the equation of the circle with centre C and radius r. Let $P(x_1, y_1)$ be any point in the plane. Then $CP^2 - r^2$ is defined as the *power of* P with respect to the circle $S = 0$

6.2.8 Note: A point $P(x_1, y_1)$ lies in the interior of the circle, on the circle or in the exterior of the circle according as the *power of* P with respect to the circle is negative, zero or positive respectively.

6.2.9 Theorem: The power of a point $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}.

Proof: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) be the equation of the given circle and $P(x_1, y_1)$ be any point in the plane. Then $C(-g, -f)$ is the centre and r is the radius of the given circle. Then $CP^2 - r^2 = (x_1+g)^2 + (y_1+f)^2 - \left(\sqrt{g^2+f^2-c}\right)^2$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_{11}.$$

Hence the power of a point $P(x_1, y_1)$ with respect to the circle $S=0$ is S_{11}.

6.2.10 Example: Let $S \equiv x^2 + y^2 + 6x + 8y - 96 = 0 ...(I)$ be the equation of the circle and $P(1,2)$ be a point in the plane. Let us find the power of a point $P(1,2)$ with respect to the circle $S=0$. Here $(x_1, y_1) = (1,2)$. $S_{11} = 1^2 + 2^2 + 6(1) + 8(2) - 96 = -69$. By Theorem 6.2.9, the power of a point $P(x_1, y_1)$ with respect to the circle $S=0$ is S_{11}. Hence, the power of a point $(1,2)$ with respect to the circle $x^2+y^2+6x+8y-96=0$ is -69.

6.2.11 Note: Let $S=0$ be a circle and $P(x_1, y_1)$ be any point in the plane. If a line through $P(x_1, y_1)$ meets the circle at A and B then the power of P is equal to $PA.PB$

6.2.11 Theorem: Let $S=0$ be a circle and $P(x_1, y_1)$ is an exterior point with respect to the circle $S=0$ then the length of the tangent from $P(x_1, y_1)$ with respect to the circle $S=0$ is $\sqrt{S_{11}}$

Proof: Let the tangent drawn from $P(x_1, y_1)$ meets the circle at A.

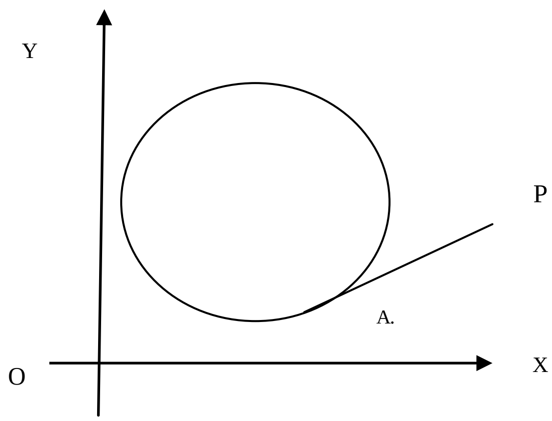
By Note 6.2.11, we have $PA.PA = S_{11}$

$$PA^2 = S_{11}$$

$\therefore PA = \sqrt{S_{11}}$.

Hence the length of the tangent from $P(x_1, y_1)$

to the circle $S=0$ is $\sqrt{S_{11}}$



6.2.12 Example: Let us find the length of the tangent from (12,17) to the circle $S \equiv x^2+y^2-6x-8y-25=0$. Here $(x_1, y_1) = (12,17)$.

$$S_{11} = 12^2 + 17^2 - 6(12) - 8(17) - 25 = 100 \Rightarrow \sqrt{S_{11}} = \sqrt{100} = 10$$

By Theorem 6.2.11, the length of the tangent from $P(x_1, y_1)$ to the circle $S=0$ is $\sqrt{S_{11}}$. Hence, the length of the tangent from $(12,17)$ to the circle $S \equiv x^2+y^2-6x-8y-25=0$ is 10

6.2.13 Solved Problems:

1. Problem: Find the position of the point $P(1,2)$ with respect to the circle $S \equiv x^2 + y^2 - 4x - 6y + 11 = 0$. Also find the power of the point.

Solution: The given point is $P(1,2)$

The given equation of the circle is $S \equiv x^2 + y^2 - 4x - 6y + 11 = 0.$ ....(I)

It is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(II)

Compare equations (I) & (II) we get $2g = -4, 2f = -6, c = 11$

$\Rightarrow g = -2, f = -3, c = 11$

We have $S_{11} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 1^2 + 2^2 + 2(-2)1 + 2(-3)2 + 11$

$\Rightarrow S_{11} = 1 + 4 - 4 - 12 + 11 = 0$

$\therefore P(1,2)$ lies on the circle $S \equiv x^2 + y^2 - 4x - 6y + 11 = 0$

The power of the point $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}

The power of the point $P(1,2)$ with respect to the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is 0.

2. Problem: Find the length of the tangent from $P(1,3)$ to the circle $S \equiv x^2 + y^2 - 2x + 4y - 11 = 0.$

Solution: The given point is $P(1,3)$

The given equation of the circle is $S \equiv x^2 + y^2 - 2x + 4y - 11 = 0$ ....(I)

It is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(II)

Compare equations (I) & (II) we get $2g = -2, 2f = 4, c = 11$

$\Rightarrow g = -1, f = 2, c = 11$

We have $S_{11} \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 1^2 + 3^2 + 2(-1)1 + 2.2.3 + 11$

$\Rightarrow S_{11} = 1 + 9 - 2 + 12 + 11 = 31$

The length of the tangent from $P(x_1, y_1)$ with respect to the circle $S = 0$ is $\sqrt{S_{11}}$

The length of the tangent from $P(1,3)$ with respect to the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is $\sqrt{31}$

3. Problem: If the length of the tangent from $P(5,4)$ to the circle $S \equiv x^2 + y^2 + 2ky = 0$ is 1, then find k

Solution: The given point is $P(5,4)$

The given equation of the circle is $S \equiv x^2 + y^2 + 2ky = 0$ ....(I)

It is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(II)

Compare equations (I) & (II) we get $2g = 0, 2f = 2k, c = 0$

$\Rightarrow g = 0, f = k, c = 0$

Given the length of the tangent from $P(5,4)$ to the circle $S \equiv x^2 + y^2 + 2ky = 0$ is 1,

The length of the tangent from $P(x_1, y_1)$ with respect to the circle $S = 0$ is $\sqrt{S_{11}}$

We have $\sqrt{S_{11}} = 1 \Rightarrow \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = 1 \Rightarrow \sqrt{5^2 + 4^2 + 2k.4} = 1$

$\Rightarrow \sqrt{41+8k} = 1 \Rightarrow 41 + 8k = 1 \Rightarrow 8k = 1 - 41 \Rightarrow 8k = -40 \Rightarrow k = -\dfrac{40}{8} \Rightarrow k = -5$

4. Problem: A point P moves such that the length of the tangents from it to the circles $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$ and $S' \equiv x^2 + y^2 + 6x + 18y + 26 = 0$ are in the ratio $2:3$ then find the locus of P

Solution: Let $P(x_1, y_1)$ be any point on the locus.

Given equation of the circles are
$S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$........(I), $S' \equiv x^2 + y^2 + 6x + 18y + 26 = 0$........(II)

Given ratio of the length of the tangents from $P(x_1, y_1)$ to the circles (I) & (II) is $2:3$

i.e $\sqrt{S_{11}} : \sqrt{S'_{11}} = 2:3$

$$\Rightarrow \frac{\sqrt{x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12}}{\sqrt{x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26}} = \frac{2}{3} \Rightarrow \frac{x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12}{x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26} = \frac{4}{9}$$

$\Rightarrow 9(x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12) = 4(x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26)$

$\Rightarrow 9x_1^2 + 9y_1^2 - 36x_1 - 54y_1 - 108 = 4x_1^2 + 4y_1^2 + 24x_1 + 72y_1 + 104$

$\Rightarrow 5x_1^2 + 5y_1^2 - 60x_1 - 126y_1 - 212 = 0$

Hence the required locus is $5x^2 + 5y^2 - 60x - 126y - 212 = 0$

6.3 Position of a straight line in the plane of a circle- Condition for a line to be tangent:

In the earlier classes, we learnt the position of a point with respect to a circle. In this section we shall learn the position of a straight line in a plane with respect to a circle.

6.3.1 Different cases of position of a straight line with respect to a circle:

Given a straight line $\mathrm{L}=0$ and a circle $\mathrm{S}=0$ we have three possibilities, namely:

(i) The line meets the circle in two distinct points

(ii) The line meets the circle in one and only one point

(iii) The line L does not meets the circle *i.e.*, L and the circle have no common points.

Now we examine under what conditions the above three situations arise.

6.3.2 Theorem: A straight line $y=mx+c$

(i) meets the circle $x^2+y^2=r^2$ in two distinct points if $\dfrac{c^2}{1+m^2}<r^2$

(ii) touches the circle $x^2+y^2=r^2$ if $\dfrac{c^2}{1+m^2}=r^2$

(iii) has no points in common with the circle $x^2+y^2=r^2$ if $\dfrac{c^2}{1+m^2}>r^2$.

Proof: The equation of the given circle is $x^2+y^2=r^2$ ...(I) and the equation of the given straight line equation is $y=mx+c$ *i.e.*, $mx-y+c=0$...(II)

If any point common with (I) and (II), the coordinates of the point satisfy both the equations (I) and (II). To solve them we eliminate y from (I) and (II).

Substitute (II) in (I) we get $x^2+(mx+c)^2=r^2$

i.e., $x^2(1+m^2)+2mcx+(c^2-r^2)=0$...(III)

The roots of (III) are real, coincident or imaginary according as

$$(2mc)^2-4(1+m^2)(c^2-r^2) >=< 0$$

$$i.e., 4m^2c^2-4(c^2+c^2m^2-r^2-r^2m^2) >=< 0$$

$$i.e., c^2-r^2(1+m^2) >=< 0$$

i.e., $\frac{c^2}{1+m^2} >=< r^2$

Case (i): If $\frac{c^2}{1+m^2} < r^2$, then the straight line meets the circle in two distinct points.

Case (ii): If $\frac{c^2}{1+m^2} = r^2$, then the straight line touches the circle.

Case (iii): If $\frac{c^2}{1+m^2} > r^2$, then the straight line does not touch or cut the circle.

6.3.3 Note: If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle then the straight line $lx + my + n = 0$

(i) meets the circle $S = 0$ in two distinct points if $g^2 + f^2 - c > \frac{(gl + mf - n)^2}{l^2 + m^2}$

(ii) touches the circle $S = 0$ if $g^2 + f^2 - c = \frac{(gl + mf - n)^2}{l^2 + m^2}$

(iii) has no points in common with the circle $S = 0$ if $g^2 + f^2 - c < \frac{(gl + mf - n)^2}{l^2 + m^2}$.

6.3.4 Note: (i) For all real values of m the straight line $y = mx \pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = r^2$ and the slope of the tangent is m.

(ii) A straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if $c = \pm r\sqrt{1+m^2}$ tangent is m.

(iii) The equation of tangent to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ having the slope m is $y + f = m(x+g) \pm r\sqrt{1+m^2}$ where r is the radius of the circle.

6.3.5 Chord joining two points on a circle: In the next section we derive the chord joining two points on a circle.

6.3.6 Theorem: The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the circle $S = 0$ is $S_1 + S_2 = S_{12}$.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then the equation of $\overline{PQ}$ is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$...(I)

Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the circle $S=0$ we have $S_{11}=0$ and $S_{22}=0$ *i.e.,* $x_1^2+y_1^2+2gx_1+2fy_1+c=0$ and $x_2^2+y_2^2+2gx_2+2fy_2+c=0$

Subtracting and simplifying we get $\frac{y_2-y_1}{x_2-x_1}=\frac{-(x_1+x_2+2g)}{(y_1+y_2+2f)}$...(II)

Substituting (II) in (I), we get $y-y_1=\frac{-(x_1+x_2+2g)}{(y_1+y_2+2f)}(x-x_1)$...(III)

$$i.e., (x-x_1)(x_1+x_2+2g)+(y-y_1)(y_1+y_2+2f)=0$$

$$i.e., xx_1+yy_1+xx_2+yy_2+2gx+2fy=x_1x_2+y_1y_2+x_1^2+y_1^2+2gx_1+2fy_1$$

By adding $g(x_1+x_2)+f(y_1+y_2)+2c$ on both sides to the above equation we obtain

$$S_1+S_2=S_{11}+S_{12}$$

$$S_1+S_2=S_{12} \quad (\because S_{11}=0)$$

$\therefore$ The equation of the chord $\overline{PQ}$ is $S_1+S_2=S_{12}$.

6.3.7 Chord joining two points on a circle: In the next section we derive the equation of tangent at a point on the circle.

6.3.8 Theorem: The equation of tangent at $P(x_1, y_1)$ to the circle $S=0$ is $S_1=0$.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the circle $S \equiv x^2+y^2+2gx+2fy+c=0$.

By Theorem 6.3.7, the equation of the chord $\overline{PQ}$ is $S_1+S_2=S_{12}$...(I)

The chord $\overline{PQ}$ becomes the tangent at P when Q approaches P.

i.e., (x_2, y_2) approaches to (x_1, y_1)

$\therefore$ The equation of the tangent at P is obtained by taking limits (x_2, y_2) tends to as (x_1, y_1) on either sides of (I)

So, the equation of the tangent at P is given by $\lim_{Q\to P}(S_1+S_2)=\lim_{Q\to P}S_{12}$

i.e., $S_1+S_1=S_{11}\left[\because S_2\to S_1, S_{12}\to S_{11} \text{ as } (x_2, y_2)\to(x_1, y_1)\right]$

$\therefore 2S_1=0 \Rightarrow S_1=0.$

$\therefore$ The equation of the tangent at the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

6.3.9 Note: The equation of the tangent at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 - r^2 = 0.$

6.3.10 Point of contact: If a straight line $lx + my + n = 0$ touches the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$ then this line is the tangent to the circle $S = 0$ at $P(x_1, y_1)$ and hence by Theorem 6.3.8 its equation is $(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$ and therefore $(x_1 + g), (y_1 + f), (gx_1 + fy_1 + c)$ are proportional to l, m, n respectively. Using these three proportions it is possible to find the point of contact of the given tangent to the circle $S = 0.$ definition we find an equation of tangent at any point in section 6.3. We also learn the position of a point with respect to a circle and power of a point. Further, we define the length of a tangent from a point and obtain a formula for it.

6.3.11 Solved Problems:

1. Problem: Find the equation of the tangent at $P(-1,1)$ to the circle

$S \equiv x^2 + y^2 - 6x + 4y - 12 = 0.$

Solution: The given point is $P(-1,1)$

The given equation of the circle is $S \equiv x^2 + y^2 - 6x + 4y - 12 = 0$..............(I)

It is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.........(II)

Compare equations (I) & (II) we get $2g = -6, 2f = 4, c = -12$

$\Rightarrow g = -3, f = 2, c = -12$

The equation of tangent at $P(x_1, y_1)$ to the circle $S = 0$ is $S_1 = 0$

The equation of tangent at $P(-1,1)$ to the circle $S \equiv x^2 + y^2 - 6x + 4y - 12 = 0$ is

$S_1 \equiv x(-1) + y(1) - 3(x - 1) + 2(y + 1) - 12 = 0 \Rightarrow -x + y - 3x + 3 + 2y + 2 - 12 = 0$

$\Rightarrow -4x + 3y - 7 = 0 \Rightarrow 4x - 3y + 7 = 0$

Hence the required equation of tangent is $4x - 3y + 7 = 0$

2. Problem: Find the point of contact of the line $x+y+\frac{3}{2}=0$ with respect to the circle $S \equiv x^2+y^2-3x+7y+14=0.$

Solution: The given line is $x+y+\frac{3}{2}=0$ ....(I)

The given equation of the circle is $S \equiv x^2+y^2-3x+7y+14=0$ ....(II)

It is of the form $x^2+y^2+2gx+2fy+c=0$ ....(III)

Compare equations (II) & (III) we get $2g=-3, 2f=7, c=14$

$\Rightarrow g=-\frac{3}{2}, f=\frac{7}{2}, c=14$

Let $P(h,k)$ be the point of contact.

The equation of tangent at $P(h,k)$ to the circle $S=0$ is $S_1=0$

The equation of tangent at $P(h,k)$ to the circle $S \equiv x^2+y^2-3x+7y+14=0$ is

$$S_1 \equiv x.h+y.k-\frac{3}{2}(x+h)+\frac{7}{2}(y+k)+14=0$$

$$\Rightarrow \left(h-\frac{3}{2}\right)x+\left(k+\frac{7}{2}\right)y+\left(-\frac{3}{2}h+\frac{7}{2}k+14\right)=0 \quad \text{....(IV)}$$

Since (I) & (IV) represents same equation we get

$$\Rightarrow \frac{\left(h-\frac{3}{2}\right)}{1}=\frac{\left(k+\frac{7}{2}\right)}{1}=\frac{\left(-\frac{3}{2}h+\frac{7}{2}k+14\right)}{\frac{3}{2}} \Rightarrow h-\frac{3}{2}=1, k+\frac{7}{2}=1, -\frac{3}{2}h+\frac{7}{2}k+14=\frac{3}{2}$$

$$\Rightarrow h=\frac{5}{2}, k=-\frac{5}{2}$$

Hence the required point of contact is $\left(\frac{5}{2},-\frac{5}{2}\right)$

6.3.12 Normal: The normal at any point P of the circle is the line passes through P and is perpendicular to the tangent at P.

6.3.13 Equation of normal: We find the equation of normal at a point lying on the circle.

6.3.14 Theorem: The equation of normal at $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0.$

Proof: Let C be the centre of the circle given by $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I)

Then $C(-g, -f)$.

We know that normal at any point passes through the centre of the circle.

The slope of $CP = \dfrac{y_1 + f}{x_1 + g}$.

Hence the equation of the normal at $P(x_1, y_1)$ is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$

$$i.e., (x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

6.3.15 Note: The equation of normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0.$

6.3.16 Example: Let us find the equation of normal to the circle $x^2 + y^2 = 25$ at $P(3, 4)$

By Note 6.3.15, the equation of normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0.$

Hence, the equation of normal to the circle $x^2 + y^2 = 25$ at $P(3, 4)$ is $4x - 3y = 0.$

Exercise 6(a)

I Find the equation of the circle having centre C and radius r are as given below:

(*i*) $C(-1, 2), r = 4$ (*ii*) $C(-7, -3), r = 4$ (*iii*) $C\left(\frac{5}{2}, -\frac{4}{3}\right), r = 6$ (*iv*) $C(1, 7), r = \frac{5}{2}$

(*v*) $C(0, 0), r = 9$ (*vi*) $C\left(-\frac{1}{2}, -9\right), r = 5$ (*vii*) $C(a, -b), r = a + b$

(*viii*) $C(a, -b), r = \sqrt{a^2 - b^2}$ (*ix*) $C(\cos\theta, \sin\theta), r = 1$ (*x*) $C(a, b), r = \sqrt{a^2 + b^2}$

II Find the centre and radius for the following circles:

$(i)\, x^2+y^2-4x-8y-41=0.$ $(ii)\, 3x^2+3y^2-6x+4y-4=0.$
$(iii)\; 3x^2+3y^2-5x-6y+4=0.$ $(iv)\, 2x^2+2y^2-4x+6y-3=0.$
$(v)\; x^2+y^2+2ax-2by+b^2=0.$

III Find the equations of the circles whose extremities of the diameter are given below:

$(i)\; (-4,3)\&(3,4)$ $(ii)\, (8,6)\&(1,2)$ $(iii)\; (4,2)\&(1,5)$ $(iv)\; (7,-3)\&(3,5)$ $(v)\, (1,1)\&(2,-1)$
$(vi)\, (3,1)\&(2,7)$ $(vii)\, (0,0)\&(8,5)$ $(viii)\; (1,2)\&(4,6)$ $(ix)\, (a,0)\&(0,b)$

IV Find the equation of the circle passing through the points as given below:
$(i)\, (3,2),(3,4),(1,4)$ $(ii)\, (2,1),(5,5),(-6,7)$ $(iii)\, (5,7),(8,1),(1,3)$ $(iv)\, (0,0),(a,0),(0,b)$

V Find the position of the point P with respect to the circle $S=0.$ Also find the power of the point.

$(i)\; P(3,4),\; S \equiv x^2+y^2-4x-6y+12=0$ $(ii)\; P(1,5),\; S \equiv x^2+y^2-2x-4y+3=0$
$(iii)\; P(2,-1), S \equiv x^2+y^2-2x-4y+3=0$ $(iv) P(4,2), S \equiv 2x^2+2y^2-5x-4y-3=0$
$(v) P(1,2), S \equiv x^2+y^2+6x+8y-96=0$ $(vi) P(5,-6), S \equiv x^2+y^2+8x+12y+15=0$
$(vii) P(2,3), S \equiv x^2+y^2-2x+8y-23=0$ $(viii) P(0,0), S \equiv x^2+y^2-14x+2y+25=0$
$(ix) P(-2,5), S \equiv x^2+y^2-25=0$

VI Find the length of the tangent from P to the circle $S=0.$

$(i)\; P(12,17),\; S \equiv x^2+y^2-6x-8y-125=0$ $(ii)\; P(0,0),\; S \equiv x^2+y^2-14x+2y+25=0$
$(iii)\; P(-2,5), S \equiv x^2+y^2-25=0$ $(iv) P(2,5), S \equiv x^2+y^2-5x+4y-5=0$

VII Find the equation of tangent at P to the circle $S=0.$

$(i)\; P(7,-5),\; S \equiv x^2+y^2-6x+4y-12=0$ $(ii)\; P(-1,0)2,\; S \equiv x^2+y^2-4x-8y+7=0$
$(iii)\; P(-6,-9), S \equiv x^2+y^2+4x+6y-39=0$ $(iv) P(3,4), S \equiv x^2+y^2-4x-6y+11=0$
$(v) P(3,2), S \equiv x^2+y^2-x-3y-4=0$ $(vi) P(1,1), S \equiv 2x^2+2y^2-2x-5y+3=0$
$(vii) P(3,-2), S \equiv x^2+y^2=13$

VIII

1. Find a if the radius of the circle $x^2+y^2-4x+6y+a=0$ is $4.$

2. If the length of the tangent from $(2,5)$ to the circle $S \equiv x^2+y^2-5x+4y+k=0$ is $\sqrt{37},$ then find k

3. A point P moves such that the length of the tangents from it to the circles $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$ and $S' \equiv x^2 + y^2 + 8x + 12y + 15 = 0$ are in the equal ratio then find the locus of P

4. A point P moves such that the length of the tangents from it to the circles $S \equiv x^2 + y^2 - 2x + 4y - 20 = 0$ and $S' \equiv x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio $2:1$ then find the locus of P

5. Find the point of contact of the line $4x - 3y + 7 = 0$ with respect to the circle $S \equiv x^2 + y^2 - 6x + 4y - 12 = 0.$

Key concepts

1. A circle is the set of points in a plane such that they are equidistant from a fixed point lying in the plane. The fixed point is called the centre and the distance from the centre to a point on the circle.
2. The equation of a circle with centre $O(0,0)$ and radius r is $x^2 + y^2 = r^2.$
3. The equation of a circle with centre $C(h,k)$ and radius r is $(x-h)^2 + (y-k)^2 = r^2.$
4. $x^2 + y^2 + 2gx + 2fy + c = 0$ is considered as general equation of circle.
5. The centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f).$
6. The radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{g^2 + f^2 - c}.$
7. Two or more circles are said to be concentric if their centres are same.
8. If the radius of a circle is 1 then it is called a unit circle.
9. If the radius of a circle is 0 then it is called a point circle.
10. If $g^2 - c > 0$ then the intercept made on the x^- axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{g^2 - c}.$
11. If $f^2 - c > 0$ then the intercept made on the y^- axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{f^2 - c}.$
12. The equation of the circle whose diameter extremities are (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$
13. The equation of a circle passing through three non-collinear points $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2)+\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}x+\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}y+\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}=0$$ where

$c_i = -(x_i^{\,2}+y_i^{\,2})\ (i=1,2,3)$

14. The centre of the circle passing through three non-collinear points

$\mathrm{P}(x_1,y_1),\mathrm{Q}(x_2,y_2)$ and $\mathrm{R}(x_3,y_3)$ is $$\left(\frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{-2\times\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}},\frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{-2\times\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}\right)$$

15. Let $\mathrm{S}=0$ be a circle in a plane and $\mathrm{P}(x_1,y_1)$ be any point in the same plane. Then

(i) P lies in the interior of the circle $\Leftrightarrow \mathrm{S}_{11}<0,$

(ii) P lies on the circle $\Leftrightarrow \mathrm{S}_{11}=0,$

(iii) P lies in the exterior of the circle $\Leftrightarrow \mathrm{S}_{11}>0.$

16. The power of a point $\mathrm{P}(x_1,y_1)$ with respect to the circle $\mathrm{S}=0$ is S_{11}.

17. The length of the tangent from $\mathrm{P}(x_1,y_1)$ with respect to the circle $\mathrm{S}=0$ is $\sqrt{\mathrm{S}_{11}}$

18. If $\mathrm{S}\equiv x^2+y^2+2gx+2fy+c=0$ represents a circle then the straight line $lx+my+n=0$

(i) meets the circle $\mathrm{S}=0$ in two distinct points if $g^2+f^2-c>\frac{(gl+mf-n)^2}{l^2+m^2}$

(ii) touches the circle $\mathrm{S}=0$ if $g^2+f^2-c=\frac{(gl+mf-n)^2}{l^2+m^2}$

(iii) has no points in common with the circle $\mathrm{S}=0$ if $g^2+f^2-c<\frac{(gl+mf-n)^2}{l^2+m^2}.$

19. The equation of the chord joining two points (x_1,y_1) and (x_2,y_2) on the circle $\mathrm{S}=0$ is $\mathrm{S}_1+\mathrm{S}_2=\mathrm{S}_{12}$.

20. The equation of tangent at $\mathrm{P}(x_1,y_1)$ to the circle $\mathrm{S}=0$ is $\mathrm{S}_1=0.$

21. The equation of the tangent at the point $\mathrm{P}(x_1,y_1)$ to the circle $x^2+y^2=r^2$ is $xx_1+yy_1-r^2=0.$

22. The equation of normal at $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0.$

23. The equation of normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0.$

Answers
Exercise 6(a)

I (i) $x^2 + y^2 + 2x - 4y - 11 = 0.$ (ii) $x^2 + y^2 + 14x + 6y + 42 = 0$

(iii) $36x^2 + 36y^2 - 180x + 96y - 1007 = 0$ (iv) $4x^2 + 4y^2 - 8x - 56y + 175 = 0$
(v) $x^2 + y^2 = 81$ (vi) $4x^2 + 4y^2 + 4x + 72y + 225 = 0$ (vii) $x^2 + y^2 - 2ax + 2by - 2ab = 0$

$(viii)$ $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ (ix) $x^2 + y^2 - 2x\cos\theta - 2y\sin\theta = 0$
(x) $x^2 + y^2 - 2ax - 2by = 0$

II $(i)\,C(2,4), r = \sqrt{61}$ $(ii)\,C\left(1, \frac{1}{3}\right), r = \frac{\sqrt{22}}{3}$ $(iii)\,C\left(\frac{5}{6}, 1\right), r = \frac{\sqrt{13}}{6}$

$(iv)\,C\left(1, -\frac{3}{2}\right), r = \frac{\sqrt{19}}{2}$ (v) $C(-a, b), r = a$.

III (i) $x^2 + y^2 + x - 7y = 0$ (ii) $x^2 + y^2 - 9x - 8y + 20 = 0$
(iii) $x^2 + y^2 - 5x - 7y + 14 = 0$ (iv) $x^2 + y^2 - 10x - 2y + 6 = 0$ (v) $x^2 + y^2 - 3x + 1 = 0$
(vi) $x^2 + y^2 - 5x - 8y + 13 = 0$ (vii) $x^2 + y^2 - 8x - 5y = 0$
$(viii)$ $x^2 + y^2 - 5x - 8y + 16 = 0$ (ix) $x^2 + y^2 - ax - by = 0.$

IV (i) $x^2 + y^2 - 4x - 6y + 11 = 0$ (ii) $x^2 + y^2 + x - 12y + 5 = 0$
(iii) $3x^2 + 3y^2 - 29x - 19y + 56 = 0$ (iv) $x^2 + y^2 - ax - by = 0$

V (i) out side (ii) out side (iii) out side (iv) out side (v) in side (vi) out side
(vii) out side $(viii)$ out side (ix) out side

VI (i) 10 (ii) 5 (iii) 2 $(iv)\sqrt{34}$

VII (i) $4x - 3y - 43 = 0$ (ii) $3x + 4y - 5 = 0$ (iii) $2x + 3y + 39 = 0$ (iv) $x + y - 7 = 0$
(v) $5x + y - 17 = 0$ (vi) $2x - y - 1 = 0$ (vii) $3x - 2y = 13$

VIII

1. $a = -3$. 2. $k = -2$ 3. $4x + 6y + 9 = 0$

4. $x^2 + y^2 - 2x - 12y + 8 = 0$ 5. $(-1, 1)$.

7. SYSTEM OF CIRCLES

Introduction:

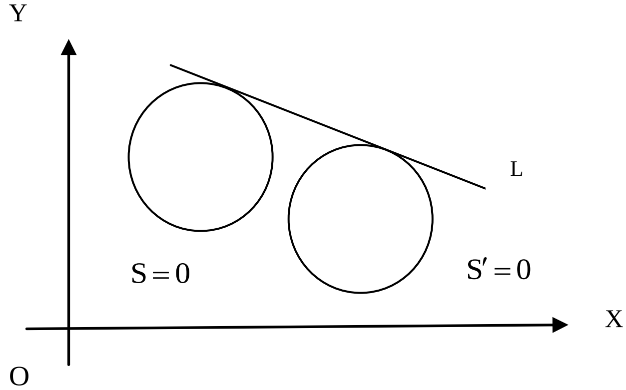
In this chapter, we shall discuss the relative position of two circles, the angle between two intersecting circles and obtain a condition for orthogonality. Also, we shall learn about the radical axis of two circles, its properties, common chord, common tangent of two circles and the radical centre.

7.1 Relative positions of two circles:

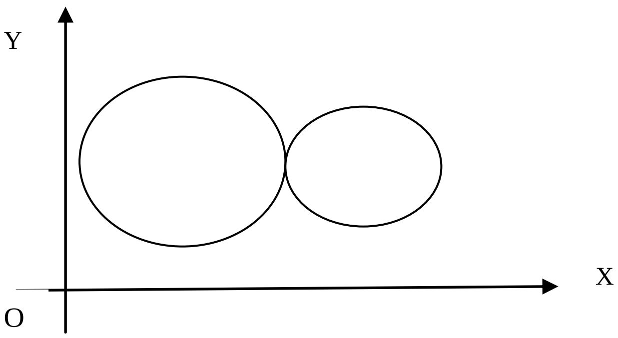
The number of common tangents that can be drawn to two given circles depend on their relative positions. We shall describe the various possible relative positions of two circles. First, let us recall that any two intersecting common tangents of two circles and the line joining the centres of the circles are concurrent, equivalently the point of intersection of two common tangents (if exists) of two circles are collinear. In this section we learn the different possible relative position of two circles and the number of common tangents exists in each case.

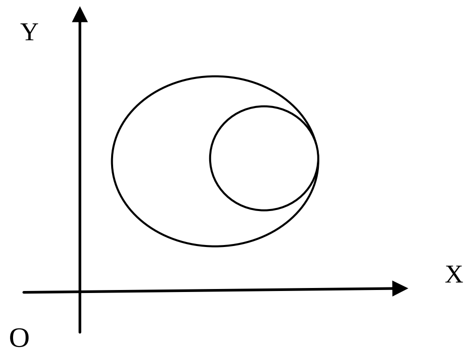
7.1.1 Definition:

A straight line L is said to be a common tangent to the circles $S = 0$ and $S' = 0$ if it is a tangent to both $S = 0$ and $S' = 0$.



7.1.2 Definition: Two circles are said to be touching each other if they have only one common point.





7.1.3 Relative positions of two circles: Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S = 0$ and $S' = 0$ respectively. Further let $\overline{C_1C_2}$ represents the line segment from C_1 to C_2. The following cases arise with regard to the relative position of two circles.

$(i)\quad C_1C_2 > r_1 + r_2$

In this case the two circles do not intersect and one circle will be away from the other circle.

$(ii)\quad C_1C_2 = r_1 + r_2$

In this case the two circles touch each other externally.

$(iii)\quad |r_1 - r_2| < C_1C_2 < r_1 + r_2$

In this case the two circles intersection two distinct points.

$(iv)\quad C_1C_2 = |r_1 - r_2|$

In this case the two circles touch each other internally.

$(v)\quad C_1C_2 < |r_1 - r_2|$

In this case the two circles do not touch or do not intersect and one circle will be completely in side the other.

7.1.4 Note: If $C_1C_2 = 0$ then the centres of the two circles coincides and they are concentric circles.

7.1.5 Note: Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S = 0$ and $S' = 0$ respectively.

(i) If $C_1C_2 > r_1 + r_2$. In this case the points P, Q are collinear with the centres C_1 and C_2 of given circles. The point of intersection of transverse pair of common tangents P is called the internal centre of similitude and the point of intersection of conjugate pair of common tangents Q is called the internal centre of similitude. Here P divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ internally and Q divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ externally. In this case the number of distinct common tangents are 4.

(ii) If $|r_1 - r_2| < C_1C_2 < r_1 + r_2$. In this case the internal centre of similitude P is the point of contact of two given circles. At P there is only one common tangent. Through Q, there will be two common tangents. In this case the number of distinct common tangents are 3.

(iii) If $C_1C_2 = r_1 + r_2$. In this case the internal centre of similitude does not exist. Only two common tangents exists at Q. In this case the number of distinct common tangents are 2.

(iv) If $C_1C_2 = |r_1 - r_2|$. In this case the internal centre of similitude does not exist. Only one common tangent exists at Q. In this case the number of distinct common tangents is 1.

(v) If $C_1C_2 < |r_1 - r_2|$. In this case the internal centre of similitude and external centre of similitudes does not exist. In this case the number of distinct common tangents is zero.

7.1.6 Solved Problems:

1. Problem: Discuss the relative position of the following pair of circles
$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$.

Solution: Let the given equations of circles be $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$.......(I)

$$S' \equiv x^2 + y^2 + 6x + 18y + 26 = 0 \text{.......(II)}$$

Let C_1, C_2 be centres and r_1, r_2 be radii of circles (I) & (II) respectively.

We have $C_1(2,3), C_2(-3,-9)$ and $r_1 = 5, r_2 = 8$.

Now $C_1C_2 = \sqrt{(2+3)^2 + (3+9)^2} = \sqrt{5^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$r_1 + r_2 = 5 + 8 = 13$.

$\therefore C_1C_2 = r_1 + r_2$.

$\therefore$ The two circles touch each other externally.

2. Problem: Show that the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $5x^2 + 5y^2 - 8x - 14y - 32 = 0$ touch each other internally. Also find the point of contact.

Solution: Let the given equations of circles be $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$.......(I)

$$S' \equiv x^2 + y^2 - \frac{8}{5}x - \frac{14}{5}y - \frac{32}{5} = 0 \text{.......(II)}$$

Let C_1, C_2 be centres and r_1, r_2 be radii of circles (I) & (II) respectively.

We have $C_1(2,3), C_2\left(\frac{4}{5}, \frac{7}{5}\right)$ and $r_1 = 5, r_2 = 3$.

Now $C_1C_2 = \sqrt{\left(2 - \frac{4}{5}\right)^2 + \left(3 - \frac{7}{5}\right)^2} = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = 2$

$r_1 - r_2 = 5 - 3 = 2$.

$\therefore C_1C_2 = r_1 - r_2$.

$\therefore$ The two circles touch each other internally.

The point of contact P divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ externally *i.e* P divides $\overline{C_1C_2}$ in the ratio $5:3$ externally.

$$P=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)=\left(\frac{5\left(\frac{4}{5}\right)-3(2)}{5-3},\frac{5\left(\frac{7}{5}\right)-3(3)}{5-3}\right)=\left(\frac{4-6}{2},\frac{7-9}{2}\right)=(-1,-1)$$

3. Problem: Show that four common tangents can be drawn for the circles given by $x^2+y^2-14x+6y+33=0$ and $x^2+y^2+30x-2y+1=0$. Also find the internal and external centre of similitudes.

Solution: Let the given equations of circles be $S \equiv x^2+y^2-14x+6y+33=0$.......(I)

$$S' \equiv x^2+y^2+30x-2y+1=0\text{.......(II)}$$

Let C_1, C_2 be centres and r_1, r_2 be radii of circles (I) & (II) respectively.

We have $C_1(7,-3), C_2(-15,1)$ and $r_1=5, r_2=15$.

Now $C_1C_2=\sqrt{(7+15)^2+(-3-1)^2}=\sqrt{22^2+(-4)^2}=\sqrt{484+16}=\sqrt{500}=10\sqrt{5}$

$r_1+r_2=5+15=20.$

$\therefore C_1C_2 > r_1+r_2.$

$\therefore$ The two circles do not touch and do not intersect.

$\therefore$ Four common tangents can be drawn for the circles.

The point P divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ internally is internal centre of similitude.

$$P=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)=\left(\frac{1(-15)+3(7)}{1+3},\frac{1(1)+3(-3)}{1+3}\right)=\left(\frac{-15+21}{4},\frac{1-9}{4}\right)$$
$$=\left(\frac{6}{4},\frac{-8}{4}\right)=\left(\frac{3}{2},-2\right)$$

The point Q divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ externally is external centre of similitude.

$$Q=\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)=\left(\frac{1(-15)-3(7)}{1-3},\frac{1(1)-3(-3)}{1-3}\right)=\left(\frac{-15-21}{-2},\frac{1+9}{-2}\right)$$
$$=\left(\frac{-36}{-2},\frac{10}{-2}\right)=(18,-5)$$

4. Problem: Show that the circles $x^2+y^2-2x+4y-4=0$ and $x^2+y^2+4x-6y-3=0$ intersect each other. Also find the number of common tangents.
Solution: Let the given equations of circles be $\mathrm{S} \equiv x^2+y^2-2x+4y-4=0$.......(I)

$$\mathrm{S}' \equiv x^2+y^2+4x-6y-3=0\text{.......(II)}$$

Let $\mathrm{C}_1, \mathrm{C}_2$ be centres and r_1, r_2 be radii of circles (I) & (II) respectively.

We have $\mathrm{C}_1(1,-2), \mathrm{C}_2(-2,3)$ and $r_1=3, r_2=4$.

Now $\mathrm{C}_1\mathrm{C}_2=\sqrt{(1+2)^2+(-2-3)^2}=\sqrt{3^2+(-5)^2}=\sqrt{34}$

$r_1+r_2=3+4=7.$

$\therefore \mathrm{C}_1\mathrm{C}_2 < r_1+r_2.$

$\therefore$ The two circles intersect each other.

$\therefore$ Two common tangents can be drawn for the circles.

7.2 Angle between two intersecting circles:

We have learnt that two circles will intersect each other if the distance between their centres lies between the absolute value of the difference of their radii and the sum of their radii. For such circles we define the angle between them.

7.2.1 Definition: The angle between two intersecting circles is defined as the angle between the tangents at the point of intersection of the two circles.

7.2.2 Note: If two circles $\mathrm{S}=0, \mathrm{S}'=0$ intersect at P and Q then the angle between the two circles at P and Q are equal.

7.2.3 Theorem: If (i) $\mathrm{C}_1, \mathrm{C}_2$ are centres of two given intersecting circles $(ii)\, d=\mathrm{C}_1\mathrm{C}_2$ $(iii)\, r_1, r_2$ are radii of these circles $(iv)\, \theta$ is the angle between these circles, then

$$\cos\theta=\frac{d^2-r_1^2-r_2^2}{2r_1r_2}.$$

Proof: Let P be a point of intersection of two given circles. Let the tangents drawn to two circles at P intersect the line joining the centres at T_1 and T_2. Then $\angle \mathrm{T}_1\mathrm{P}\mathrm{T}_2=\theta.$

Consider $\angle \mathrm{C}_1\mathrm{P}\mathrm{C}_2=\angle \mathrm{C}_1\mathrm{P}\mathrm{T}_2+\angle \mathrm{T}_2\mathrm{P}\mathrm{C}_2=90^0+90^0-\theta=180^0-\theta.$

From $\Delta \mathrm{C}_1\mathrm{P}\mathrm{C}_2$, we have $\mathrm{C}_1\mathrm{C}_2{}^2=\mathrm{C}_1\mathrm{P}^2+\mathrm{C}_2\mathrm{P}^2-2(\mathrm{C}_1\mathrm{P})(\mathrm{C}_2\mathrm{P})\cos\angle \mathrm{C}_1\mathrm{P}\mathrm{C}_2$

$$i.e.,\ d^2=r_1^2+r_2^2-2r_1r_2\cos(180^0-\theta)$$

$$i.e., \; d^2 = r_1^2 + r_2^2 - 2r_1r_2(-\cos\theta) \left[\because \cos(180^0 - \theta) = -\cos\theta\right]$$

$$\therefore \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

Note that $\cos\theta$ is independent of the point of intersection (coordinates of the point of intersection are not involved). Therefore, the angle at Q is also equal to θ.

7.2.4 Theorem: If θ is the angle between the intersecting circles $x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II) then

$$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}.$$

Proof: Let the given equations of circles be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I)

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \text{...(II)}$$

Let C_1, C_2 be centres and r_1, r_2 be radii of circles (I) & (II) respectively. Then $C_1(-g, -f), C_2(-g', -f')$

and $r_1 = \sqrt{g^2 + f^2 - c}, r_2 = \sqrt{g'^2 + f'^2 - c'}$.

Let θ be angle between circles (I) & (II)

By Theorem 7.2.3, we have

$$\cos\theta = \left(\frac{(g' - g)^2 + (f' - f)^2 - (g^2 + f^2 - c) - (g'^2 + f'^2 - c')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}\right)$$

$$\therefore \cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}.$$

7.2.5 Definition: Two intersecting circles are said to be orthogonal if the angle between them is a right angle $(i.e., 90^0)$

7.2.6 Condition for orthogonality: Let the two given circles be given by $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II)

These two circles are orthogonal, By Theorem 7.2.4,

$$\Leftrightarrow \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}} = 0$$

$\Leftrightarrow c+c'-2gg'-2ff'=0 \Leftrightarrow 2gg'+2ff'=c+c'$ Thus, the condition for orthogonality of the two intersecting circles (I) & (II) is $2gg'+2ff'=c+c'$.

7.2.7 Note: (i) Two intersecting circles are orthogonal if and only if the square of the distance between their centres is equal to the sum of the squares of their radii. In this case, a tangent of one circle at the point of intersection will be normal to the other circle and hence it passes through the centre of the other circle.

(ii) If two circles are orthogonal, then $d^2 = r_1^2 + r_2^2$ where d is the distance between the centres of the circles and r_1, r_2 are their radii.

7.2.8 Theorem: (i) If $S=0, S'=0$ are two circles intersecting at two distinct points, then $S-S'=0$ (or $S'-S=0$) represents a common chord of these circles.

(ii) If $S=0, S'=0$ are two circles touching each other, then $S-S'=0$ (or $S'-S=0$) represents a common chord of these circles.

Proof: Let the given equations of circles be $S \equiv x^2+y^2+2gx+2fy+c=0$...(I)

$$S' \equiv x^2+y^2+2g'x+2f'y+c'=0 \text{...(II)}$$

(i) Let $P=(x_1,y_1), Q=(x_2,y_2)$ be the point of intersection of (I) and (II)

Consider $S-S'=0$

i.e., $2(g-g')x+2(f-f')y+(c-c')=0$...(III)

Clearly the points P, Q lie on (III), since $S_{11}=0, S_{22}=0, S'_{11}=0, S'_{22}=0$.

Further, the equation (III) is linear in x and y and hence it represents a line. Therefore $S-S'=0$ is the equation of common chord of circles (I) and (II)

(ii) Let (I) and (II) touch each other at $P(x_1,y_1)$.

Consider $S-S'=0$

i.e., $2(g-g')x+2(f-f')y+(c-c')=0$

$P(x_1,y_1)$ is a point on (III) and it represents a line and the slope of (III) is $=-\dfrac{g-g'}{f-f'}$ The slope of the line joining the centres of the circles $=\dfrac{f-f'}{g-g'}$.

Thus the line given by (III) is perpendicular to the line of centres and it passes through the point of contact of the two circles. Hence it is a common tangent.

7.3 Radical axis of two circles:

In this section we shall define the radical axis of two circles and study its properties. Also we discuss about the common chord, common tangent of two circles.

7.3.1 Definition: The radical axis of two circles is defined to be the locus of a point which moves so that its powers with respect to the two circles are equal.

7.3.2 Theorem: If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \ldots (\text{I})$ and

$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \ldots (\text{II})$ are two non-concentric circles, then the radical axis of (I) and (II) is a straight line represented by $S - S' = 0$ (or $S' - S = 0$) *i.e.*, $2(g - g')x + 2(f - f')y + (c - c') = 0 \ldots (\text{III})$

Proof: Let $P(x_1, y_1)$ be a point on the radical axis. Then by the definition of the radical axis, we have that the powers of $P(x_1, y_1)$ with respect to (I) and (II) are equal. $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1 = x_1^2 + y_1^2 + 2g'x_1 + 2f'y_1 + c_1'$ ($\because$ The powers of $P(x_1, y_1)$ with respect to $S = 0$ is S_{11})

i.e., $2(g - g')x + 2(f - f')y + (c - c') = 0$

Hence the equation of the locus of $P(x_1, y_1)$ is $2(g - g')x + 2(f - f')y + (c - c') = 0$

Note that this equation represents a straight line, since the circles are non-concentric and therefore $g \neq g'$ or $f \neq f'$. Equation (III) can be written as $S - S' = 0$.

7.3.3 Note: (i) For the concentric circles with distinct radii, the radical axis does not exist, since there is no point whose powers with respect to two distinct concentric circles are equal. However if their radii are equal then the locus is the whole plane.

(ii) While using the formula $S - S' = 0$ to find the equation of the radical axis, first reduce the equations of the circles to general form (if they are not in general form)

(iii) Whenever we consider the radical axis of two circles, it means that two circles are non-concentric.

7.3.4 Examples:

1. Example: Let us find the equation of the radical axis of the circles $S \equiv x^2 + y^2 - 5x + 6y + 12 = 0 \ldots (\text{I})$ and $S' \equiv x^2 + y^2 + 6x - 4y - 14 = 0 \ldots (\text{II})$

The given equations of circles are in general form. Therefore their radical axis is $S - S' = 0 \Rightarrow (x^2 + y^2 - 5x + 6y + 12) - (x^2 + y^2 + 6x - 4y - 14) = 0$ *i.e.*, $11x - 10y - 26 = 0$

2. Example: Let us find the equation of the radical axis of the circles $S \equiv 2x^2 + 2y^2 + 3x + 6y - 5 = 0 \ldots (\text{I})$ and $S' \equiv 3x^2 + 3y^2 - 7x + 8y - 11 = 0 \ldots (\text{II})$

Here, the given equations of circles are not in general form. Reducing them into general form, we get $S \equiv x^2 + y^2 + \frac{3}{2}x + 3y - \frac{5}{2} = 0$...(III) and $S' \equiv x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y - \frac{11}{3} = 0$...(IV)

Therefore their radical axis is $S - S' = 0$

$$\Rightarrow \left(x^2 + y^2 + \frac{3}{2}x + 3y - \frac{5}{2} \right) - \left(x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y - \frac{11}{3} \right) = 0$$

$$\Rightarrow \left(\frac{3}{2} + \frac{7}{3} \right)x - \left(3 - \frac{8}{3} \right)y + \left(-\frac{5}{2} + \frac{11}{3} \right) = 0$$

i.e., $23x + 2y + 7 = 0$

7.3.5 Theorem: The radical axis of any two circles is perpendicular to the line joining their centres.

Proof: Let the equations of two non-concentric circles be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II)

Then $(-g, -f) \neq (-g', -f')$. The equation of the radical axis is $2(g - g')x + 2(f - f')y + (c - c') = 0$...(III)

$\therefore$ The slope of the radical axis $= -\dfrac{g - g'}{f - f'}$

The slope of the line joining the centres of the circles $= \dfrac{f - f'}{g - g'}$.

Since (the slope of the radical axis) $\times$ (slope of the line joining the centres) = -1,

the radical axis is perpendicular to the line joining the centres.

7.3.6 Theorem: The radical axis of two circles is

(i) the common chord when the two circles intersect at two distinct pints.

(ii) the common tangent at the point of contact when the two circles touch each other.

Proof: Let the equations of two circles be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I), $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II) and the radical axis of these circles be L, then $L \equiv S - S' = 2(g - g')x + 2(f - f')y + (c - c') = 0$...(III)

(i) Let the circles given by (I) and (II) intersect at two distinct points P and Q.

By theorem 7.2.8, $S - S' = 0$ is the common chord. Hence is (i) proved.

(ii) Let the circles given by (I) and (II) touch each other at P

By theorem 7.2.8, $S - S' = 0$ is the common tangent. Hence is (ii) proved.

7.3.7 Solved problems:

1. Problem: Find the angle between the pair of circles $x^2 + y^2 + 4x - 14y + 28 = 0$ and $x^2 + y^2 + 4x - 5 = 0$

Solution: Let the given equations of circles be $S \equiv x^2 + y^2 + 4x - 14y + 28 = 0$.......(I)

$S' \equiv x^2 + y^2 + 4x - 5 = 0$.......(II)

Here $g = 2,\ f = -7, c = 28; g' = 2,\ f' = 0, c' = -5$

Let θ be angle between circles (I) & (II) then $\cos\theta = \dfrac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$.

$$\Rightarrow \cos\theta = \frac{28 - 5 - 2(2)(2) - 2(-7)(0)}{2\sqrt{(-2)^2 + (-7)^2 - 28}\sqrt{(-2)^2 + (0)^2 - (-5)}} \Rightarrow \cos\theta = \frac{28 - 5 - 8 - 0}{2\sqrt{4 + 49 - 28}\sqrt{4 + 0 + 5}}$$

$$\Rightarrow \cos\theta = \frac{15}{2\sqrt{25}\sqrt{9}} \Rightarrow \cos\theta = \frac{15}{2(5)(3)} \Rightarrow \cos\theta = \frac{1}{2}$$

$\therefore \theta = 60^0$.

2. Problem: If the angle between the circles $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + kx + 6y - 59 = 0$ is 45^0 then find k.

Solution: Let the given equations of circles be $S \equiv x^2 + y^2 - 12x - 6y + 41 = 0$.......(I)

$S' \equiv x^2 + y^2 + kx + 6y - 59 = 0$.......(II)

Here $g = 6,\ f = 3, c = 41; g' = k/2,\ f' = 3, c' = -59$

Let θ be angle between circles (I) & (II) then $\cos\theta = \dfrac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$.

Given $\theta = 45^0$.

$$\Rightarrow \cos 45^0 = \frac{41 - 59 - 2(-6)(k/2) - 2(-3)(3)}{2\sqrt{(-6)^2 + (-3)^2 - 41}\sqrt{(k/2)^2 + (3)^2 - (-59)}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{41 - 59 + 6k + 18}{2\sqrt{36 + 9 - 41}\sqrt{k^2/4 + 9 + 59}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{3k}{\sqrt{4}\sqrt{k^2/4+68}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{3k}{\sqrt{k^2+272}} \Rightarrow k^2+272 = 18k^2 \Rightarrow 17k^2 = 272$$

$$\Rightarrow k^2 = 16 \Rightarrow k = \pm 4$$

3. Problem: Find the equation of radical axis of the pair of circles $x^2+y^2-3x-4y+5=0$ and $3x^2+3y^2-7x+8y+11=0$

Solution: Let the given equations of circles be $S \equiv x^2+y^2-3x-4y+5=0$.......(I)

$$S' \equiv x^2+y^2-\frac{7}{3}x+\frac{8}{3}y+\frac{11}{3}=0\text{.......(II)}$$

The equation of radical axis of the pair of circles is $S - S' = 0$

$$\Rightarrow \left(x^2+y^2-3x-4y+5\right)-\left(x^2+y^2-\frac{7}{3}x+\frac{8}{3}y+\frac{11}{3}\right)=0 \Rightarrow -\frac{2}{3}x-\frac{20}{3}y+\frac{4}{3}=0$$

$$\Rightarrow x+10y-2=0$$

$\therefore$ The required equation of radical axis is $x+10y-2=0$

4. Problem: Find the equation of common chord of the pair of circles $x^2+y^2-4x-4y+3=0$ and $x^2+y^2-5x-6y+4=0$

Solution: Let the given equations of circles be $S \equiv x^2+y^2-4x-4y+3=0$.......(I)

$$S' \equiv x^2+y^2-5x-6y+4=0\text{.......(II)}$$

The equation of common chord of the pair of circles is $S - S' = 0$

$$\Rightarrow \left(x^2+y^2-4x-4y+3\right)-\left(x^2+y^2-5x-6y+4\right)=0 \Rightarrow -x+2y-1=0$$

$$\Rightarrow x-2y+1=0$$

$\therefore$ The required equation of common chord is $x-2y+1=0$

5. Problem: Find the equation of common tangent of the pair of circles $x^2+y^2+10x-2y+22=0$ and $x^2+y^2+2x-8y+8=0$

Solution: Let the given equations of circles be $S \equiv x^2+y^2+10x-2y+22=0$.......(I)

$$S' \equiv x^2+y^2+2x-8y+8=0\text{.......(II)}$$

The equation of common tangent of the pair of circles is $S - S' = 0$

$$\Rightarrow \left(x^2+y^2+10x-2y+22\right)-\left(x^2+y^2+2x-8y+8=0\right)=0 \Rightarrow 8x+6y+14=0$$

$\Rightarrow 3x+4y+7=0$

$\therefore$ The required equation of common tangent is $3x+4y+7=0$

Exercise 7(a)

I Discuss the relative positions of the following pairs of circles:

$(i)\, x^2+y^2-8x-6y+21=0$ and $x^2+y^2-2y-15=0$

$(ii)\; x^2+y^2+6x+6y+14=0$ and $x^2+y^2-2x-4y-4=0$

$(iii)\left(x-2\right)^2+\left(y+1\right)^2=9$ and $\left(x+1\right)^2+\left(y-3\right)^2=4$

$(iv) x^2+y^2-4x-2y+1=0$ and $x^2+y^2-6x-4y+4=0$

$(v)\; x^2+y^2=4$ and $x^2+y^2-6x-8y+16=0$

$(vi)\, x^2+y^2-4x+2y-4=0$ and $x^2+y^2+2x-6y+6=0$

$(vii)\, x^2+y^2+4x-6y-3=0$ and $x^2+y^2+4x-2y+4=0$

$(viii)\, x^2+y^2+6x-2y+1=0$ and $x^2+y^2-2x-6y+9=0$

$(ix)\, x^2+y^2=4$ and $x^2+y^2-2x-6y+9=0$

II Find the internal and external centre of similitudes for the following circles:

$(i)\, x^2+y^2-8x-6y+21=0$ and $x^2+y^2-2y-15=0$

$(ii)\; x^2+y^2+6x+6y+14=0$ and $x^2+y^2-2x-4y-4=0$

$(iii)\left(x-2\right)^2+\left(y+1\right)^2=9$ and $\left(x+1\right)^2+\left(y-3\right)^2=4$

$(iv) x^2+y^2-4x-2y+1=0$ and $x^2+y^2-6x-4y+4=0$

$(v)\; x^2+y^2=4$ and $x^2+y^2-6x-8y+16=0$

$(vi)\, x^2+y^2-4x+2y-4=0$ and $x^2+y^2+2x-6y+6=0$

$(vii)\, x^2+y^2+4x-6y-3=0$ and $x^2+y^2+4x-2y+4=0$

$(viii)\, x^2+y^2+6x-2y+1=0$ and $x^2+y^2-2x-6y+9=0$

$(ix)\, x^2+y^2=4$ and $x^2+y^2-2x-6y+9=0$

III Find the angle between the following pair of circles:

$(i)\, x^2+y^2-12x-6y+41=0$ and $x^2+y^2+4x+6y-59=0$

$(ii)\; x^2+y^2+2ax+8=0$ and $x^2+y^2+2by-8=0$

$(iii)\, x^2+y^2+4x+8=0$ and $x^2+y^2-16y-8=0$

$(iv)\, x^2+y^2-6x-8y+12=0$ and $x^2+y^2-4x+6y-24=0$

$(v)\; x^2+y^2-5x-14y-34=0$ and $x^2+y^2+2x+4y+1=0$

$(vi)\, x^2+y^2-4x+14y-116=0$ and $x^2+y^2+6x-10y-135=0$

IV Find the equations of the radical axis of the following pair of circles:

$(i)\, x^2+y^2+2x+4y+1=0$ and $x^2+y^2+4x+y=0$

$(ii)\; x^2+y^2+4x+6y-7=0$ and $4x^2+4y^2+8x+12y-9=0$

$(iii)\, x^2+y^2-2x-4y-1=0$ and $x^2+y^2-4x-6y+5=0$

V Find the equations of the common chords of the following pair of circles:

$(i)\, x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$

$(ii)\; x^2+y^2-6x-4y+9=0$ and $x^2+y^2-8x-6y+23=0$

$(iii)\, (x-a)^2+(y-b)^2=c^2$ and $(x-b)^2+(y-a)^2=c^2$ $(a\neq b)$

VI Find the equations of the common tangents of the following pair of circles:

$(i)\, x^2+y^2-2x-4y=0$ and $x^2+y^2-8y-4=0$

$(ii)\; x^2+y^2-8x-2y+8=0$ and $x^2+y^2-2x+6y+6=0$

$(iii)\, x^2+y^2-2x=0$ and $x^2+y^2+6x-6y+2=0$

$(iv)\, x^2+y^2-2x-4y-20=0$ and $x^2+y^2+6x+2y-90=0$

Key concepts

1. A straight line L is said to be a common tangent to the circles $\mathrm{S}=0$ and $\mathrm{S}'=0$ if it is a tangent to both $\mathrm{S}=0$ and $\mathrm{S}'=0$.

2. Two circles are said to be touching each other if they have only one common point.

3. Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S=0$ and $S'=0$ respectively. Further let $\overline{C_1C_2}$ represents the line segment from C_1 to C_2. The following cases arise with regard to the relative position of two circles.

(i) If $C_1C_2 > r_1 + r_2$ then the two circles do not intersect and one circle will be away from the other circle and the number of distinct common tangents are 4.

(ii) If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally and the number of distinct common tangents are 3.

(iii) If $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ then the two circles intersection two distinct points the number of distinct common tangents are 2.

(iv) If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally the number of distinct common tangents are 1.

(v) If $C_1C_2 < |r_1 - r_2|$ then the two circles do not touch or do not intersect and one circle will be completely in side the other the number of distinct common tangents are 0.

4. If $C_1C_2 = 0$ then the centres of the two circles coincides and they are concentric circles.

(iii) If $C_1C_2 = r_1 + r_2$. In this case the internal centre of similitude does not exist. Only two common tangents exists at Q. In this case the number of distinct common tangents are 2.

(iv) If $C_1C_2 = |r_1 - r_2|$. In this case the internal centre of similitude does not exist. Only one common tangent exists at Q. In this case the number of distinct common tangents is 1.

(v) If $C_1C_2 < |r_1 - r_2|$. In this case the internal centre of similitude and external centre of similitudes does not exist. In this case the number of distinct common tangents is zero.

5. If (i) C_1, C_2 are centres of two given intersecting circles $(ii)\, d = C_1C_2$ $(iii)\, r_1, r_2$ are radii of these circles $(iv)\, \theta$ is the angle between these circles, then

$$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

6. If θ is the angle between the intersecting circles $x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II) then

$$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}.$$

7. Two intersecting circles are said to be orthogonal if the angle between them is a right angle $(i.e., 90^0)$

8. Let the two given circles be given by $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II) Thus, the condition for orthogonality of the two intersecting circles (I) & (II) is $2gg' + 2ff' = c + c'$.

(ii) If two circles are orthogonal, then $d^2 = r_1^2 + r_2^2$ where d is the distance between the centres of the circles and r_1, r_2 are their radii.

9. (i) If $S = 0, S' = 0$ are two circles intersecting at two distinct points, then $S - S' = 0$ (or $S' - S = 0$) represents a common chord of these circles.

(ii) If $S = 0, S' = 0$ are two circles touching each other, then $S - S' = 0$ (or $S' - S = 0$) represents a common chord of these circles.

10. If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$...(I) and

$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$...(II) are two non-concentric circles, then the radical axis of (I) and (II) is a straight line represented by $S - S' = 0$ (or $S' - S = 0$) *i.e.*, $2(g - g')x + 2(f - f')y + (c - c') = 0$...(III)

11. The radical axis of any two circles is perpendicular to the line joining their centres.

12. The radical axis of two circles is

(i) the common chord when the two circles intersect at two distinct pints.

(ii) the common tangent at the point of contact when the two circles touch each other.

Answers
Exercise 7(a)

I (i) intersect each other (ii) each lies on the exterior of the other

(iii) touch each other (iv) intersect each other (v) touch each other externally

(vi) touch each other externally (vii) one lies interior on the other

$(viii)$ touch each other externally (ix) touch each other internally

II $(i)\left(\frac{8}{3}, \frac{7}{3}\right)$ and $(8,5)$ $(ii)\left(-\frac{7}{5}, -1\right)$ and $(-11, -13)$ $(iii)\left(\frac{1}{5}, \frac{7}{5}\right)$ (iv) $(0, -1)$

$(v)\left(\frac{6}{5},\frac{8}{5}\right)$ and $(-6,-8)$ $(vi)\left(\frac{1}{5},\frac{7}{5}\right)$ and $(-7,11)$ $(vii)\left(-2,\frac{1}{3}\right)$ $(viii)\left(0,\frac{5}{2}\right)$ and $(3,4)$

$(ix)\left(\frac{2}{3},2\right)$ and $(2,6)$

III (i) $\theta = 45^0$ (ii) $\theta = 90^0$ (iii) $\theta = 90^0$ (iv) $\theta = 90^0$ (v) $\theta = 90^0$ (vi) $\theta = 120^0$

IV $(i)\,2x-3y-1=0$ $(ii)\,8x+12y-19=0$ $(iii)\,x+y-3=0$

V $(i)\,2x+1=0$ $(ii)\,x+y-7=0$ $(iii)\,x-y=0$

VI $(i)\,x+2y-2=0$ $(ii)\,3x+4y-1=0$ $(iii)\,4x-3y+1=0$ $(iv)\,4x+3y-35=0$

8. PARABOLA

Introduction:

In the preceding chapters, we have studied various forms of the equations of circles and system of circles. In this chapter we shall study about parabola. The name “parabola” (the shape described when you throw a ball in air) was given by Apollonius (Ca. 262 B.C.-Ca. 190 B.C).

8.1 Conic sections- Equation of a parabola in standard form:

In fact circle, parabola, ellipse, hyperbola, a pair of straight lines, a straight line and a point are called as conic sections because each is a section of a double napped right circular cone with a plane. These curves have a very wide range of applications in planetary motion, design of telescopes and antennas, reflectors in flash lights etc.

More generally, suppose the cutting plane makes an angle β with the axis of the cone and suppose the generating angle of the cone is α. Then the section is

(i) a circle if $\beta = \pi/2$

(ii) an ellipse if $\alpha < \beta < \pi/2$

(iii) a parabola if $\alpha = \beta$

(iv) a hyperbola if $0 \le \beta < \alpha$

(v) We get the degenerated sections when the plane passes through the vertex of the cone.

(a) a point when $\alpha < \beta \le \pi/2$

(b) a straight line when $\alpha = \beta$ a generator of the cone

(c) a pair of intersecting straight lines when $0 \le \beta < \alpha$. It is the degenerated cases of a hyperbola.

Note: A pair of parallel straight lines, however, is not a conic section as there is no plane which cuts the cone along two parallel lines.

A conic section can also be defined as the locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line are in constant ratio.

It can be proved that these two approaches to define a conic section (as plane section of a cone and as locus) are equivalent. But it is beyond the scope of this book. Further, in view of the analytic approach of the second definition, we shall adopt the same throughout this book.

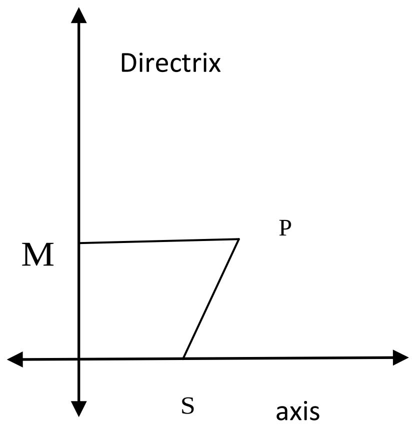
8.1.1 Conic: The locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line in the plane are in a constant ratio e, is called a *conic*.

The fixed point is called the *focus* and is usually denoted by S.

The fixed straight line is called the *directrix*.

The constant ratio e is called the *eccentricity*.

The straight line of the plane passing through the focus and perpendicular to the directrix is called the axis. Therefore the locus of a point P moving on a plane such that $\frac{\mathrm{SP}}{\mathrm{PM}} = e$ (constant) where PM is the perpendicular distance from P to the directrix, is called a *conic*.



usually denoted by S.

If $e = 1$, the conic is called a *parabola*.

If $0 < e < 1$, the conic is called an *ellipse*.

If $e > 1$, the conic is called a *hyperbola*.

8.1.2 Equation of a parabola:

In this section we derive the equation of a parabola in general form.

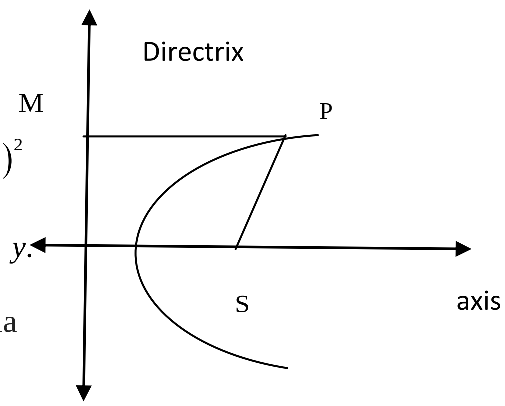
Let $\mathrm{S}(x_1, y_1)$ be the focus and the directrix be $ax + by + c = 0$. Thus, by definition of the parabola, the equation of the parabola having is

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$ or

$$(a^2 + b^2)\left[(x - x_1)^2 + (y - y_1)^2\right] = (ax + by + c)^2$$

a general equation of second degree in x and y.

The equation of the axis of the above parabola is $b(x - x_1) - a(y - y_1) = 0.$



8.1.3 Equation of a parabola in standard form:

To study the nature of the curve, we prefer its equation in the simple possible form. We proceed as follows to derive such an equation.

Let S be the focus, l be the directrix. Let Z be the projection of S on l and A be the midpoint of SZ. A lies on the parabola because $\mathrm{SA} = \mathrm{AZ}$, A is called the vertex of the parabola. Let $\mathrm{YAY'}$ be the straight line through A and parallel to the directrix. Now take ZX as the X- axis and $\mathrm{YY'}$ as the Y- axis.

Then A is the origin $(0,0)$. Let $\mathrm{S} = (a,0), a > 0$. Then $\mathrm{Z} = (-a,0)$ and the equation of the directrix is $x + a = 0$.

If $\mathrm{P}(x, y)$ is a point on the parabola and PM is the perpendicular distance from P to the directrix l, then $\dfrac{\mathrm{SP}}{\mathrm{PM}} = e = 1$. M

$$\therefore \mathrm{SP}^2 = \mathrm{PM}^2$$

$$\Rightarrow (x-a)^2 + y^2 = (x+a)^2$$

$$\therefore y^2 = 4ax.$$

Conversely if $\mathrm{P}(x, y)$ is any point such that $y^2 = 4ax$ then

$$\mathrm{SP} = \sqrt{(x-a)^2 + y^2} = \sqrt{x^2 - 2ax + a^2 + 4ax} = \sqrt{x^2 + 2ax + a^2}$$
$$= \sqrt{(x+a)^2} = |x+a| = \mathrm{PM}.$$

Hence $\mathrm{P}(x, y)$ is on the locus. In other words, a necessary and sufficient condition for the $\mathrm{P}(x, y)$ point to be on the parabola is that $y^2 = 4ax$.

Thus the equation of the parabola is $y^2 = 4ax$.

8.1.4 Remark:

(i) If the focus is situated on the left side of the directrix, the equation of the parabola with the vertex as X- axis is $y^2 = -4ax$. (since in this case the focus S is $(-a,0)$)

(ii) The vertex being the origin, if the axis of the parabola is taken as Y- axis, equation of parabola is $x^2 = 4ay$ or $x^2 = -4ay$ according as the focus is above or below the X- axis.

(iii) If S lies on l, then the locus is a straight line passing through S and perpendicular to l. We take this case as the degenerated parabola.

8.1.5 Nature of the curve:

In this section we shall study the nature of the parabola or trace the curve represented by the equation $y^2 = 4ax, a > 0.$

(i) If $y = 0,$ then $4ax = 0$ and $x = 0.$

$\therefore$ The curve passes though the origin $(0,0).$

(ii) If $x = 0,$ then $y^2 = 0$ which gives $y = 0,0.$ Hence Y- axis is a tangent to the parabola at the origin

(iii) Let $\mathrm{P}(x, y)$ be any point on the parabola. Since $a > 0$ and $y^2 = 4ax,$ we have $x \geq 0$ and $y = \pm\sqrt{4ax}.$

$\therefore$ For any positive real value of $x,$ we obtain two values of y of equal magnitude but opposite in signs. This shows that the curve is symmetric about X- axis and lies in first and fourth quadrants. The curve does not exist on the left side of the Y- axis (*i.e.,* second and the third quadrants) since $x \geq 0$ for any point (x, y) on the parabola.

(iv) As x increases infinitely, the two values of y also increases infinitely in magnitude. So the two branches of the parabola lying on opposite sides of the X- axis extended to infinity towards the positive direction of the X- axis. Hence it is an open curve.

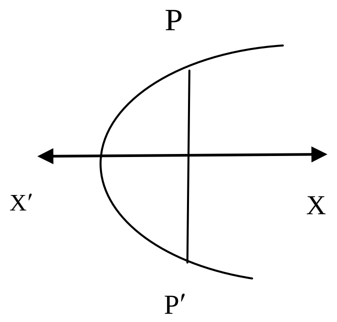
8.1.6 Note: As noted earlier, S is called the focus and the line l is called the directrix of the parabola. For the parabola $y^2 = 4ax, a > 0,$ the focus is $\mathrm{S} = (a,0),$ directrix is $x + a = 0$ and axis is $y = 0.$ The point $\mathrm{A} = (0,0)$ is called the vertex of the parabola.

8.1.7 Definitions (Chord, focal chord, double ordinate and latusrectum):

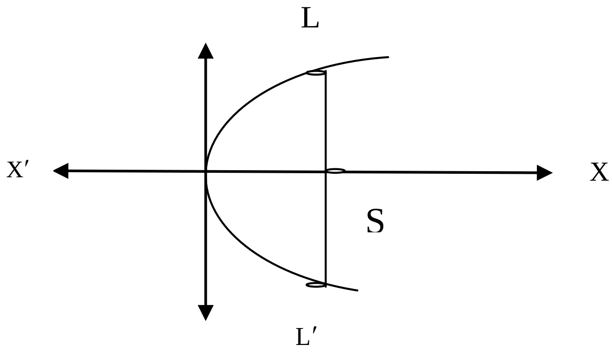
The line joining two points of a parabola is called a *chord* of a parabola. A chord passing through focus is called a *focal chord*. A chord through a point P on the parabola, which is perpendicular to the axis of the parabola, is called the *double ordinate* of the point P. The double ordinate passing though the focus is called the *latusrectum* of the parabola.

8.1.8 Remark:

From $y^2 = 4ax,$ for any positive $x,$ $\mathrm{P}(x, 2\sqrt{ax}), \mathrm{P}'(x, -2\sqrt{ax})$ are points on the parabola $y^2 = 4ax$ and PP' is perpendicular to the axis and hence is the double ordinate through P.



8.1.9 Length of the latusrectum:



The equation of the parabola is $y^2 = 4ax.$

Let LSL' be the latusrectum of the parabola.

Let $SL = l,$ then coordinates of l are (a, l)

$SL = l,$ lies on the parabola.

$\therefore l^2 = 4a.a = 4a^2$

$\therefore l = 2a$ and so, $LSL' = 2(SL) = 2l = 2(2a) = 4a,$ which is the length of the latusrectum.

8.1.10 Note: When latusrectum is known, the equation of parabola is known in its standard form, the size and the shape of the curve are determined accordingly.

8.1.11 Various forms of the parabola:

(i) The equation of parabola is of the form $y^2 = 4ax$

Vertex $A = (0,0)$

Focus $S = (a,0)$

Equation of directrix is $x + a = 0$

Axis is $x-$axis, hence the equation of axis is $y = 0.$

(ii) The equation of parabola is of the form $y^2 = -4ax$

Vertex $A = (0,0)$

Focus $S = (-a,0)$

Equation of directrix is $x - a = 0$

Axis is $x-$axis, hence the equation of axis is $y = 0.$

(iii) The equation of parabola is of the form $x^2 = 4ay$

Vertex $A = (0,0)$

Focus $S = (0, a)$

Equation of directrix is $y + a = 0$

Axis is $y-$axis, hence the equation of axis is $x = 0.$

(iv) The equation of parabola is of the form $x^2 = -4ay$

Vertex $\text{A} = (0,0)$

Focus $\text{S} = (0,-a) = (0,-2)$

Equation of directrix is $y - a = 0$

Axis is $y-$axis, hence the equation of axis is $x = 0.$

(v) The equation of parabola is of the form $(y-k)^2 = 4a(x-h)$

Vertex $\text{A} = (h,k)$

Focus $\text{S} = (h+a,k)$

Equation of directrix is $x + a = h$

Axis is parallel to $x-$axis, hence the equation of axis is $y = k.$

(vi) The equation of parabola is of the form $(y-k)^2 = -4a(x-h)$

Vertex $\text{A} = (h,k)$

Focus $\text{S} = (h-a,k)$

Equation of directrix is $x - a = h$

Axis is parallel to $x-$axis, hence the equation of axis is $y = k.$

(vii) The equation of parabola is of the form $(x-h)^2 = 4a(y-k)$

Vertex $\text{A} = (h,k)$

Focus $\text{S} = (h,k+a)$

Equation of directrix is $y + a = k$

Axis is parallel to $y-$axis, hence the equation of axis is $x = h.$

$(viii)$ The equation of parabola is of the form $(x-h)^2 = -4a(y-k)$

Vertex $\text{A} = (h,k)$

Focus $\text{S} = (h,k-a)$

Equation of directrix is $y - a = k$

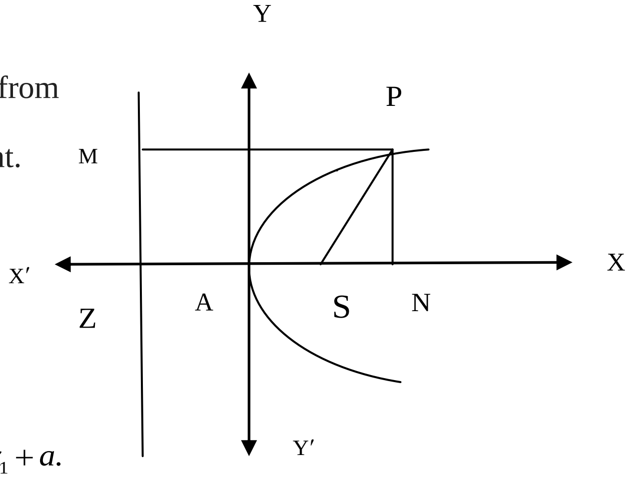
Axis is parallel to $y-$axis, hence the equation of axis is $x = h$.

8.1.12 Note: We may conclude that the equation of a parabola whose axis is parallel to the $x-$ axis is $x = ay^2 + by + c$ and whose axis is parallel to the $y-$ axis is $y = ax^2 + bx + c$, where a, b, c are real numbers, $a \neq 0$.

8.1.13 Definition (Focal distance):

The distance of a point on the parabola from its focus is called the *focal distance* of the point.

If $\mathrm{P}(x_1, y_1)$ is a point on the parabola $y^2 = 4ax$ whose focus is $\mathrm{S}(a, 0)$ then



Focal distance of $\mathrm{P} = \mathrm{SP}$

$$= \mathrm{PM} = \mathrm{NZ} = \mathrm{NA} + \mathrm{AZ} = x_1 + a.$$

8.1.14 Notation: Here after the following notation will be adapted through out this chapter.

(i) $\mathrm{S} \equiv y^2 - 4ax$

(ii) $\mathrm{S}_1 \equiv yy_1 - 2a(x + x_1)$

(iii) $\mathrm{S}_{12} \equiv y_1 y_2 - 2a(x_1 + x_2)$

(iv) $\mathrm{S}_{11} \equiv y_1^2 - 4ax_1$, where (x_1, y_1) and (x_2, y_2) are the points in the plane of the parabola $y^2 = 4ax$.

8.1.15 Parabola and a point in the plane of the parabola:

A parabola divides the plane into two disjoint parts, one containing the focus is called the interior of the parabola and the other is called the exterior of the parabola.

Let $\mathrm{P}(x_1, y_1)$ be a point in the plane of the parabola. Draw PM perpendicular to the $x-$axis to meet the parabola $y^2 = 4ax$ at $\mathrm{Q}(x_1, 2a\sqrt{x_1})$ and $\mathrm{M}(x_1, 0)$.

$$\therefore \mathrm{PM}^2 = y_1^2, \mathrm{MQ}^2 = 4ax_1.$$

(i) P lies outside the parabola (*i.e.*, P is an external point) $\Leftrightarrow (\mathrm{MP})^2 > (\mathrm{MQ})^2$

$$\Leftrightarrow y_1^2 > 4ax_1 \Leftrightarrow \mathrm{S}_{11} > 0.$$

If $x_1 < 0,$ then the point P lies in Quadrant II or in Quadrant III in which case the point P clearly lies outside the parabola and $y_1^2 - 4ax_1 > 0$ in this case also.

(ii) P lies on the parabola $\Leftrightarrow \text{MP} = \text{MQ} \Leftrightarrow (\text{MP})^2 = (\text{MQ})^2 \Leftrightarrow y_1^2 = 4ax_1 \Leftrightarrow \text{S}_{11} = 0.$

(iii) P lies inside the parabola (*i.e.*, P is an internal point) $\Leftrightarrow (\text{MP})^2 < (\text{MQ})^2$

$$\Leftrightarrow y_1^2 < 4ax_1 \Leftrightarrow \text{S}_{11} < 0.$$

Thus P lies outside, on or inside the parabola $\text{S} \equiv y^2 - 4ax = 0$ according as $\text{S}_{11} >=< 0.$

8.1.16 Solved Problems:

1. Problem: Find the coordinates of the vertex and focus, and the equations of directrix and axis of the following parabolas:

(i) $y^2 = 8x$ (ii) $y^2 = -4x$ (iii) $x^2 = 4y$ (iv) $x^2 = -8y$ (v) $3x^2 + 9x + 5y - 2 = 0$
(vi) $y^2 - x - 4y + 5 = 0$ (vii) $x^2 + 2x - 4y - 3 = 0$ $(viii)$ $4y^2 + 12x + 20y + 67 = 0$

Solution: (i) The given equation of parabola is $y^2 = 8x$ ...(I)

It is of the form $y^2 = 4ax$ ...(II)

Compare equations (I) and (II) we get $a = 2.$

Vertex $\text{A} = (0,0)$

Focus $\text{S} = (a,0) = (2,0)$

Equation of directrix is $x + a = 0 \Rightarrow x + 2 = 0$

Axis is $x-$axis, hence the equation of axis is $y = 0.$

(ii) The given equation of parabola is $y^2 = -4x$ ...(I)

It is of the form $y^2 = -4ax$ ...(II)

Compare equations (I) and (II) we get $a = 1.$

Vertex $\text{A} = (0,0)$

Focus $\text{S} = (-a,0) = (-1,0)$

Equation of directrix is $x - a = 0 \Rightarrow x - 1 = 0$

Axis is $x-$axis, hence the equation of axis is $y = 0.$

(iii) The given equation of parabola is $x^2 = 4y$ ...(I)

It is of the form $x^2 = 4ay$ ...(II)

Compare equations (I) and (II) we get $a = 1.$

Vertex $\text{A} = (0,0)$

Focus $\text{S} = (0, a) = (0,1)$

Equation of directrix is $y + a = 0 \Rightarrow y + 1 = 0$

Axis is $y -$ axis, hence the equation of axis is $x = 0.$

(iv) The given equation of parabola is $x^2 = -8y$ ...(I)

It is of the form $x^2 = -4ay$ ...(II)

Compare equations (I) and (II) we get $a = 2.$

Vertex $\text{A} = (0,0)$

Focus $\text{S} = (0, -a) = (0, -2)$

Equation of directrix is $y - a = 0 \Rightarrow y - 2 = 0$

Axis is $y -$ axis, hence the equation of axis is $x = 0.$

(v) The given equation of parabola is $3x^2 + 9x + 5y - 2 = 0 \Rightarrow 3(x^2 + 3x) = -5y + 2$

$$\Rightarrow 3\left[x^2 + 2.x.\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2\right] = -5y + 2 + \frac{27}{4} \Rightarrow 3\left(x + \frac{3}{2}\right)^2 = -5y + \frac{35}{4}$$

$$\Rightarrow 3\left(x + \frac{3}{2}\right)^2 = -5\left(y - \frac{7}{4}\right) \Rightarrow \left(x + \frac{3}{2}\right)^2 = -\frac{5}{3}\left(y - \frac{7}{4}\right) \quad \text{...(I)}$$

It is of the form $\left(x - h\right)^2 = -4a\left(y - k\right)$ ...(II)

Compare equations (I) and (II) we get $h = -\frac{3}{2}, k = \frac{7}{4}, a = \frac{5}{3}.$

Vertex $\text{A} = \left(h, k\right) = \left(-\frac{3}{2}, \frac{7}{4}\right)$

Focus $\text{S} = \left(h, k - a\right) = \left(-\frac{3}{2}, \frac{7}{4} - \frac{5}{3}\right) = \left(-\frac{3}{2}, \frac{1}{12}\right)$

Equation of directrix is $y-a=k \Rightarrow y-\frac{5}{3}=\frac{7}{4} \Rightarrow 12y-20=21$

Axis is parallel to $y-$axis, hence the equation of axis is $x=h \Rightarrow x=-\frac{3}{2} \Rightarrow 2x+3=0$

(vi) The given equation of parabola is $y^2-x-4y+5=0 \Rightarrow y^2-4y=x-5$

$\Rightarrow \left[y^2-2.y.2+2^2\right]=x-5+2^2 \Rightarrow (y-2)^2=x-1 \Rightarrow (y-2)^2=4.\frac{1}{4}(x-1)$ ...(I)

It is of the form $(y-k)^2=4a(x-h)$ ...(II)

Compare equations (I) and (II) we get $h=1, k=2, a=\frac{1}{4}$

Vertex $\text{A}=(h,k)=(1,2)$

Focus $\text{S}=(h+a,k)=\left(1+\frac{1}{4},2\right)=\left(\frac{5}{4},2\right)$

Equation of directrix is $x+a=h \Rightarrow x+\frac{1}{4}=1 \Rightarrow 4x-3=0$

Axis is parallel to $x-$axis, hence the equation of axis is $y=k \Rightarrow y=2 \Rightarrow y-2=0$

(vii) The given equation of parabola is $x^2+2x-4y-3=0 \Rightarrow x^2+2x=4y+3$

$\Rightarrow \left[x^2+2.x.1+1^2\right]=4y+3+1^2 \Rightarrow (x+1)^2=4y+4 \Rightarrow (x+1)^2=4(y+1)$ ...(I)

It is of the form $(x-h)^2=4a(y-k)$ ...(II)

Compare equations (I) and (II) we get $h=-1, k=-1, a=1$

Vertex $\text{A}=(h,k)=(-1,-1)$

Focus $\text{S}=(h,k+a)=(-1,-1+1)=(-1,0)$

Equation of directrix is $y+a=k$

$y+a=k \Rightarrow y+1=-1 \Rightarrow y+2=0$

Axis is parallel to $y-$axis, hence the equation of axis is

$$x=h \Rightarrow x=-1 \Rightarrow x+1=0$$

$(viii)$ The given equation of parabola is $4y^2+12x+20y+67=0$

$$\Rightarrow 4y^2+20y=-12x-67 \Rightarrow 4\left[y^2+2.y.\frac{5}{2}+\left(\frac{5}{2}\right)^2\right]=-12x-67+\left(\frac{5}{2}\right)^2$$

$$\Rightarrow 4\left[y^2+2.y.\frac{5}{2}+\left(\frac{5}{2}\right)^2\right]=\frac{-48x-243}{4} \Rightarrow \left(y+\frac{5}{2}\right)^2=-3\left(x+\frac{81}{16}\right) \qquad ...(I)$$

It is of the form $(y-k)^2=-4a(x-h)$ ...(II)

Compare equations (I) and (II) we get $h=-\frac{81}{16}, k=-\frac{5}{2}, a=\frac{3}{4}$

Vertex $A=(h,k)=\left(-\frac{81}{16},-\frac{5}{2}\right)$

Focus $S=(h-a,k)=\left(-\frac{81}{16}-\frac{3}{4},-\frac{5}{2}\right)=\left(-\frac{93}{16},-\frac{5}{2}\right)$

Equation of directrix is $x-a=h \Rightarrow x-\frac{3}{4}=-\frac{81}{16} \Rightarrow 16x+69=0$

Axis is parallel to $x-$axis, hence the equation of axis is $y=k \Rightarrow y=-\frac{5}{2} \Rightarrow 2y+5=0$

2. Problem: Find the equation of the parabola whose focus is $(3,-4)$ and the directrix is the line $x-y+5=0$.

Solution: Let P (x,y) be any point on the locus.

Given focus $S=(3,-4)$ and the equation of directrix is $M \equiv x-y+5=0$.

The equation of parabola having focus (x_1,y_1) and the directrix $ax+by+c=0$ is

$$(a^2+b^2)\left[(x-x_1)^2+(y-y_1)^2\right]=(ax+by+c)^2$$

The equation of parabola having focus $S=(3,-4)$ and the directrix $M \equiv x-y+5=0.$ is
$(1^2+(-1)^2)\left[(x-3)^2+(y-(-4))^2\right]=(x-y+5)^2$

$$\Rightarrow 2\left[x^2-6x+9+y^2+8y+16\right]=x^2+y^2+25-2xy+10x-10y$$

$$\Rightarrow 2x^2-12x+2y^2+16y+50=x^2+y^2+25-2xy+10x-10y$$

$\Rightarrow x^2+y^2+2xy-22x+26y+25=0$ which is the required equation of parabola.

3. Problem: Find the equation of the parabola whose vertex is $(3,-2)$ and focus is $(3,2)$.

Solution: Let the given vertex be $(h,k)=(3,-2)$ and focus be $(h,k+a)=(3,2)$.

We have $h=3, k=-2.$

Also $k+a=2 \Rightarrow -2+a=2 \Rightarrow a=4$

The equation of parabola having vertex (h,k) and focus $(h,k+a)$ is $(x-h)^2=4a(y-k)$

$\Rightarrow (x-3)^2 = 4.4(y-(-2))$

$\Rightarrow (x-3)^2 = 16(y+2)$ which is the required equation of parabola.

4. Problem: Find the coordinates of the points on the parabola $y^2=16x$ whose focal distance is 5.

Solution: The given equation of parabola is $y^2=16x$ ...(I)

It is of the form $y^2=4ax$ ...(II)

Compare equations (I) and (II) we get $a=4.$

Let $P(x_1, y_1)$ be any point on the parabola.

We have focal distance is $x_1+a=5 \Rightarrow x_1+4=5 \Rightarrow x_1=1$

Since $P(x_1, y_1)$ lies on the parabola $y^2=16x$

$\Rightarrow y_1^2=16x_1 \Rightarrow y_1^2=16.1 \Rightarrow y_1^2=16 \Rightarrow y_1=\pm 4$

The required points on the parabola $y^2=16x$ are $(1,\pm 4)$

5. Problem: Find the equation of the parabola passing through the points $(1,2),(-1,1),$ $(2,-1)$ and having the axis parallel to $x-$ axis.

Solution: Let the given points be $A(1,2), B(-1,1), C(2,-1)$

The equation of the parabola having the axis parallel to x^- axis is $x=ay^2+by+c$.....(I)

Since $A(1,2)$ lies on equation (I) we have $1=a.2^2+b.2+c \Rightarrow 4a+2b+c=1$......(II)

Since $B(-1,1)$ lies on equation (I) we have $-1=a.1^2+b.1+c \Rightarrow a+b+c=-1$......(III)

Since $C(2,-1)$ lies on equation (I) we have $2=a.(-1)^2+b.(-1)+c \Rightarrow a-b+c=2$....(IV)

By solving (III) and (IV) we get $b=-\frac{3}{2}$

Substitute the value of b in equations (II) & (III) we get

$$4a+c=4......(V),\ a+c=\frac{1}{2}......(VI)$$

By solving (V) and (VI) we get $a=\frac{7}{6}, c=-\frac{2}{3}$

Now substitute the values of a, b, c in equation (I) we get

$$x=\left(\frac{7}{6}\right)y^2+\left(-\frac{3}{2}\right)y+\left(-\frac{2}{3}\right)$$

$\Rightarrow 6x=7y^2-9y-4$ which is the required equation of the parabola

6. Problem: Find the equation of the parabola passing through the points $(5,4),(11,-2),$ $(21,-4)$ and having the axis parallel to $y-$axis.

Solution: Let the given points be $A(5,4), B(11,-2), C(21,-4)$

The equation of the parabola having the axis parallel to y^- axis is $y=ax^2+bx+c$.....(I)

Since $A(5,4)$ lies on equation (I) we have $4=a.5^2+b.5+c \Rightarrow 25a+5b+c=4$....(II)

Since $B(11,-2)$ lies on equation (I) we have $-2=a.11^2+b.11+c$

$\Rightarrow 121a+11b+c=-2$ ...(III)

Since $C(21,-4)$ lies on equation (I) we have $-4=a.21^2+b.21+c$

$\Rightarrow 441a+21b+c=-4$ ...(IV)

By solving (II) and (III) we get $96a+6b=-6 \Rightarrow 16a+b=-1$ ...(V)

By solving (III) and (IV) we get $320a+10b=-2 \Rightarrow 32a+b=-\frac{1}{5}$ ...(VI)

By solving (V) and (VI) we get $a=\frac{1}{20}, b=-\frac{9}{5}$

Substitute the value of $a=\frac{1}{20}, b=-\frac{9}{5}$ in equations (II) we get $c=\frac{47}{4}$

Now substitute the values of a, b, c in equation (I) we get

$$y=\left(\frac{1}{20}\right)x^2+\left(-\frac{9}{5}\right)x+\left(\frac{47}{4}\right)$$

$\Rightarrow 20y = x^2 - 36x + 235$ which is the required equation of the parabola

7. Problem: Find the position (exterior or interior or on) of the following points with respect to the parabola $y^2 = 4x$ (i) $(3,-3)$ (ii) $(4,4)$ (iii) $(3,4)$.

Solution: The given equation of parabola is $S \equiv y^2 - 4x = 0$.......(I)

(i) Let $P(3,-3)$ be the given point.

Now $S_{11} = y_1^2 - 4x_1 = (-3)^2 - 4(3) = 9 - 12 = -3 < 0$

$\therefore P(3,-3)$ lies inside the parabola $y^2 = 4x$

(ii) Let $P(4,4)$ be the given point.

Now $S_{11} = y_1^2 - 4x_1 = 4^2 - 4(4) = 16 - 16 = 0$

$\therefore P(4,4)$ lies on the parabola $y^2 = 4x$

(iii) Let $P(3,4)$ be the given point.

Now $S_{11} = y_1^2 - 4x_1 = 4^2 - 4(3) = 16 - 12 = 3 > 0$

$\therefore P(3,4)$ lies outside the parabola $y^2 = 4x$

Exercise 8(a)

1. Find the coordinates of the vertex and focus, and the equations of directrix and axis of the following parabolas:

(i) $y^2 = 16x$ (ii) $x^2 = -4y$ (iii) $3x^2 - 9x + 5y - 2 = 0$ (iv) $y^2 - x + 4y + 5 = 0$

(v) $y^2 + 4x + 4y - 3 = 0$ (vi) $x^2 - 2x + 4y - 3 = 0$ (vii) $4y^2 + 12x - 20y + 67 = 0$

$(viii)$ $x^2 - 6x - 6y + 6 = 0$

2. Find the equation of the parabola whose vertex is $(3,-2)$ and focus is $(3,1)$.
3. Find the equation of the parabola whose vertex is $(1,-7)$ and focus is $(1,-2)$.

4. Find the equation of the parabola whose focus is $(3,5)$ and vertex is $(1,3)$.

5. Find the coordinates of the points on the parabola $y^2 = 8x$ whose focal distance is 10.
6. Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is $\frac{5}{2}$.

7. Find the equation of the parabola passing through the points $(-1,2),(1,-1),(2,1)$ and having its axis parallel to the $x-$axis.

8. Find the equation of the parabola passing through the points $(-2,1),(1,2),(-1,3)$ and having its axis parallel to the $x-$axis.

9. Find the equation of the parabola passing through the points $(4,5),(-2,11),(-4,21)$ and having its axis parallel to the $y-$axis.

10. Find the position (exterior or interior or on) of the following points with respect to the parabola $y^2 = 6x$ (i) $(6,-6)$ (ii) $(0,1)$ (iii) $(2,3)$.

8.2 Tangent and normal at a point on the parabola:

In this section, the condition for a straight line to be a tangent to a given parabola is obtained. The Cartesian equations of the tangent and the normal at a given point on the parabola are derived.

8.2.1 Point of intersection of the parabola $y^2 = 4ax, a > 0$ and the line $y = mx + c, m \neq 0$:

Let $y^2 = 4ax, a > 0$ .....(I)

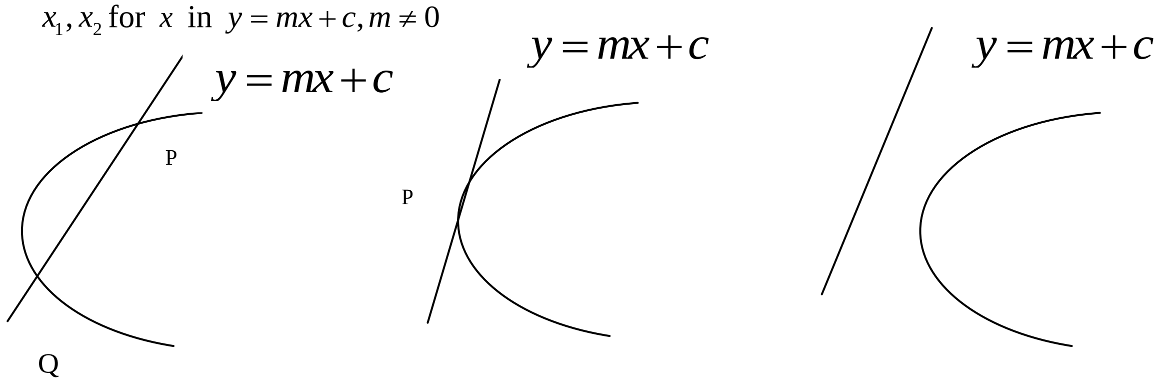
and the straight line $y = mx + c, m \neq 0$ be given that .....(II)

The coordinates of the point of the intersection of the straight line and the parabola satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$(mx+c)^2 = 4ax, \quad i.e., \; m^2x^2 + 2x(mc-2a) + c^2 = 0 \qquad .....(III)$$

This is a quadratic equation in x and therefore has two roots which are distinct real equal or imaginary according as the discriminant of equation (I) is positive or zero or negative respectively. $y = mx + c$

The ordinates of the points of intersection y_1, y_2 can be obtained by substituting x_1, x_2 for x in $y = mx + c, m \neq 0$



8.2.2 Theorem: The condition for a straight line $y = mx + c, m \neq 0$ to be a tangent to the parabola $y^2 = 4ax$ is $c = \frac{a}{m}$ or $cm = a.$

Proof: Let $y^2 = 4ax, a > 0$ .....(I)

and the straight line $y = mx + c, m \neq 0$ be given that .....(II)

The coordinates of the point of the intersection of the straight line and the parabola satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$(mx + c)^2 = 4ax, \quad i.e., \; m^2x^2 + 2x(mc - 2a) + c^2 = 0 \qquad .....(III)$$

The given line will touch the parabola $\Leftrightarrow$ the two points coincide.

$\Leftrightarrow$ discriminant of (III) is zero.

$\Leftrightarrow 4(mc - 2a)^2 - 4m^2c^2 = 0$

$\Leftrightarrow a(a - mc) = 0$

$\Leftrightarrow c = \frac{a}{m}$ or $cm = a.$

8.2.3 Note:

(i) When $m = 0,$ the line $y = c$ is parallel to the axis of the parabola $y^2 = 4ax,$ $i.e.,$ $x-$axis. Further $y = c \Rightarrow x = \frac{c^2}{4a}.$

$\therefore$ The straight line intersects the parabola at the point $\left(\frac{c^2}{4a}, c\right).$

(ii) We have seen $y = mx + c, m \neq 0$ is a tangent to the parabola $y^2 = 4ax$ when $c = \frac{a}{m}$ or $cm = a.$ Hence $y = mx + \frac{a}{m}$ is always a tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{c}{m}, 2c\right)$ when $m \neq 0.$

(iii) If $m \neq 0$ and $c = 0,$ then the line $y = mx$ is non vertical and passes through the origin which intersects the parabola in two points $(0, 0)$ and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right).$ Hence it is not a tangent.

(iii) Observe that for a parabola $y^2 = 4ax$, there is one and only one tangent, parallel to $y-$axis ($i.e.,$ $y-$itself) and there is no tangent parallel to $x-$axis.

8.2.4 Theorem: Two tangents can be drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$

Proof: Let $P(x_1, y_1)$ be an external point to the parabola $y^2 = 4ax$ then

$$S_{11} = y_1^2 - 4ax_1 > 0 \qquad .....(I)$$

We have $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all nonzero values of m. If it passes through the point $P(x_1, y_1)$ then $y_1 = mx_1 + \frac{a}{m}$ or $m^2x_1 - my_1 + a = 0$ and its discriminant $y_1^2 - 4ax_1 > 0$.

The equation being a quadratic in m, has two distinct real roots, say m_1 and m_2.

Then $y = m_1x + \frac{a}{m_1}$ and $y = m_2x + \frac{a}{m_2}$ are the two distinct tangents through (x_1, y_1)

8.2.5 Theorem: The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on $S = 0$ is $S_1 + S_2 = S_{12}$.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola $S \equiv y^2 - 4ax = 0$, then

$S_{11} = 0$ and $S_{22} = 0$. Consider the second degree equation $S_1 + S_2 = S_{12}$.

$i.e.,$ $[yy_1 - 2a(x + x_1)] + [yy_2 - 2a(x + x_2)] = [y_1y_2 - 2a(x_1 + x_2)]$

$i.e.,$ $4ax - (y_1 + y_2)y + y_1y_2 = 0$ which represents a straight line.

Substituting $P(x_1, y_1)$ it becomes $S_{11} + S_{12} = 0 + S_{12} = S_{12}$

$\therefore$ $P(x_1, y_1)$ satisfies the equation $S_1 + S_2 = S_{12}$.

Similarly $Q(x_2, y_2)$ satisfies the equation $S_1 + S_2 = S_{12}$.

$\therefore$ $S_1 + S_2 = S_{12}$ is a straight line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$

$\therefore$ The equation of the chord PQ is $S_1 + S_2 = S_{12}$.

8.2.6 Theorem: The equation of tangent at (x_1, y_1) to the parabola $S = 0$ is $S_1 = 0$.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola $S \equiv y^2 - 4ax = 0$, then $S_{11} = 0$ and $S_{22} = 0$. By Theorem 8.2.5, the equation of the chord PQ is $S_1 + S_2 = S_{12}$...(I)

The chord PQ becomes the tangent at P when Q approaches P.

i.e., (x_2, y_2) approaches to (x_1, y_1)

$\therefore$ The equation of the tangent at P is obtained by taking limits (x_2, y_2) tends to as (x_1, y_1) on either sides of (I)

So, the equation of the tangent at P is given by $\lim_{Q \to P}(S_1 + S_2) = \lim_{Q \to P} S_{12}$

i.e., $S_1 + S_1 = S_{11}\left[\because S_2 \to S_1, S_{12} \to S_{11} \text{ as } (x_2, y_2) \to (x_1, y_1)\right]$

$\therefore 2S_1 = 0 \Rightarrow S_1 = 0.$

$\therefore$ The equation of the tangent to the parabola $S \equiv y^2 - 4ax = 0$ at (x_1, y_1) is $S_1 \equiv yy_1 - 2a(x + x_1) = 0.$

8.2.7 Theorem: The equation of normal at (x_1, y_1) on the parabola $S = 0$ is $y - y_1 = -\frac{y_1}{2a}(x - x_1) = 0.$

Proof: By Theorem 8.2.6, the equation of tangent to the parabola $S \equiv y^2 - 4ax = 0$ at (x_1, y_1) is $S_1 \equiv yy_1 - 2a(x + x_1) = 0.$

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ is $\frac{2a}{y_1}$.

$\therefore$ Slope of the normal at $P(x_1, y_1)$ is $-\frac{y_1}{2a}$.

Hence the equation of normal at (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1) = 0.$

8.2.8 Solved Problems:

1. Problem: Show that the line $4x + y + 1 = 0$ is a tangent to the parabola $y^2 = 16x$ and find the point of contact

Solution: The given equation of the parabola is $S \equiv y^2 = 16x$ ...(I)

The given line equation is $4x + y + 1 = 0$ ...(II)

Let $P(x_1, y_1)$ be any point on the parabola

Equation of tangent at $P(x_1, y_1)$ to the parabola $S \equiv 0$ is $S_1 \equiv 0$

i.e., $yy_1 - 8(x + x_1) = 0$

$\Rightarrow 8x - y_1 y + 8x_1 = 0$ ...(III)

Since (II) and (III) represents the same line equations we have $\frac{8}{4} = \frac{y_1}{1} = \frac{8x_1}{1}$

$$\Rightarrow x_1 = \frac{1}{4}, y_1 = 2$$

Hence the line $4x + y + 1 = 0$ is a tangent to the parabola $y^2 = 16x$ and the point of contact is $\left(\frac{1}{4}, 2\right)$

2. Problem: Find the value of k if the line $2x - y + k = 0$ is a tangent to the parabola $y^2 = 8x.$.

Solution: The given equation of the parabola is $S \equiv y^2 = 8x$ ...(I)

The given line equation is $2x - y + k = 0$ ...(II)

The condition that the line $lx + my + n = 0$ is a tangent to the parabola $y^2 = 4ax$.

is $nl = am^2$.

We have $a = 2, l = 2, m = -1, n = k$

$k.2 = 2(-1)^2 \Rightarrow k = 1$

3. Problem: Find the equations of the tangent and normal to the parabola $x^2 + 4x - y - 2 = 0$ at $(1, 3)$.

Solution: The given equation of the parabola is $S \equiv x^2 + 4x - y - 2 = 0$ ...(I)

Let $P(1, 3)$ be any point on the parabola

Equation of tangent at $P(x_1, y_1)$ to the parabola $S \equiv 0$ is $S_1 \equiv 0$

i.e., $xx_1 + 2(x + x_1) - \frac{1}{2}(y + y_1) - 2 = 0 \Rightarrow x.1 + 2(x + 1) - \frac{1}{2}(y + 3) - 2 = 0$

$\Rightarrow x+2x+2-\frac{1}{2}y-\frac{3}{2}-2=0 \Rightarrow 6x-y-3=0$ ...(II)

Since equation of normal is perpendicular to the tangent and passing through $P(1,3)$

equation of the normal is $(x-1)+6(y-3)=0$

$\Rightarrow x+6y-19=0$

4. Problem: Find the equations of the tangent to the parabola $y^2=8x$ which are parallel and perpendicular respectively to the line $2x+y+5=0$.

Solution: The given equation of the parabola is $S \equiv y^2 = 8x$ ...(I)

The given line equation is $2x+y+5=0$ ...(II)

The equation of a line parallel to the line (II) is $2x+y+k=0$ ...(III)

Since the line (III) is a tangent to the parabola (I).

We have $2k = 2.1^2 \Rightarrow k=1$

$\therefore$ The equation of the tangent to the parabola $y^2=8x$ which is parallel to the line $2x+y+5=0$ is $2x+y+1=0$

The equation of a line perpendicular to the line (II) is $x-2y+k=0$ ...(IV)

Since the line (IV) is a tangent to the parabola (I).

We have $1.k = 2.(-2)^2 \Rightarrow k=8$

$\therefore$ The equation of the tangent to the parabola $y^2=8x$ which is perpendicular to the line $2x+y+5=0$ is $x-2y+8=0$

Exercise 8(b)

1. Show that the line $7x+6y=13$ is a tangent to the parabola $y^2-7x-8y+14=0$ and find the point of contact.

2. Show that the line $2x-y+2=0$ is a tangent to the parabola $y^2=16x$ and find the point of contact.

3. Find the value of k if the line $2y=5x+k$ is a tangent to the parabola $y^2=6x$..

4. Find the equations of the tangent and normal to the parabola $y^2=6x$ at the positive end of the latusrectum.

5. Find the equations of the tangent and normal to the parabola $x^2 - 4x - 8y + 12 = 0$ at $(4, 3/2)$.

6. Find the equations of the normal to the parabola $y^2 = 4x$ which is parallel to the line $y - 2x + 5 = 0$.

7. Find the equation of tangents to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$, also find the coordinates of their point of contact.

Key concepts

1. The locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line in the plane are in a constant ratio e, is called a *conic*. The fixed point is called the *focus* and is usually denoted by S. The fixed straight line is called the *directrix*. The constant ratio e is called the *eccentricity*. The straight line of the plane passing through the focus and perpendicular to the directrix is called the axis.

   Therefore the locus of a point P moving on a plane such that $\frac{SP}{PM} = e$ (constant)

   where PM is the perpendicular distance from P to the directrix, is called a *conic*. usually denoted by S.

   If $e = 1$, the conic is called a *parabola*.

   If $0 < e < 1$, the conic is called an *ellipse*.

   If $e > 1$, the conic is called a *hyperbola*.

2. Let $S(x_1, y_1)$ be the focus and the directrix be $ax + by + c = 0$. The equation of the parabola is $(a^2 + b^2)\left[(x - x_1)^2 + (y - y_1)^2\right] = (ax + by + c)^2$ and the equation of the axis of the above parabola is $b(x - x_1) - a(y - y_1) = 0$.

3. The equation of parabola in standard form is $y^2 = 4ax$. For the parabola $y^2 = 4ax, a > 0$, the focus is $S = (a, 0)$, directrix is $x + a = 0$ and axis is $y = 0$.

   The point $A = (0,0)$ is called the vertex of the parabola.

4. The line joining two points of a parabola is called a *chord* of a parabola.

5. A chord passing through focus is called a *focal chord*.

6. A chord through a point P on the parabola, which is perpendicular to the axis of the parabola, is called the *double ordinate* of the point P.

7. The double ordinate passing though the focus is called the focus is called the *latusrectum* of the parabola.

8. The length of the latusrectum of the parabola $y^2 = 4ax$ is $4a$.

9. (i) If the equation of parabola is of the form $y^2 = 4ax$, then vertex is $A = (0,0)$, Focus is $S = (a,0)$, Equation of directrix is $x + a = 0$ and axis is $x-$axis, hence the equation of axis is $y = 0$.

(ii) If the equation of parabola is of the form $y^2 = -4ax$, then vertex is $A = (0,0)$, Focus is $S = (-a,0)$, Equation of directrix is $x - a = 0$ and axis is $x-$axis, hence the equation of axis is $y = 0$.

(iii) If the equation of parabola is of the form $x^2 = 4ay$, then vertex is $A = (0,0)$, Focus is $S = (0,a)$, Equation of directrix is $y + a = 0$ and axis is $y-$axis, hence the equation of axis is $x = 0$.

(iv) If the equation of parabola is of the form $x^2 = -4ay$, then vertex is $A = (0,0)$, Focus is $S = (0,-a)$, Equation of directrix is $y - a = 0$ and axis is $y-$axis, hence the equation of axis is $x = 0$.

(v) If the equation of parabola is of the form $(y-k)^2 = 4a(x-h)$, then vertex is $A = (h,k)$, Focus is $S = (h+a,k)$, Equation of directrix is $x + a = h$ and axis is parallel to $x-$axis, hence the equation of axis is $y = k$.

(vi) If the equation of parabola is of the form $(y-k)^2 = -4a(x-h)$, then vertex is $A = (h,k)$, Focus is $S = (h-a,k)$, Equation of directrix is $x - a = h$ and axis is parallel to $x-$axis, hence the equation of axis is $y = k$.

(vii) If the equation of parabola is of the form $(x-h)^2 = 4a(y-k)$, then vertex is $A = (h,k)$, Focus is $S = (h,k+a)$, Equation of directrix is $y + a = k$ and axis is parallel to $y-$axis, hence the equation of axis is $x = h$.

$(viii)$ If the equation of parabola is of the form $(x-h)^2 = -4a(y-k)$, then vertex is $A = (h,k)$, Focus is $S = (h,k-a)$, Equation of directrix is $y - a = k$ and axis is parallel to $y-$axis, hence the equation of axis is $x = h$.

10. The equation of a parabola whose axis is parallel to the $x-$axis is $x = ay^2 + by + c$ and whose axis is parallel to the y^- axis is $y = ax^2 + bx + c,$ where a, b, c are real numbers, $a \neq 0.$

11. The distance of a point on the parabola from its focus is called the *focal distance* of the point. If $\text{P}(x_1, y_1)$ is a point on the parabola $y^2 = 4ax$ whose focus is $\text{S}(a,0)$ then Focal distance of $\text{P} = \text{SP} = x_1 + a.$

12. The following notation will be adapted.

(i) $\text{S} \equiv y^2 - 4ax$ (ii) $\text{S}_1 \equiv yy_1 - 2a(x + x_1)$ (iii) $\text{S}_{12} \equiv y_1 y_2 - 2a(x_1 + x_2)$

(iv) $\text{S}_{11} \equiv y_1^2 - 4ax_1,$ where (x_1, y_1) and (x_2, y_2) are the points in the plane of the parabola

$y^2 = 4ax.$

13. The condition that the point P lies outside, on or inside the parabola

$\text{S} \equiv y^2 - 4ax = 0$ according as $\text{S}_{11} >=< 0.$

14. The condition for a straight line $y = mx + c, m \neq 0$ to be a tangent to the parabola

$y^2 = 4ax$ is $c = \dfrac{a}{m}$ or $cm = a.$ The point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

15. The condition for a straight line $lx + my + n = 0$ to be a tangent to the parabola

$y^2 = 4ax$ is $nl = am^2.$ The point of contact is $\left(\dfrac{n}{l}, -\dfrac{2am}{l}\right)$

16. Two tangents can be drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$

17. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on $\text{S} = 0$ is $\text{S}_1 + \text{S}_2 = \text{S}_{12}.$

18. The equation of tangent at (x_1, y_1) to the parabola $\text{S} = 0$ is $\text{S}_1 = 0.$

19. The equation of normal at (x_1, y_1) on the parabola $\text{S} = 0$ is $y - y_1 = -\dfrac{y_1}{2a}(x - x_1).$

Exercise 8(a)

1. (i) vertex $= (0,0)$, focus $= (4,0)$, directrix : $x + 4 = 0,$ axis: $y = 0.$

(ii) vertex $= (0,0)$, focus $= (0,-1)$, directrix : $y - 1 = 0,$ axis : $x = 0.$

(iii) vertex $=\left(\frac{3}{2},\frac{7}{4}\right)$, focus $=\left(\frac{3}{2},\frac{4}{3}\right)$, directrix : $6y-13=0$, axis : $2x-3=0$.

(iv) vertex $=(1,-2)$, focus $=\left(\frac{5}{4},-2\right)$, directrix : $4x-3=0$, axis : $y+2=0$.

(v) vertex $=\left(\frac{7}{4},-2\right)$, focus $=\left(\frac{3}{4},-2\right)$, directrix : $4x-11=0$, axis : $y+2=0$.

(vi) vertex $=(1,1)$, focus $=(1.0)$, directrix : $y-2=0$, axis : $x-1=0$.

(vii) vertex $=\left(\frac{-7}{2},\frac{5}{2}\right)$, focus $=\left(\frac{-17}{4},\frac{5}{2}\right)$, directrix : $4x+11=0$, axis : $2y-5=0$.

$(viii)$ vertex $=\left(3,\frac{-1}{2}\right)$, focus $=(3,1)$, directrix : $y+2=0$, axis : $x-3=0$.

2. $(x-3)^2=12(y+2)$ 3. $(x-1)^2=20(y+7)$ 4. $x^2-2xy+y^2-12x-20y+68=0$
5. $(8,\pm 8)$ 6. $(2,\pm 2)$ 7. $7y^2+6x-3y-16=0$ 8. $5y^2+2x-21y+20=0$
9. $x^2-4x-2y+10=0$

10 (i) on the parabola (ii) exterior to the parabola (iii) interior to the parabola

Exercise 8(b)

1. $(1,1)$ 2. $(1,4)$ 3. $k=6/5$. 4. $2x-2y+3=0$, $2x+2y-9=0$.

5. $x-2y-1=0$, $4x+2y-19=0$.

6. Find the equation of the normal to the parabola $y^2=4x$ which is parallel to the line $2x-y-12=0$.

7. $2x-y+2=0$ point of contact $(1,4)$, $x+2y+16=0$ point of contact $(16,-16)$.

9. ELLIPSE

Introduction:

We study the ellipse in this chapter. We also discuss, about the standard form of equation of ellipse, condition for a line to be a tangent to the ellipse, chord of contact in this chapter.

9.1 Equation of an ellipse in standard form:

In this section, we study the equation of an ellipse in the standard form.

9.1.1 Definition (Ellipse): A conic with eccentricity less than unity is called an *ellipse.* Hence an ellipse is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio $e,$ which is less than unity. The fixed point is called the *focus* and the fixed line is called the *directrix* of the ellipse.

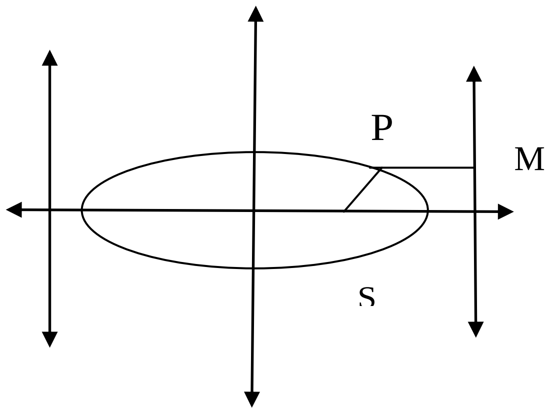
9.1.2 Equation of an ellipse:

In this section we derive the equation of an ellipse in general form.

Let $S(x_1, y_1)$ be the focus and the directrix be $ax+by+c=0.$ Thus, by definition of the ellipse, the equation of the ellipse is

$$\sqrt{(x-x_1)^2+(y-y_1)^2} = e\frac{|ax+by+c|}{\sqrt{a^2+b^2}} \text{ or}$$

$$(a^2+b^2)\left[(x-x_1)^2+(y-y_1)^2\right]=e^2(ax+by+c)^2$$



a general equation of second degree in x and $y.$

The equation of the axis of the above ellipse is $b(x-x_1)-a(y-y_1)=0.$

9.1.3 Equation of an ellipse in standard form:

To study the nature of the curve, we prefer its equation in the simple possible form. We proceed as follows to derive such an equation.

Let S be the focus, l be the corresponding directrix and e be the eccentricity. Let Z be the foot of the perpendicular from S on directrix $l.$ Let A and A' be the points which divide SZ in the ratio $e:1,$ internally and externally respectively.

Consider C midpoint of AA' as origin, consider the line CZ extended as $x-$axis and a line perpendicular to it C as $y-$axis.

Let $CA=a=CA'$ so that $A=(a,0)$ and $A'=(-a,0).$

But $\dfrac{SA}{AZ} = e = \dfrac{SA'}{A'Z} \Rightarrow SA = e(AZ)$ and $SA' = e(A'Z)$

$\therefore CA - CS = e(CZ - CA) \Rightarrow a - CS = e(CZ - a).....(I)$

$CS + CA' = e(CA' + CZ) \Rightarrow CS + a = e(CZ + a).....(II)$

Adding (I) and (II) above, we get $2a = 2e(CZ) \Rightarrow CZ = \dfrac{a}{e}$

$\therefore$ Equation of directrix is $x = \dfrac{a}{e}......(III)$

Subtracting (I) from (II), we get $2(CS) = 2ae \Rightarrow CS = ae.$

Coordinates of focus S are $(ae, 0).$

Now let $P(x, y)$ be a point on the ellipse and PM be the perpendicular distance from P to the direcrix. Then by definition $PS = e(PM).$

$PS^2 = e^2(PM)^2$

i.e., $\left(x - ae\right)^2 + y^2 = e^2\left(x^2 + \dfrac{a^2}{e^2} - \dfrac{2ax}{e}\right) \quad \left[\because PM = x - \dfrac{a}{e}\right]$

i.e., $x^2\left(1 - e^2\right) + y^2 = a^2\left(1 - e^2\right)$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2\left(1 - e^2\right)} = 1$

Since $0 < e < 1 \Rightarrow 1 - e^2 > 0 \Rightarrow a^2(1 - e^2) > 0.$

$\therefore \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, (a > 0, b > 0).....(IV)$

$\therefore$ We can choose a real number $b > 0$ such that $a^2(1 - e^2) = b^2.$

We have shown that coordinates of P must satisfy (IV) if P satisfies the geometric condition $SP = e(PM).$ Conversely, if x, y satisfy the algebraic equation (IV) with $b^2 = a^2(1 - e^2)$ and $0 < e < 1,$ then

i.e., $y^2 = b^2\left(1 - \dfrac{x^2}{a^2}\right) = b^2\left(\dfrac{a^2 - x^2}{a^2}\right) = \dfrac{a^2\left(1 - e^2\right)\left(a^2 - x^2\right)}{a^2} = \left(1 - e^2\right)\left(a^2 - x^2\right)$

$\therefore SP = \sqrt{\left(x - ae\right)^2 + y^2} = \sqrt{x^2 + a^2e^2 - 2aex + \left(1 - e^2\right)\left(a^2 - x^2\right)}$

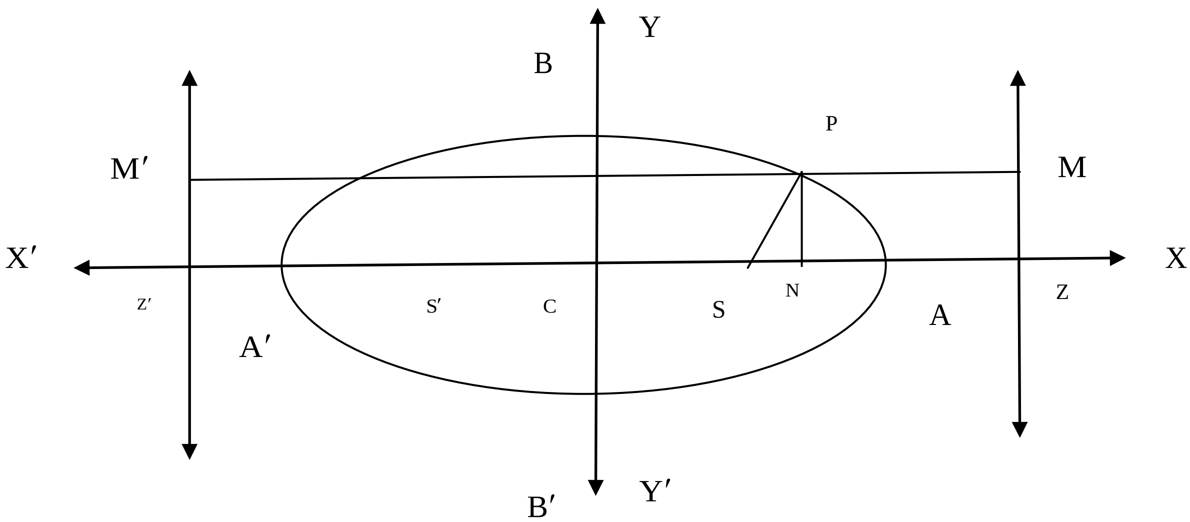
$\therefore \text{SP} = \sqrt{x^2e^2 - 2aex + a^2} = |xe - a| = e\left|x - \frac{a}{e}\right| = e(\text{PM}).$

If P satisfies the algebraic condition then P satisfies the geometric condition and vice versa.

Thus the locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ the equation of ellipse in standard form.

Now let S' be the image of S and $\text{Z}'\text{M}'$ be the image of ZM with respect to $y-$axis, taking S' as focus and $\text{Z}'\text{M}'$ as corresponding directrix, it can be seen that the corresponding equation of ellipse is also $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ Hence for every ellipse, there are two foci and two corresponding directrices.

We have $b^2 = a^2(1 - e^2)$ and $0 < e < 1 \Rightarrow b^2 < a^2 \Rightarrow b < a.$



9.1.4 Nature of the curve:

In this section we shall study the nature of the ellipse or trace the curve represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > 0, b > 0)$

(i) Point of intersection with coordinate axes:

If $y = 0,$ then $x = \pm a$ *i.e.,* the curve intersects the $x-$axis at $\text{A} = (a, 0)$ and $\text{A}' = (-a, 0).$ Hence $\text{AA}' = 2a.$

If $x = 0,$ then $y = \pm b$ *i.e.,* the curve intersects the $y-$axis at $\text{B} = (0, b)$ and $\text{B}' = (0, -b).$ Hence $\text{BB}' = 2b.$

$\therefore$ The curve passes though the origin $(0,0)$.

(ii) From (i) we have $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$ and $x=\pm\frac{a}{b}\sqrt{b^2-y^2}$.

From (ii), y is real $\Leftrightarrow a^2-x^2\geq 0 \Leftrightarrow -a\leq x\leq a \Leftrightarrow |x|\leq a.$

From (ii), x is real $\Leftrightarrow b^2-y^2\geq 0 \Leftrightarrow -b\leq y\leq b \Leftrightarrow |y|\leq b.$

$\therefore$ Corresponding to every real value of x, with $|x|\leq a$, there are two real values of y, equal in magnitude but opposite in sign. Similarly corresponding to every real value of y, with $|y|\leq b$, there are two real values of x, equal in magnitude but opposite in sign. Hence ellipse is symmetric about both the axes.

(iii) The curve lies inside the rectangle bounded by the lines $x=a, x=-a, y=b, y=-b.$

(iv) Any chord through $C(0,0)$ of the ellipse is bisected at the point C, for the points $(x,y),(-x,-y)$ simultaneously lie on the curve. The centre of an ellipse is defined as the point of intersection of its axes of symmetry. The centre of the ellipse is the point C.

9.1.5 Definition (Major and Minor axes): The line segment AA' and BB' of lengths $2a$ and $2b$ respectively are called axes of the ellipse. If $a>b$, AA' is called major axis and BB' is called minor axis and vice versa if $a<b$.

9.1.6 Definitions (Chord, focal chord and latusrectum):

The line joining two points on the ellipse is called a *chord* of the ellipse. A chord passing through one of the foci is called a *focal chord*. A focal chord perpendicular to the major axis of the ellipse is called a *latusrectum* of the ellipse. An ellipse has two latusrecta.

9.1.7 Length of the latusrectum:

Let L, L' be the ends of the latusrectum passing through the one of the foci $S(ae,0)$ of the ellipse equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$.....(I)

Since LL' is perpendicular to $x-$axis, the coordinates of L and L' are equal to ae.

This $L(ae,y_1)$ is on (I), we have $\frac{(ae)^2}{a^2}+\frac{y_1^{\,2}}{b^2}=1\Rightarrow\frac{y_1^{\,2}}{b^2}=1-e^2\Rightarrow y_1^{\,2}=b^2(1-e^2)$

$\Rightarrow y_1^{\,2}=b^2\left(\frac{b^2}{a^2}\right)$ $\qquad\left[\because b^2=a^2(1-e^2)\right]$

$\therefore y_1 = \pm\frac{b^2}{a}.$

Hence $L\left(ae, \frac{b^2}{a}\right)$ and $L' = \left(ae, -\frac{b^2}{a}\right)$

$\therefore$ Length of the latusrectum $LL' = \frac{2b^2}{a}.$

9.1.8 Note: (i) The coordinates of the four ends of the latusrecta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$ are $L\left(ae, \frac{b^2}{a}\right), L' = \left(ae, -\frac{b^2}{a}\right)$ and $L_1\left(-ae, \frac{b^2}{a}\right), L_1' = \left(-ae, -\frac{b^2}{a}\right)$

(ii) Length of the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (b>a)$ is $\frac{2a^2}{b}$ and the coordinates of the four ends of the latusrecta of the ellipse are $L\left(\frac{a^2}{b}, be\right), L' = \left(-\frac{a^2}{b}, be\right)$ and $L_1\left(\frac{a^2}{b}, -be\right), L_1' = \left(-\frac{a^2}{b}, -be\right).$

(iii) The equation of the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$ through S is $x = ae$ and through S$'$ is $x = -ae.$ The equation of the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (b>a)$ through S is $y = be$ and through S$'$ is $y = -be.$

9.1.9 Various forms of the ellipse:

If $a = b,$ then the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a circle $x^2 + y^2 = a^2$ with centre at origin and having radius a and we are familiar with circles. We assumed $a \neq b$ and in the following discussion, we describe different forms of the ellipse.

(i) The equation of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b>0)$

Major axis is along $x-$axis and minor axis is along $y-$axis.

Length of major axis is $AA' = 2a$ and length of minor axis is $BB' = 2b.$

Eccentricity $e = \sqrt{1-\frac{b^2}{a^2}}$

Centre $C = (0,0)$

Foci $S = (ae,0), S' = (-ae,0)$

Equation of directrices $x = a/e, x = -a/e$

(ii) The equation of ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (b > a > 0)$

Major axis is along $y-$axis and minor axis is along $x-$axis.

Length of major axis is $\mathrm{BB'} = 2b$ and length of minor axis is $\mathrm{AA'} = 2a$

Eccentricity $e = \sqrt{1 - \frac{a^2}{b^2}}$

Centre $\mathrm{C} = (0,0)$

Foci $\mathrm{S} = (0, be), \mathrm{S'} = (0, -be)$

Equation of directrices $y = b/e, y = -b/e$

(iii) The equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, (a > b > 0)$

Major axis is parallel to $x-$axis *i.e.*, along $y = k$ and minor axis is parallel to $y-$axis *i.e.*, along $x = h.$

Length of major axis is $\mathrm{AA'} = 2a$ and length of minor axis is $\mathrm{BB'} = 2b.$

Eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$

Centre $\mathrm{C} = (h, k)$

Foci $\mathrm{S} = (h + ae, k), \mathrm{S'} = (h - ae, k)$

Equation of directrices $x = h + a/e, x = h - a/e$

(iv) The equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, (b > a > 0)$

Major axis is parallel to $y-$axis *i.e.*, along $x = h$ and minor axis is parallel to $x-$axis *i.e.*, along $y = k$

Length of major axis is $\mathrm{BB'} = 2b$ and length of minor axis is $\mathrm{AA'} = 2a$

Eccentricity $e = \sqrt{1 - \frac{a^2}{b^2}}$

Centre $\mathrm{C} = (h, k)$

Foci $S=(h,k+be), S'=(h,k-be)$

Equation of directrices $y=k+b/e, y=k-b/e$

9.1.10 Definition (Auxiliary circle): The circle described on the major axis of an ellipse as diameter is called *auxiliary circle* of the ellipse. The equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$.

9.1.11 Notation: Here after the following notation will be adapted through out this chapter.

(i) $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

(ii) $S_1\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

(iii) $S_{12}\equiv\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$

(iv) $S_{11}\equiv\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1$ where (x_1,y_1) and (x_2,y_2) are the points in the plane of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

9.1.12 Ellipse and a point in the plane of the ellipse:

An ellipse divides the $xy-$plane into two disjoint regions, one containing the foci is called the interior region of the ellipse and the other is called the exterior region of the ellipse.

Let $P(x_1,y_1)$ be a point in the plane of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, |x_1|\le a$......(I). Draw PN, perpendicular to the major axis of the ellipse (I), which meets the ellipse in t Q.

Then $N(x_1,0)$, $Q\left(x_1,\frac{b}{a}\sqrt{a^2-x_1^{\,2}}\right)$ or $\left(x_1,-\frac{b}{a}\sqrt{a^2-x_1^{\,2}}\right)$ and $PN=|y_1|$

consider $\frac{PN^2-QN^2}{b^2}=\frac{y_1^{\,2}-\frac{b^2}{a^2}(a^2-x_1^{\,2})}{b^2}=\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1=S_{11}.$

(i) P lies outside the ellipse (*i.e.*, P is an external point) $\Leftrightarrow (PN)>(QN)$

$$\Leftrightarrow (\mathrm{PN})^2 > (\mathrm{QN})^2 \quad \Leftrightarrow \mathrm{S}_{11} = \frac{\mathrm{PN}^2 - \mathrm{QN}^2}{b^2} > 0.$$

If $|x_1| > a,$ then the point P lies in Quadrant II or in Quadrant III in which case the point P clearly lies outside the ellipse and $\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} - 1 > 0$ in this case also.

(ii) P lies on the ellipse $\Leftrightarrow \mathrm{PN} = \mathrm{QN} \Leftrightarrow (\mathrm{PN})^2 = (\mathrm{QN})^2 \Leftrightarrow (\mathrm{PN})^2 - (\mathrm{QN})^2 = 0$

$$\Leftrightarrow \mathrm{S}_{11} = \frac{\mathrm{PN}^2 - \mathrm{QN}^2}{b^2} = \frac{0}{b^2} = 0.$$

(iii) P lies inside the ellipse ($i.e.$, P is an internal point) $\Leftrightarrow (\mathrm{PN}) < (\mathrm{QN})$

$$\Leftrightarrow (\mathrm{PN})^2 < (\mathrm{QN})^2 \quad \Leftrightarrow \mathrm{S}_{11} = \frac{\mathrm{PN}^2 - \mathrm{QN}^2}{b^2} < 0.$$

Thus P lies outside, on or inside the ellipse $\mathrm{S} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ according as $\mathrm{S}_{11} >=< 0.$

9.1.13 Solved Problems:

1. Problem: Find the centre, eccentricity, coordinates of foci, Length of major axis, Length of minor axis, Length of latusrectum and equations of directrices of the following ellipses:

(i) $9x^2 + 25y^2 = 225$ (ii) $16x^2 + 9y^2 = 144$ (iii) $9x^2 + 16y^2 + 36x - 32y - 92 = 0$

(iv) $x^2 + 3y^2 - 2x - 6y - 5 = 0$ (v) $x^2 + 2y^2 + 4x + 12y + 18 = 0$

(vi) $4x^2 + y^2 + 8x + 2y + 1 = 0$

Solution: (i) The given equation of ellipse is $9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$.......(I)

It is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(II)

Compare equations (I) and (II) we get $a^2 = 25, b^2 = 9 \Rightarrow a = 5, b = 3$

Here $a > b.$

Centre $\mathrm{C} = (0,0)$

eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

Focii $S=(\pm ae,0)=\left(\pm 5\left(\frac{4}{5}\right),0\right)=(\pm 4,0)$

Length of major axis $=2a=2(5)=10$

Length of minor axis $=2b=2(4)=8$

Length of latusrectum $=\frac{2b^2}{a}=\frac{2(4)^2}{5}=\frac{32}{5}$

Equation of directrices is $x=\pm\frac{a}{e}\Rightarrow x=\pm\frac{5}{3/5}\Rightarrow x=\pm\frac{25}{3}\Rightarrow 3x=\pm 25$

(ii) The given equation of ellipse is $16x^2+9y^2=144\Rightarrow\frac{x^2}{9}+\frac{y^2}{16}=1$.......(I)

It is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(II)

Compare equations (I) and (II) we get $a^2=9, b^2=16\Rightarrow a=3, b=4$

Here $a<b$.

Centre $C=(0,0)$

eccentricity $e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{9}{16}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

Focii $S=(0,\pm be)=\left(0,\pm 4\left(\frac{\sqrt{7}}{4}\right)\right)=(0,\pm\sqrt{7})$

Length of major axis $=2b=2(4)=8$

Length of minor axis $=2a=2(3)=6$

Length of latusrectum $=\frac{2a^2}{b}=\frac{2(3)^2}{4}=\frac{9}{2}$

Equation of directrices is $y=\pm\frac{b}{e}\Rightarrow y=\pm\frac{4}{\sqrt{7}/4}\Rightarrow y=\pm\frac{16}{\sqrt{7}}\Rightarrow\sqrt{7}y=\pm 16$

(iii) The given equation of ellipse is $9x^2+16y^2+36x-32y-92=0$

$\Rightarrow 9(x^2+4x)+16(y^2-2y)=92$

$\Rightarrow 9(x^2+2.x.2+2^2)+16(y^2-2.y.1+1^2)=92+9.2^2+16.1^2$

$$\Rightarrow 9(x+2)^2+16(y-1)^2=144$$

$$\Rightarrow \frac{(x+2)^2}{16}+\frac{(y-1)^2}{9}=1 \quad ...(I)$$

It is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ...(II)

Compare equations (I) and (II) we get $h=-2, k=1$

$a^2=16, b^2=9 \Rightarrow a=4, b=3$

Here $a>b.$

Centre $C=(h,k)=(-2,1)$

eccentricity $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{16}}=\sqrt{\frac{16-9}{16}}=\sqrt{\frac{7}{16}}=\frac{\sqrt{7}}{4}$

Focii $S=(h\pm ae,k)=\left(-2\pm 4\left(\frac{\sqrt{7}}{4}\right),1\right)=\left(-2\pm\sqrt{7},1\right)$

Length of major axis $=2a=2(4)=8$

Length of minor axis $=2b=2(3)=6$

Length of latusrectum $=\frac{2b^2}{a}=\frac{2(3)^2}{4}=\frac{9}{2}$

Equation of directrices is $x=h\pm\frac{a}{e}\Rightarrow x=-2\pm\frac{4}{\sqrt{7}/4}\Rightarrow x+2=\pm\frac{16}{\sqrt{7}}\Rightarrow\sqrt{7}(x+2)=\pm 16$

(iv) The given equation of ellipse is $x^2+3y^2-2x-6y-5=0$

$$\Rightarrow (x^2-2x)+3(y^2-2y)=5$$

$$\Rightarrow (x^2-2.x.1+1^2)+3(y^2-2.y.1+1^2)=5+1^2+3.1^2$$

$$\Rightarrow (x-1)^2+3(y-1)^2=9$$

$$\Rightarrow \frac{(x-1)^2}{9}+\frac{(y-1)^2}{3}=1 \quad ...(I)$$

It is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ...(II)

Compare equations (I) and (II) we get $h=1, k=1$

$a^2=9, b^2=3 \Rightarrow a=3, b=\sqrt{3}$

Here $a>b.$

Centre $\mathrm{C}=(h,k)=(1,1)$

eccentricity $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{3}{9}}=\sqrt{\frac{9-3}{9}}=\sqrt{\frac{6}{9}}=\frac{\sqrt{6}}{3}$

Focii $\mathrm{S}=(h\pm ae,k)=\left(1\pm 3\left(\frac{\sqrt{6}}{3}\right),1\right)=\left(1\pm\sqrt{6},1\right)$

Length of major axis $=2a=2(3)=6$

Length of minor axis $=2b=2(\sqrt{3})=2\sqrt{3}$

Length of latusrectum $=\frac{2b^2}{a}=\frac{2(\sqrt{3})^2}{3}=2$

Equation of directrices is $x=h\pm\frac{a}{e}\Rightarrow x=1\pm\frac{3}{\sqrt{6}/3}\Rightarrow x-1=\pm\frac{9}{\sqrt{6}}\Rightarrow\sqrt{6}(x-1)=\pm 9$

(v) The given equation of ellipse is $x^2+2y^2+4x+12y+18=0$

$$\Rightarrow (x^2+4x)+2(y^2+6y)+18=0$$

$$\Rightarrow (x^2+2.x.2+2^2)+2(y^2+2.y.3+3^2)+18=2^2+2.3^2$$

$$\Rightarrow (x+1)^2+2(y+3)^2=4$$

$$\Rightarrow \frac{(x+1)^2}{4}+\frac{(y+3)^2}{2}=1 \qquad ...(\mathrm{I})$$

It is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ...(II)

Compare equations (I) and (II) we get $h=-1, k=-3$

$a^2=4, b^2=2 \Rightarrow a=2, b=\sqrt{2}$

Here $a>b.$

Centre $\mathrm{C}=(h,k)=(-1,-3)$

eccentricity $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{2}{4}}=\sqrt{\frac{4-2}{2}}=\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$

Focii $\mathrm{S}=(h\pm ae,k)=\left(-1\pm 2\left(\frac{1}{\sqrt{2}}\right),-3\right)=\left(-1\pm\sqrt{2},-3\right)$

Length of major axis $=2a=2(2)=4$

Length of minor axis $=2b=2(\sqrt{2})=2\sqrt{2}$

Length of latusrectum $=\frac{2b^2}{a}=\frac{2(\sqrt{2})^2}{2}=2$

Equation of directrices is $x=h\pm\frac{a}{e}\Rightarrow x=-1\pm\frac{2}{1/\sqrt{2}}\Rightarrow x+1=\pm 2\sqrt{2}$

(vi) The given equation of ellipse is $4x^2+y^2+8x+2y+1=0$

$$\Rightarrow 4(x^2+2x)+(y^2+2y)+1=0$$

$$\Rightarrow 4(x^2+2.x.1+1^2)+(y^2+2.y.1+1^2)+1=4.1^2+1^2$$

$$\Rightarrow 4(x+1)^2+(y+1)^2=4$$

$$\Rightarrow \frac{(x+1)^2}{1}+\frac{(y+1)^2}{4}=1 \qquad ...(\mathrm{I})$$

It is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ ...(II)

Compare equations (I) and (II) we get $h=-1, k=-1$

$a^2=1, b^2=4\Rightarrow a=1, b=2$

Here $a<b.$

Centre $\mathrm{C}=(h,k)=(-1,-1)$

eccentricity $e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{4-1}{4}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$

Focii $\mathrm{S}=(h,k\pm be)=\left(-1,-1\pm 2\left(\frac{\sqrt{3}}{2}\right)\right)=\left(-1,-1\pm\sqrt{3}\right)$

Length of major axis $=2b=2(2)=4$

Length of minor axis $= 2a = 2(1) = 2$

Length of latusrectum $= \frac{2a^2}{b} = \frac{2(1)^2}{2} = 1$

Equation of directrices is $y = k \pm \frac{b}{e} \Rightarrow y = -1 \pm \frac{2}{\sqrt{3}/2} \Rightarrow y + 1 = \pm 4/\sqrt{3}$

2. Problem: Find the equation of the ellipse referred to its major and minor axes $x-, y-$ axes respectively with latusrectum of length 4 and distance between foci $2\sqrt{3}.$

Solution: Given length of latusrectum is 4 $i.e., \frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$ ...(I)

distance between foci is $i.e., 2ae = 2\sqrt{3} \Rightarrow ae = \sqrt{3}$

We have eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow a^2e^2 = a^2 - b^2 \Rightarrow (\sqrt{3})^2 = a^2 - 2a$

$\Rightarrow a^2 - 2a - 3 = 0 \Rightarrow a = 3, a = -1$

Since a need not be negative, therefore $a = 3.$

From equation (I), $\Rightarrow b^2 = 2.3 \Rightarrow b^2 = 6$

The required equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{6} = 1$

3. Problem: Find the equation of the ellipse in standard form, if it passes through the points $(2, -2)$ and $(-1, 3).$

Solution: Let the given points be $\mathrm{A}(2, -2), \mathrm{B}(-1, 3)$

Let the required equation of ellipse be $ax^2 + by^2 = 1$ ....(I)

Since $\mathrm{A}(2, -2)$ lies on equation (I) we have $a.2^2 + b.(-2)^2 = 1 \Rightarrow 4a + 4b = 1$....(II)

Since $\mathrm{B}(-1, 3)$ lies on equation (I) we have $a.(-1)^2 + b.(3)^2 = 1 \Rightarrow a + 9b = 1$....(III)

By solving (II) and (III) we get $a = \frac{5}{32}, b = \frac{3}{32}$

Hence the required equation of ellipse is $5x^2 + 3y^2 = 32$

4. Problem: If the length of the major axis of an ellipse is twice the length of its minor axis then find the eccentricity of the ellipse.

Solution: Length of the major axis of an ellipse in standard form is $2a$

Length of the minor axis of an ellipse in standard form is $2b.$

Given length of the major axis of an ellipse is twice the length of its minor axis

$i.e., 2a = 2(2b) \Rightarrow a = 2b$

We have eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 - \frac{b^2}{(2b)^2}} \Rightarrow e = \sqrt{1 - \frac{1}{4}} \Rightarrow e = \sqrt{\frac{4-1}{4}}$

$\Rightarrow e = \sqrt{\frac{3}{4}} \Rightarrow e = \frac{\sqrt{3}}{2}$

5. Problem: Find the equation of ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$ given the following data

(i) centre $(-1, 2),$ one end of major axis is $(5, 2), e = \frac{1}{2}.$

(ii) centre $(1, 3),$ $e = \frac{2}{3},$ semi-major axis is $4.$

(iii) centre $(2, 1),$ $e = \frac{1}{2},$ length of latusrectum is $4.$

(iv) centre $(-3, 1),$ one end of major axis is $(-1, 1)$ and passes through $(-3, 0).$

Solution: Let the required equation of ellipse be $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

(i) Given centre of the ellipse is $(h, k) = (-1, 2) \Rightarrow h = -1, k = 2.$

One end of major axis is $(h + a, k) = (5, 2) \Rightarrow h + a = 5 \Rightarrow -1 + a = 5 \Rightarrow a = 6.$

Also given that eccentricity $e = \frac{1}{2}.$

We have eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow a^2e^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - a^2e^2$

$\Rightarrow b^2 = 6^2 - 6^2\left(\frac{1}{2}\right)^2 \Rightarrow b^2 = 36 - 36\left(\frac{1}{4}\right) \Rightarrow b^2 = 36 - 9 \Rightarrow b^2 = 27$

Hence the required equation of ellipse is $\frac{(x+1)^2}{36} + \frac{(y-2)^2}{27} = 1$

(ii) Given centre of the ellipse is $(h,k)=(1,3) \Rightarrow h=1, k=3.$

Semi major axis is $a=4.$

Also given that eccentricity $e=\dfrac{2}{3}.$

We have eccentricity $e=\sqrt{1-\dfrac{b^2}{a^2}} \Rightarrow e^2=1-\dfrac{b^2}{a^2} \Rightarrow a^2e^2=a^2-b^2 \Rightarrow b^2=a^2-a^2e^2$

$\Rightarrow b^2=4^2-4^2\left(\dfrac{2}{3}\right)^2 \Rightarrow b^2=16-16\left(\dfrac{4}{9}\right) \Rightarrow b^2=\dfrac{80}{9}$

Hence the required equation of ellipse is $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{80/9}=1$

$\Rightarrow 5(x-1)^2+9(y-3)^2=80$

(iii) Given centre of the ellipse is $(h,k)=(2,1) \Rightarrow h=2, k=1.$

length of latusrectum is 4 $i.e., \dfrac{2b^2}{a}=4 \Rightarrow b^2=2a$

Also given that eccentricity $e=\dfrac{1}{2}.$

We have eccentricity $e=\sqrt{1-\dfrac{b^2}{a^2}} \Rightarrow e^2=1-\dfrac{b^2}{a^2} \Rightarrow \left(\dfrac{1}{2}\right)^2=1-\dfrac{2a}{a^2} \Rightarrow \dfrac{1}{4}=1-\dfrac{2}{a}$

$\Rightarrow \dfrac{2}{a}=\dfrac{3}{4} \Rightarrow a=\dfrac{8}{3}$

$b^2=2a \Rightarrow b^2=2\left(\dfrac{8}{3}\right) \Rightarrow b^2=\dfrac{16}{3}$

Hence the required equation of ellipse is $\dfrac{(x-2)^2}{64/9}+\dfrac{(y-1)^2}{16/3}=1$

$\Rightarrow 9(x-2)^2+12(y-1)^2=64$

(iv) Given centre of the ellipse is $(h,k)=(-3,1) \Rightarrow h=-3, k=1.$

One end of major axis is $(h+a,k)=(-1,1) \Rightarrow h+a=-1 \Rightarrow -3+a=-1 \Rightarrow a=2.$

The equation of ellipse is $\dfrac{(x+3)^2}{4}+\dfrac{(y-1)^2}{b^2}=1$

Since it passes through $(-3,0)$, we have $\frac{(-3+3)^2}{4}+\frac{(0-1)^2}{b^2}=1 \Rightarrow b^2=1$

Hence the required equation of ellipse is $\frac{(x+3)^2}{4}+\frac{(y-1)^2}{1}=1$

6. Problem: Find the equation of ellipse whose focus is $(2,1)$, eccentricity $\frac{3}{4}$ and directrix $2x-y+3=0$.

Solution: Let $\mathrm{P}\left(x,y\right)$ be any point on the locus.

Given focus $\mathrm{S}=(2,1)$, eccentricity $e=\frac{3}{4}$ and the equation of directrix is $\mathrm{M}\equiv 2x-y+3=0$.

The equation of ellipse having focus $\left(x_1,y_1\right)$, eccentricity $e=\frac{3}{4}$ and the directrix $ax+by+c=0$ is $\left(a^2+b^2\right)\left[\left(x-x_1\right)^2+\left(y-y_1\right)^2\right]=e^2\left(ax+by+c\right)^2$

The equation of ellipse having focus $\mathrm{S}=(2,1)$, eccentricity $e=\frac{3}{4}$ and the directrix $\mathrm{M}\equiv 2x-y+3=0$ is $\left(2^2+(-1)^2\right)\left[\left(x-2\right)^2+\left(y-1\right)^2\right]=\left(\frac{3}{4}\right)^2\left(2x-y+3\right)^2$

$\Rightarrow 80\left[x^2-4x+4+y^2-2y+1\right]=9\left[4x^2+y^2+9-4xy+12x-6y\right]$

$\Rightarrow 80x^2-320x+80y^2-160y+400=36x^2+9y^2+81-36xy+108x-54y$

$\Rightarrow 44x^2+71y^2+36xy-428x-106y+319=0$ which is the required equation of ellipse

7. Problem: Find the position (exterior or interior or on) of the following points with respect to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ $\quad (i)\ \ (3/2,-1)\quad (ii)\ \ (0,2)\quad (iii)\ \ (3,4)$.

Solution: The given equation of ellipse is $\mathrm{S}\equiv y^2-4x=0$.......(I)

(i) Let $\mathrm{P}\left(3/2,-1\right)$ be the given point.

Now $\mathrm{S}_{11}=\frac{x_1^{\ 2}}{9}+\frac{y_1^{\ 2}}{4}-1=\frac{(3/2)^2}{9}+\frac{(-1)^2}{4}-1=\frac{1}{4}+\frac{1}{4}-1=-\frac{1}{2}<0$

$\therefore \mathrm{P}\left(3/2,-1\right)$ lies inside the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

(ii) Let $P(0,2)$ be the given point.

Now $S_{11} = \frac{x_1^{\ 2}}{9} + \frac{y_1^{\ 2}}{4} - 1 = \frac{0^2}{9} + \frac{2^2}{4} - 1 = 0 + \frac{4}{4} - 1 = 1 - 1 = 0$

$\therefore P(0,2)$ lies on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(iii) Let $P(3,4)$ be the given point.

Now $S_{11} = \frac{x_1^{\ 2}}{9} + \frac{y_1^{\ 2}}{4} - 1 = \frac{(3)^2}{9} + \frac{(4)^2}{4} - 1 = 1 + 4 - 1 = 4 > 0$

$\therefore P(3,4)$ lies outside the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Exercise 9(a)

1. Find the centre, eccentricity, coordinates of foci, Length of major axis, Length of minor axis, Length of latusrectum and equations of directrices of the following ellipses:
(i) $9x^2 + 16y^2 - 36x + 32y - 92 = 0$ (ii) $3x^2 + y^2 - 6x - 2y - 5 = 0$ (iii) $9x^2 + 16y^2 = 144$
(iv) $4x^2 + y^2 - 8x + 2y + 1 = 0$ (v) $x^2 + 2y^2 - 4x + 12y + 14 = 0$ (vi) $16x^2 + 25y^2 = 400$

2. Find the equation of the ellipse referred to its major and minor axes x^-, y^- axes respectively with latusrectum of length 4 and distance between foci $4\sqrt{2}$.

3. Find the equation of the ellipse in the standard form whose distance between foci is 2 and the length of the latusrectum is $\frac{15}{2}$.

4. Find the equation of the ellipse in the standard form whose distance between foci is 8 and the distance between directrices is 32.

5. Find the equation of ellipse in standard form, if it passes through the points $(-2,2)$)and $(3,-1)$.

6. If the ends of major axis of an ellipse are $(5,0), (-5,0)$. Find the equation of the ellipse in the standard form if its focus lies on the line $3x - 5y - 9 = 0$.

7. If the length of the major axis of an ellipse is three times the length of its minor axis then find the eccentricity of the ellipse.

8. If the length of the latusrectum is equal to half of its minor axis of an ellipse in the standard from then find the eccentricity of the ellipse.

9. If the length of the latusrectum is equal to half of its major axis of an ellipse in the standard from then find the eccentricity of the ellipse.

10. Find the equation of ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,$ given the following data

(i) centre $(2,-1),$ one end of major axis is $(2,-5), e=\frac{1}{3}.$

(ii) centre $(4,-1),$ one end of major axis is $(-1,-1)$ and passes through $(8,0).$

(iii) centre $(0,-3),$ $e=\frac{2}{3},$ semi-major axis is $5.$

(iv) centre $(2,-1),$ $e=\frac{1}{2},$ length of latusrectum is $4.$

11. Find the equation of ellipse whose focus is $(1,-1),$ eccentricity $\frac{2}{3}$ and directrix $x+y+2=0.$

12. Find the position (exterior or interior or on) of the following points with respect to the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ (i) $(4,5)$ (ii) $(0,3)$ (iii) $(4,3).$

9.2 Tangent and normal at a point on the ellipse:

In this section, the condition for a straight line to be a tangent to a given ellipse is obtained. The Cartesian equations of the tangent and the normal at a given point on the ellipse are derived.

9.2.1 Point of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $y=mx+c$:

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse .....(I)

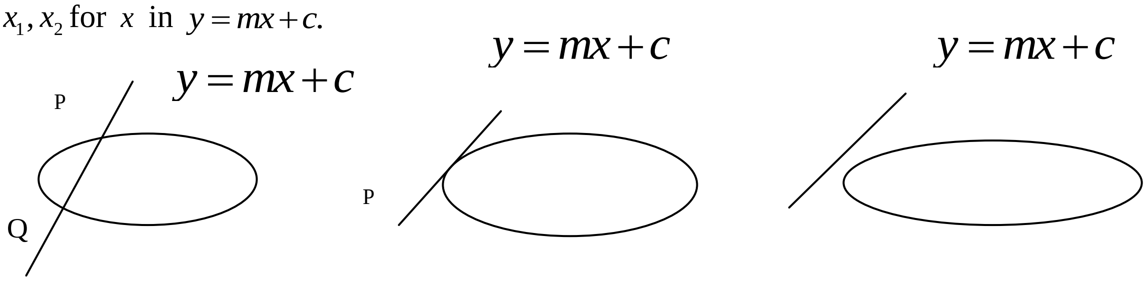
and the straight line $y=mx+c$ be given. .....(II)

The coordinates of the point of the intersection of the straight line and the ellipse satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1 \;\; i.e., x^2\,(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \qquad .....(III)$$

This is a quadratic equation in x and therefore has two roots which are distinct real equal or imaginary according as the discriminant of equation (I) is positive or zero or negative respectively. $y=mx+c$

The ordinates of the points of intersection y_1, y_2 can be obtained by substituting x_1, x_2 for x in $y = mx + c.$



9.2.2 Theorem: The condition for a straight line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$

Proof: Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse .....(I)

and the straight line $y = mx + c$ be given .....(II)

The coordinates of the point of the intersection of the straight line and the parabola satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \text{ i.e., } x^2(a^2m^2 + b^2) + 2a^2mcx + a^2(c^2 - b^2) = 0 \quad .....(III)$$

The given line will touch the ellipse $\Leftrightarrow$ the two points coincide.

$\Leftrightarrow$ discriminant of (III) is zero.

$$\Leftrightarrow (2a^2mc)^2 - 4a^2(a^2m^2 + b^2)(c^2 - b^2) = 0$$

$$\Leftrightarrow 4a^4m^2c^2 - 4a^2(a^2m^2c^2 + b^2c^2 - a^2m^2b^2 - b^4) = 0$$

$$\Leftrightarrow c^2 = a^2m^2 + b^2 \quad \Leftrightarrow c = \sqrt{a^2m^2 + b^2}$$

9.2.3 Note:

(i) In view of the above theorem 9.2.2, the equation of any tangent to the ellipse $\text{S} = 0$ can be taken as $y = mx \pm \sqrt{a^2m^2 + b^2}$.

(ii) For every real value of $m,$ there are two parallel tangents to the ellipse.

(iii) The points of contact of these tangents are $\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right) = \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$ and $\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{-b^2}{\sqrt{a^2m^2+b^2}}\right) = \left(\frac{a^2m}{c}, \frac{-b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$

(iv) The condition for a straight line $lx + my + n = 0$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $n^2 = a^2l^2 + b^2m^2$

9.2.4 Theorem: The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on $\mathrm{S} = 0$ is $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

Proof: Let $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$ be two points on the ellipse $\mathrm{S} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, then

$\mathrm{S}_{11} = 0$ and $\mathrm{S}_{22} = 0$. Consider the second degree equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

$$i.e., \left[\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right] + \left[\frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1\right] = \left[\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right]$$

which represents a straight line.

Substituting $\mathrm{P}(x_1, y_1)$ it becomes $\mathrm{S}_{11} + \mathrm{S}_{12} = 0 + \mathrm{S}_{12} = \mathrm{S}_{12}$

$\therefore$ $\mathrm{P}(x_1, y_1)$ satisfies the equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

Similarly $\mathrm{Q}(x_2, y_2)$ satisfies the equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

$\therefore$ $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$ is a straight line passing through $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$

$\therefore$ The equation of the chord PQ is $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

9.2.5 Theorem: The equation of tangent at (x_1, y_1) to the ellipse $\mathrm{S} = 0$ is $\mathrm{S}_1 = 0$.

Proof: Let $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$ be two points on the parabola $\mathrm{S} \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, then $\mathrm{S}_{11} = 0$ and $\mathrm{S}_{22} = 0$. By Theorem 9.2.4, the equation of the chord PQ is $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$...(I)

The chord PQ becomes the tangent at P when Q approaches P.

i.e., (x_2, y_2) approaches to (x_1, y_1)

$\therefore$ The equation of the tangent at P is obtained by taking limits (x_2, y_2) tends to as (x_1, y_1) on either sides of (I)

So, the equation of the tangent at P is given by $\lim\limits_{Q\to P}(S_1 + S_2) = \lim\limits_{Q\to P} S_{12}$

i.e., $S_1 + S_1 = S_{11}\left[\because S_2 \to S_1, S_{12} \to S_{11} \text{ as } (x_2, y_2) \to (x_1, y_1)\right]$

$\therefore 2S_1 = 0 \Rightarrow S_1 = 0.$

$\therefore$ The equation of the tangent to the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$ at (x_1, y_1) is $S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

9.2.6 Theorem: The equation of normal at (x_1, y_1) on the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$ is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2, (x_1 \neq 0, y_1 \neq 0).$

Proof: By Theorem 9.2.5, the equation of tangent to the parabola $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at (x_1, y_1) is $S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ is $= \frac{-x_1 / a^2}{y_1 / b^2} = \frac{-b^2 x_1}{a^2 y_1}.$

$\therefore$ Slope of the normal at $P(x_1, y_1)$ is $= \frac{-x_1 / a^2}{y_1 / b^2} = \frac{a^2 y_1}{b^2 x_1}.$

Hence the equation of normal at (x_1, y_1) is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2.$

9.2.7 Note:

(i) If $x_1 = 0$ and $y_1 \neq 0$ then the equation of the normal at $P(x_1, y_1) = (0, y_1) = (0, \pm b)$ is the $y-$ axis.

(ii) If $y_1 = 0$ and $x_1 \neq 0$ then the equation of the normal at $P(x_1, y_1) = (x_1, 0) = (\pm a, 0)$ is the $x-$ axis.

9.2.8 Solved Problems:

1. Problem: Find the equations of the tangent and normal to the ellipse $3x^2 + 4y^2 = 17$ at $(1, 2)$.

Solution: The given equation of the ellipse is $S \equiv 3x^2 + 4y^2 = 17$ ...(I)

Let $P(1, 2)$ be any point on the ellipse

Equation of tangent at $P(x_1, y_1)$ to the ellipse $S \equiv 0$ is $S_1 \equiv 0$

i.e., $3xx_1 + 4yy_1 - 17 = 0 \Rightarrow 3x.1 + 4y.2 - 17 = 0$

$\Rightarrow 3x + 8y - 17 = 0$ ...(II)

Since equation of normal is perpendicular to the tangent and passing through $P(1, 2)$

equation of the normal is $8(x-1) - 3(y-2) = 0$

$\Rightarrow 8x - 3y - 2 = 0$

2. Problem: Find the equations of the tangent and normal to the ellipse $x^2 + 4y^2 - 8x - 6y - 7 = 0$ at $(-1,1)$.

Solution: The given equation of the ellipse is $S \equiv x^2 + 4y^2 - 8x - 6y - 7 = 0$....(I)

Let $P(-1,1)$ be any point on the ellipse

Equation of tangent at $P(x_1, y_1)$ to the ellipse $S \equiv 0$ is $S_1 \equiv 0$

i.e., $xx_1 + 4yy_1 - 4(x + x_1) - 3(y + y_1) - 7 = 0 \Rightarrow x(-1) + 4y.1 - 4(x-1) - 3(y+1) - 7 = 0$

$\Rightarrow 5x - y + 6 = 0$ ...(II)

Since equation of normal is perpendicular to the tangent and passing through $P(-1,1)$

equation of the normal is $(x+1) + 5(y-1) = 0$

$\Rightarrow x + 5y - 4 = 0$

3. Problem: Find the value of k if the line $2x - y + k = 0$ is a tangent to the ellipse $x^2 + 2y^2 = 5$.

Solution: The given equation of the ellipse is $S \equiv x^2 + 2y^2 = 5 \Rightarrow \frac{x^2}{5} + \frac{y^2}{5/2} = 1$....(I)

The given line equation is $2x - y + k = 0$ ...(II)

The condition that the line $lx+my+n=0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

is $n^2=a^2l^2+b^2m^2$.

We have $a^2=5, b^2=5/2, l=2, m=-1, n=k$

$$k^2=5.2^2+\frac{5}{2}(-1)^2 \Rightarrow k^2=20+\frac{5}{2} \Rightarrow k^2=\frac{45}{2} \Rightarrow k=\pm\sqrt{\frac{45}{2}}$$

4. Problem: Find the equations of the tangent to the ellipse $x^2+3y^2=4$ which are (i) parallel to the line $x+3y+5=0$ and (ii) perpendicular to the line $2x+y+4=0$.

Solution: The given equation of the ellipse is $S \equiv x^2+3y^2=4 \Rightarrow \frac{x^2}{4}+\frac{y^2}{4/3}=1$....(I)

(i) The given line equation is $x+3y+5=0$ ...(II)

The equation of a line parallel to the line (II) is $x+3y+k=0$ ...(III)

Since the line (III) is a tangent to the ellipse (I).

We have $k^2=4.1^2+\frac{4}{3}.3^2 \Rightarrow k^2=4+12 \Rightarrow k^2=16 \Rightarrow k=\pm 4$

$\therefore$ The equation of the tangent to the ellipse $x^2+3y^2=4$ which is parallel to the line $x+3y+5=0$ is $x+3y+\pm 4=0$

(ii) The equation of a line perpendicular to the line $2x+y+4=0$ is

$$x-2y+k=0 \qquad \text{...(IV)}$$

Since the line (IV) is a tangent to the ellipse (I).

We have $k^2=4.1^2+\frac{4}{3}.(-2)^2 \Rightarrow k^2=4+\frac{16}{3} \Rightarrow k^2=\frac{28}{3} \Rightarrow k=\pm\sqrt{\frac{28}{3}}$

$\therefore$ The equation of the tangent to the ellipse $x^2+3y^2=4$ which is perpendicular to the line $2x+y+4=0$ is $x-2y\pm\sqrt{\frac{28}{3}}=0$

5. Problem: Find the equations of the tangent and normal to the ellipse $x^2+4y^2=16$ at the point whose ordinate is 2.

Solution: The given equation of the ellipse is $S \equiv x^2+4y^2=16$....(I)

Given ordinate is 2, *i.e.*, $y = 2 \Rightarrow x^2 + 4.2^2 = 16 \Rightarrow x^2 + 16 = 16 \Rightarrow x^2 = 0 \Rightarrow x = 0$

Let $P(0,2)$ be any point on the ellipse

Equation of tangent at $P(x_1, y_1)$ to the ellipse $S \equiv 0$ is $S_1 \equiv 0$

i.e., $xx_1 + 4yy_1 - 16 = 0 \Rightarrow x.0 + 4y.2 - 16 = 0 \Rightarrow 8y - 16 = 0$

$\Rightarrow y - 2 = 0$ ...(II)

Since equation of normal is perpendicular to the tangent and passing through $P(0,2)$

equation of the normal is $(x-0) - 0(y-2) = 0$

$\Rightarrow x = 0$

Exercise 9(b)

1. Find the equations of the tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$.

2. Find the equations of the tangent and normal to the ellipse $x^2 + 2y^2 - 4x + 12y + 14 = 0$ at $(2, -1)$.

3. Find the equation of the tangent to the ellipse $9x^2 + 16y^2 = 144$ which makes equal intercepts on the coordinate axis.

4. Find the coordinates of the points on the ellipse $x^2 + 3y^2 = 37$ at which the normal is parallel to the line $6x - 5y = 2$.

5. Find the value of k if the line $4x + y + k = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 3$..

6. Find the equations of the tangent to the ellipse $2x^2 + y^2 = 8$ which are (i) parallel to the line $x - 2y - 4 = 0$ (ii) perpendicular to the line $x + y + 2 = 0$ and

7. Find the equations of the tangent and normal to the ellipse $2x^2 + 3y^2 = 11$ at the point whose ordinate is 1.

8. Find the equations of the tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latusrectum in the first quadrant.

Key concepts

1. A conic with eccentricity less than unity is called an *ellipse*. Hence an ellipse is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio e, which is less than unity. The fixed point is called the *focus* and the fixed line is called the *directrix* of the ellipse.

2. Let $S(x_1, y_1)$ be the focus, e be the eccentricity and the directrix be $ax+by+c=0.$ The equation of the ellipse is $\left(a^2+b^2\right)\left[\left(x-x_1\right)^2+\left(y-y_1\right)^2\right]=e^2\left(ax+by+c\right)^2$ and the equation of the axis of the above ellipse is $b\left(x-x_1\right)-a\left(y-y_1\right)=0.$

3. The equation of ellipse in standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ For the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b>0)$ the focus is $S=(ae,0),$ directrix is $x=\frac{a}{e}$ and axis is $y=0.$

The point $A=(0,0)$ is called the vertex of the ellipse.

4. The line joining two points of a ellipse is called a *chord* of a ellipse.

5. A chord passing through focus is called a *focal chord*.

6. A chord through a point P on the ellipse, which is perpendicular to the axis of the parabola, is called the *double ordinate* of the point P.

7. The double ordinate passing though the focus is called the focus is called the *latusrectum* of the ellipse.

8. The length of the latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}.$

9. (i) If the equation of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b>0)$ then Eccentricity is $e=\sqrt{1-\frac{b^2}{a^2}},$ Centre $C=(0,0),$ vertices are $A=(a,0), A'=(-a,0)$ Focii are $S=(ae,0), S'=(-ae,0)$ Equation of directrices are $x=\pm\frac{a}{e},$ Length of major axis is $AA'=2a$ and length of minor axis is $BB'=2b$

(ii) If the equation of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (b>a>0)$ then Eccentricity is $e=\sqrt{1-\frac{a^2}{b^2}},$ Centre $C=(0,0),$ vertices are $B=(0,b), B'=(0,-b)$ Focii are $S=(0,be), S'=(0,-be)$ Equation of directrices are $y=\pm\frac{b}{e},$ Length of major axis is $BB'=2b$ and length of minor axis is $AA'=2a$ axis

(iii) If the equation of ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,(a>b>0)$ then

Eccentricity is $e=\sqrt{1-\frac{b^2}{a^2}}$, Centre $C=(h,k)$, vertices are $A=(h+a,k), A'=(h-a,k)$

Focii are $S=(h+ae,k), S'=(h-ae,k)$ Equation of directrices are $x=h\pm\frac{a}{e}$, Length of major axis is $AA'=2a$ and length of minor axis is $BB'=2b$

(iv) If the equation of ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,(b>a>0)$ then

Eccentricity is $e=\sqrt{1-\frac{a^2}{b^2}}$, Centre $C=(h,k)$, vertices are $B=(h,k+b), B'=(h,k-b)$

Focii are $S=(h,k+be), S'=(h,k-be)$ Equation of directrices are $y=k\pm\frac{b}{e}$, Length of major axis is $BB'=2b$ and length of minor axis is $AA'=2a$ axis

10. The circle described on the major axis of an ellipse as diameter is called *auxiliary circle* of the ellipse. The equation of the auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$.

11. The distance of a point on the ellipse from its focus is called the *focal distance* of the point.

12. The following notation will be adapted.

(i) $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ (ii) $S_1\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ (iii) $S_{12}\equiv\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$

(iv) $S_{11}\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ where (x_1,y_1) and (x_2,y_2) are the points in the plane of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

13. The condition that the point P lies outside, on or inside the ellipse

$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ according as $S_{11} >=< 0$.

14. The condition for a straight line $y=mx+c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$

15. The condition for a straight line $lx + my + n = 0$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $n^2 = a^2l^2 + b^2m^2$

16. Two tangents can be drawn from an external point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

17. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on $S = 0$ is $S_1 + S_2 = S_{12}$.

18. The equation of tangent at (x_1, y_1) to the ellipse $S = 0$ is $S_1 = 0$ or the equation of the tangent to the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$ at (x_1, y_1) is $S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

19. The equation of normal at (x_1, y_1) on the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2, (x_1 \neq 0, y_1 \neq 0).$

Answers
Exercise 9(a)

1.

(*i*) centre: $(2,-1)$, eccentricity: $\sqrt{7}/4$, focii: $\left(2 \pm \sqrt{7}, -1\right)$, length of major axis: 8,
length of minor axis: 6, length of latusrectum: 9/2, eq. of directrices: $\sqrt{7}x = 2\sqrt{7} \pm 16$

(*ii*) centre: $(1,1)$, eccentricity: $\sqrt{2/3}$, focii: $\left(1, 1 \pm \sqrt{6}\right)$, length of major axis: 6,
length of minor axis: $2\sqrt{3}$, length of latusrectum: 2, eq. of directrices: $\sqrt{2}y = \sqrt{2} \pm 3\sqrt{3}$

(*iii*) centre: $(0,0)$, eccentricity: $\sqrt{7}/4$, focii: $\left(\pm\sqrt{7}, 0\right)$, length of major axis: 8,
length of minor axis: 6, length of latusrectum: 9/2, eq. of directrices: $\sqrt{7}x = \pm 16$

(*iv*) centre: $(1,-1)$, eccentricity: $\sqrt{3}/2$, focii: $\left(1, -1 \pm \sqrt{3}\right)$, length of major axis: 4,
length of minor axis: 2, length of latusrectum: 1, eq. of directrices: $\sqrt{3}y + \sqrt{3} \pm 4 = 0$

(*v*) centre: $(2,-3)$, eccentricity: $1/\sqrt{2}$, focii: $(4,-3), (0,-3)$, length of major axis: $4\sqrt{2}$,
length of minor axis: 4, length of latusrectum: $2\sqrt{2}$, eq. of directrices: $x = 6, x = -2.$

(*vi*) centre: $(0,0)$, eccentricity: $3/5$, focii: $(\pm 3, 0)$, length of major axis: 10,
length of minor axis: 6, length of latusrectum: 32/5, eq. of directrices: $3x = \pm 25$

2. $\frac{x^2}{16}+\frac{y^2}{8}=1$ 3. $\frac{x^2}{16}+\frac{y^2}{15}=1$ 4. $\frac{x^2}{64}+\frac{y^2}{48}=1$ 5. $3x^2+5y^2=32$

6. $16x^2+25y^2=400$ 7. $\frac{2\sqrt{2}}{3}$ 8. $\frac{\sqrt{3}}{2}$ 9. $\frac{1}{\sqrt{2}}$

10. (i) $9(x-2)^2+8(y+1)^2=128,$ (ii) $(x-4)^2+9(y+1)^2=25,$

(iii) $\frac{x^2}{25}+\frac{(y+3)^2}{45}=1$ or $\frac{x^2}{45}+\frac{(y+3)^2}{25}=1$

(iv) $9(x-2)^2+12(y+1)^2=64,$ or $12(x-2)^2+9(y+1)^2=64.$

11. $7x^2+7y^2-4xy-26x+10y+10=0.$

12. (i) exterior (ii) interior (iii) exterior

Exercise 9(b)

1. $x-16y+33=0,\ 16x+y+44=0.$ 2. $y+1=0,\ x-2=0.$ 3. $x+y\pm 5=0.$

4. $(5,2),(-5,-2).$ 5. $k=\pm 7$ 6. (i) $x-2y\pm 6=0$ (ii) $x-y\pm 2\sqrt{3}=0$

7. $4x+3y-11=0, 4x-3y+11=0;\ 3x-4y-2=0, 3x+4y+2=0;$

8. $\sqrt{7}x+4y-16=0, 16x-4\sqrt{7}y-7\sqrt{7}=0;$

10. HYPERBOLA

Introduction:

We study the hyperbola in this chapter. We also discuss, about the standard form of equation of hyperbola, condition for a line to be a tangent to the ellipse, chord of contact in this chapter.

10.1 Equation of a hyperbola in standard form:

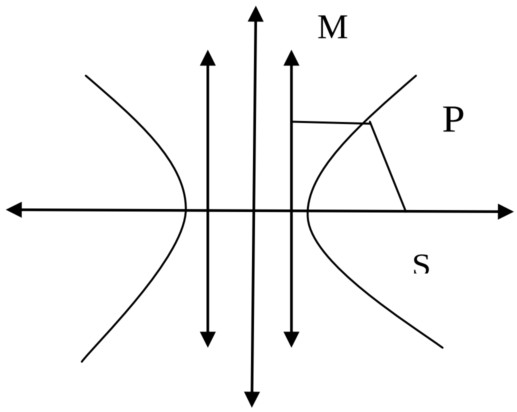
In this section, we study the equation of hyperbola in the standard form.

10.1.1 Definition (Hyperbola): A conic with eccentricity greater than unity is called a *hyperbola*. Hence hyperbola is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio e, which is greater than unity. The fixed point is called the *focus* and the fixed line is called the *directrix* of the hyperbola.

10.1.2 Equation of hyperbola:

In this section we derive the equation of hyperbola in general form.

Let $\mathrm{S}(x_1, y_1)$ be the focus and the directrix be $ax+by+c=0$. Thus, by definition of the ellipse, the equation of the hyperbola is



$$\sqrt{(x-x_1)^2+(y-y_1)^2} = e\frac{|ax+by+c|}{\sqrt{a^2+b^2}} \text{ or}$$

$$(a^2+b^2)\left[(x-x_1)^2+(y-y_1)^2\right] = e^2(ax+by+c)^2$$

a general equation of second degree in x and y.

The equation of the axis of the above hyperbola is $b(x-x_1)-a(y-y_1)=0.$

10.1.3 Equation of hyperbola in standard form:

To study the nature of the curve, we prefer its equation in the simple possible form. We proceed as follows to derive such an equation.

Let S be the focus, l be the corresponding directrix and e be the eccentricity. Let Z be the foot of the perpendicular from S on directrix l. Let A and A' be the points which divide SZ in the ratio $e:1,$ internally and externally respectively.

Consider C midpoint of AA' as origin, consider the line CZ extended as $x-$axis and a line perpendicular to it C as $y-$axis.

Let $\mathrm{CA} = a = \mathrm{CA}'$ so that $\mathrm{A} = (a, 0)$ and $\mathrm{A}' = (-a, 0).$

But $\frac{\text{SA}}{\text{AZ}} = e = \frac{\text{SA}'}{\text{A}'\text{Z}} \Rightarrow \text{SA} = e(\text{AZ})$ and $\text{SA}' = e(\text{A}'\text{Z})$

$\therefore \text{CA} - \text{CS} = e(\text{CZ} - \text{CA}) \Rightarrow a - \text{CS} = e(\text{CZ} - a)$.....(I)

$\text{CS} + \text{CA}' = e(\text{CA}' + \text{CZ}) \Rightarrow \text{CS} + a = e(\text{CZ} + a)$.....(II)

Adding (I) and (II) above, we get $2a = 2e(\text{CZ}) \Rightarrow \text{CZ} = \frac{a}{e}$

$\therefore$ Equation of directrix is $x = \frac{a}{e}$......(III)

Subtracting (I) from (II), we get $2(\text{CS}) = 2ae \Rightarrow \text{CS} = ae.$

Coordinates of focus S are $(ae, 0)$.

Now let $\text{P}(x, y)$ be a point on the ellipse and PM be the perpendicular distance from P to the direcrix. Then by definition $\text{PS} = e(\text{PM})$.

$$\text{PS}^2 = e^2(\text{PM})^2$$

$$i.e., (x - ae)^2 + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2ax}{e}\right) \quad \left[\because \text{PM} = x - \frac{a}{e}\right]$$

$$i.e., x^2\left(e^2 - 1\right) - y^2 = a^2\left(e^2 - 1\right)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2\left(e^2 - 1\right)} = 1,$$

Since $e > 1 \Rightarrow e^2 - 1 > 0 \Rightarrow b^2 = a^2(1 - e^2) > 0.$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > 0, b > 0).....(IV)$$

$\therefore$ We can choose a real number $b > 0$ such that $a^2(e^2 - 1) = b^2$.

We have shown that coordinates of P must satisfy (IV) if P satisfies the geometric condition $\text{SP} = e(\text{PM})$. Conversely, if x, y satisfy the algebraic equation (IV) with $b^2 = a^2(e^2 - 1)$ and $e > 1$, then

$$i.e., y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = b^2\left(\frac{x^2 - a^2}{a^2}\right) = \frac{a^2\left(e^2 - 1\right)\left(x^2 - a^2\right)}{a^2} = \left(e^2 - 1\right)\left(x^2 - a^2\right)$$

$$\therefore \text{SP} = \sqrt{(x - ae)^2 + y^2} = \sqrt{x^2 + a^2e^2 - 2aex + \left(e^2 - 1\right)\left(x^2 - a^2\right)}$$

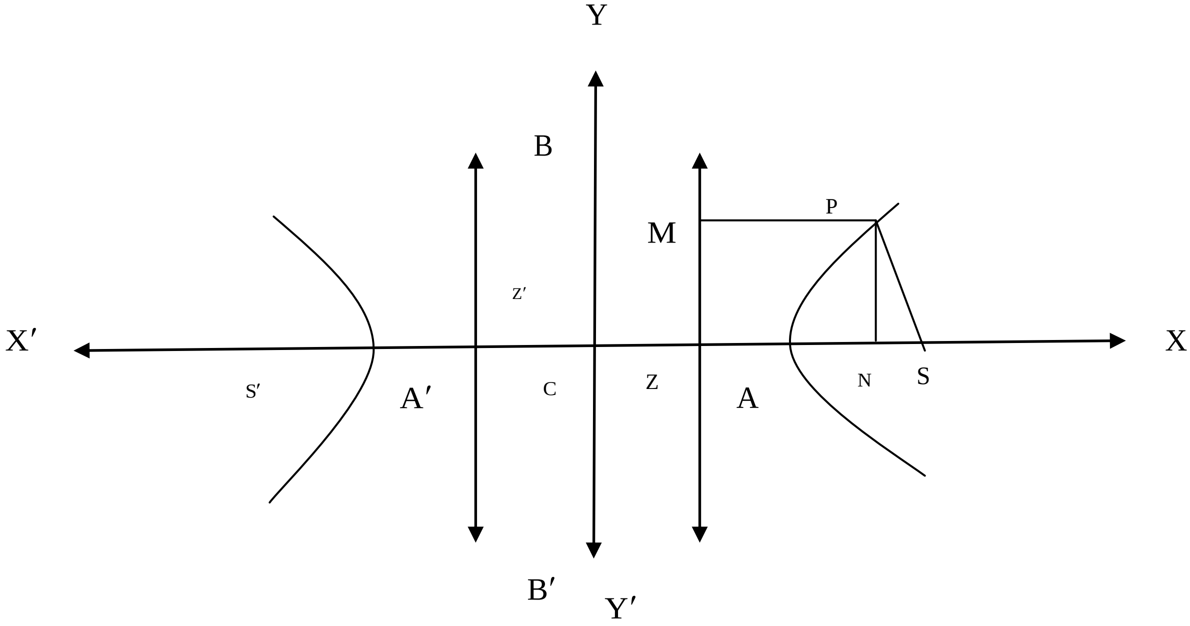
$$\therefore \text{SP} = \sqrt{x^2e^2 - 2aex + a^2} = |xe - a| = e\left|x - \frac{a}{e}\right| = e(\text{PM}).$$

If P satisfies the algebraic condition then P satisfies the geometric condition and vice versa.

Thus the locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ the equation of hyperbola in standard form.

Now let S′ be the image of S and Z′M′ be the image of ZM with respect to $y-$axis, taking S′ as focus and Z′M′ as corresponding directrix, it can be seen that the corresponding equation of hyperbola is also $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ Hence for every hyperbola, there are two foci and two corresponding directrices.

We have $b^2 = a^2(e^2 - 1)$ and $e > 1\ e = \sqrt{1 + \frac{b^2}{a^2}}$



10.1.4 Nature of the curve:

In this section we shall study the nature of the hyperbola or trace the curve represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, (a > 0, b > 0)$

(i) Point of intersection with coordinate axes:

If $y = 0,$ then $x = \pm a$ *i.e.,* the curve intersects the $x-$axis at $\text{A} = (a, 0)$ and $\text{A}' = (-a, 0).$ Hence $\text{AA}' = 2a.$

If $x = 0$, then $y = \pm\sqrt{-b^2}$ does not exist in the Cartesian plane. Hence, the curve does not intersect the $y-$axis.

(ii) From (i) we have $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ then y is real $\Leftrightarrow x^2 - a^2 \geq 0 \Leftrightarrow x \leq -a$ or $x \geq a$ *i.e.,* the curve does not exist between the vertical lines $x = a$ and $x = -a$ further from $x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$ then x is real for all values of y and hence each horizontal line $y = k$ intersect the hyperbola in exactly two points. Also $x \to \pm\infty$ when $y \to \pm\infty$ *i.e.,* the curve is unbounded.

(iii) For any value of y, we have two values of $x = \pm\frac{a}{b}\sqrt{y^2 + b^2}$ equal but opposite in sign.

$\therefore$ The curve is symmetric about $y-$axis.

(iv) For each real value of y belonging $\mathrm{R} - (-a, a)$, we have two values of $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ equal but opposite in sign.

$\therefore$ The curve is symmetric about $x-$axis.

$\therefore$ The curve consists of two symmetrical branches each extending to infinity in two directions.

(v) Any chord through $\mathrm{C}(0,0)$ of the hyperbola is bisected at the point C, for the points $(x, y), (-x, -y)$ simultaneously lie on the curve. The centre of hyperbola is defined as the point of intersection of its axes of symmetry. The centre of the hyperbola is the point C.

10.1.5 Definition (Transverse and conjugate axes): The line segment $\mathrm{AA'}$ and $\mathrm{BB'}$ of lengths $2a$ and $2b$ respectively are called axes of the hyperbola. The $\mathrm{AA'}$ along $x-$axis is called the transverse axis of the hyperbola. $\mathrm{BB'}$ is called the conjugate axis.

10.1.6 Definition (Conjugate hyperbola): The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of the given hyperbola, is called the conjugate hyperbola of the given hyperbola.

10.1.7 Note: (i) The equation of the hyperbola conjugate to $\mathrm{S} \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is $\mathrm{S'} \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$

(ii) For $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ the transverse axis lies along $x-$axis and its length $2a$ and the conjugate axis lies along $y-$axis and its length $2b$.

(ii) For $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ the transverse axis lies along $y-$axis and its length $2b$ and the conjugate axis lies along $x-$axis and its length $2a$.

10.1.8 Definitions (Chord, focal chord and latusrectum):

The line joining two points on the hyperbola is called a *chord* of the hyperbola. A chord passing through one of the foci is called a *focal chord*. A focal chord perpendicular to the transverse axis of the hyperbola is called a *latusrectum* of the hyperbola. A hyperbola has two latusrecta.

10.1.9 Length of the latusrectum:

Let L, L' be the ends of the latusrectum passing through the one of the foci $S(ae, 0)$ of the hyperbola equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, (a > b)$.....(I)

Since LL' is perpendicular to x^-axis, the coordinates of L and L' are equal to ae.

This $L(ae, y_1)$ is on (I), we have $\frac{(ae)^2}{a^2} - \frac{y_1^{\ 2}}{b^2} = 1 \Rightarrow \frac{y_1^{\ 2}}{b^2} = e^2 - 1 \Rightarrow y_1^{\ 2} = b^2(e^2 - 1)$

$$\Rightarrow y_1^{\ 2} = b^2\left(\frac{b^2}{a^2}\right) \qquad \left[\because b^2 = a^2(e^2 - 1)\right]$$

$$\therefore y_1 = \pm \frac{b^2}{a}.$$

Hence $L\left(ae, \frac{b^2}{a}\right)$ and $L' = \left(ae, -\frac{b^2}{a}\right)$

$\therefore$ Length of the latusrectum $LL' = \frac{2b^2}{a}$.

10.1.10 Note: (i) The coordinates of the four ends of the latusrecta of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $L\left(ae, \frac{b^2}{a}\right), L' = \left(ae, -\frac{b^2}{a}\right)$ and $L_1\left(-ae, \frac{b^2}{a}\right), L_1{}' = \left(-ae, -\frac{b^2}{a}\right)$

(ii) Length of the latusrectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is $\frac{2a^2}{b}$ and the coordinates of the four ends of the latusrecta of the hyperbola are $\mathrm{L}\left(\frac{a^2}{b},be\right),\mathrm{L}'=\left(-\frac{a^2}{b},be\right)$ and $\mathrm{L}_1\left(\frac{a^2}{b},-be\right),\mathrm{L}_1{}'=\left(-\frac{a^2}{b},-be\right)$.

(iii) The equation of the latusrectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ through focus S is $x=ae$ and through S′ is $x=-ae.$ The equation of the latusrectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ through S is $y=be$ and through S′ is $y=-be.$

10.1.11 Rectangular hyperbola:

If in a hyperbola the length of transverse axis ($2a$) is equal to the length of conjugate axis ($2b$), the hyperbola is called the rectangular hyperbola. The equation of the rectangular hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2-y^2=a^2$. In this case the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

We assumed $a\neq b$ and in the following discussion, we describe different forms of the hyperbola.

10.1.12 Various forms of the hyperbola:

(i) The equation of hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Transverse axis is along $x-$axis and conjugate axis is along $y-$axis.

Length of transverse axis is $\mathrm{AA}'=2a$ and length of conjugate axis is $\mathrm{BB}'=2b.$

Eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}$

Centre $\mathrm{C}=(0,0)$

Foci $\mathrm{S}=(ae,0),\mathrm{S}'=(-ae,0)$

Equation of directrices $x=a/e, x=-a/e$

(ii) The equation of hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Transverse axis is along $y-$axis and conjugate axis is along $x-$axis.

Length of transverse axis is $\mathrm{BB}' = 2b$ and length of conjugate axis is $\mathrm{AA}' = 2a$

Eccentricity $e = \sqrt{1+\frac{a^2}{b^2}}$

Centre $\mathrm{C} = (0,0)$

Foci $\mathrm{S} = (0, be), \mathrm{S}' = (0, -be)$

Equation of directrices $y = b/e, y = -b/e$

(iii) The equation of hyperbola is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Transverse axis is parallel to $x-$axis *i.e.,* along $y = k$ and conjugate axis is parallel to $y-$axis *i.e.,* along $x = h.$

Length of transverse axis is $\mathrm{AA}' = 2a$ and length of conjugate axis is $\mathrm{BB}' = 2b.$

Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$

Centre $\mathrm{C} = (h, k)$

Foci $\mathrm{S} = (h+ae, k), \mathrm{S}' = (h-ae, k)$

Equation of directrices $x = h + a/e, x = h - a/e$

(iv) The equation of hyperbola is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Transverse axis is parallel to $y-$axis *i.e.,* along $x = h$ and conjugate axis is parallel to $x-$axis *i.e.,* along $y = k$

Length of transverse axis is $\mathrm{BB}' = 2b$ and length of conjugate axis is $\mathrm{AA}' = 2a$

Eccentricity $e = \sqrt{1+\frac{a^2}{b^2}}$

Centre $\mathrm{C} = (h, k)$

Foci $\mathrm{S} = (h, k+be), \mathrm{S}' = (h, k-be)$

Equation of directrices $y = k + b/e, y = k - b/e$

10.1.13 Definition (Auxiliary circle): The circle described on the transverse axis of a hyperbola as diameter is called *auxiliary circle* of the hyperbola. The equation of the auxiliary circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$.

10.1.14 Notation: Here after the following notation will be adapted through out this chapter.

(i) $S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1$

(ii) $S_1 \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(iii) $S_{12} \equiv \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$

(iv) $S_{11} \equiv \frac{x_1^{\,2}}{a^2}-\frac{y_1^{\,2}}{b^2}-1$ where (x_1, y_1) and (x_2, y_2) are the points in the plane of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

10.1.15 Hyperbola and a point in the plane of the hyperbola:

A hyperbola divides the xy – plane into two disjoint regions, one containing the foci is called the interior region of the hyperbola and the other is called the exterior region of the hyperbola.

Let $P(x_1, y_1)$ be a point in the plane of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$......(I).

Consider $S_{11}=\frac{x_1^{\,2}}{a^2}-\frac{y_1^{\,2}}{b^2}-1$

(i) P lies outside the hyperbola (*i.e.*, P is an external point) $\Leftrightarrow S_{11}>0$.

(ii) P lies on the hyperbola $\Leftrightarrow S_{11}=0$.

(iii) P lies inside the hyperbola (*i.e.*, P is an internal point) $\Leftrightarrow S_{11}<0$.

Thus P lies outside, on or inside the hyperbola $S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1$ according as $S_{11} >=< 0$.

10.2 Tangent and normal at a point on the hyperbola:

In this section, the condition for a straight line to be a tangent to a given hyperbola is obtained. The Cartesian equations of the tangent and the normal at a given point on the hyperbola are derived.

9.2.1 Point of intersection of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **and the line** $y=mx+c$ **:**

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola ...(I)

and the straight line $y=mx+c$ be given. .....(II)

The coordinates of the point of the intersection of the straight line and the hyperbola satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1 \ \ i.e., x^2(b^2-a^2m^2)-2a^2mcx-a^2(c^2+b^2)=0 \qquad .....(III)$$

This is a quadratic equation in x and therefore has two roots which are distinct real equal or imaginary according as the discriminant of equation (I) is positive or zero or negative respectively. $y=mx+c$

The ordinates of the points of intersection y_1, y_2 can be obtained by substituting x_1, x_2 for x in $y=mx+c.$

10.2.2 Theorem: The condition for a straight line $y=mx+c$ to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$

Proof: Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola .....(I)

and the straight line $y=mx+c$ be given .....(II)

The coordinates of the point of the intersection of the straight line and the parabola satisfy both the equations (I) and (II) and, therefore, can be found by solving them. Substituting the values of y from (II) in (I), we have

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1 \ \ i.e., x^2(b^2-a^2m^2)-2a^2mcx-a^2(c^2+b^2)=0 \qquad .....(III)$$

The given line will touch the hyperbola $\Leftrightarrow$ the two points coincide.

$\Leftrightarrow$ discriminant of (III) is zero.

$\Leftrightarrow (-2a^2mc)^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$

$\Leftrightarrow 4a^4m^2c^2 + 4a^2(-a^2m^2c^2 + b^2c^2 - a^2m^2b^2 + b^4) = 0$

$\Leftrightarrow c^2 = a^2m^2 - b^2 \quad \Leftrightarrow c = \sqrt{a^2m^2 - b^2}$

10.2.3 Note:

(i) In view of the above theorem 10.2.2, the equation of any tangent to the hyperbola $\mathrm{S} = 0$ can be taken as $y = mx \pm \sqrt{a^2m^2 - b^2}$.

(ii) For every real value of m, there are two parallel tangents to the hyperbola.

(iii) The points of contact of these tangents are $\left(\dfrac{-a^2m}{\sqrt{a^2m^2-b^2}}, \dfrac{b^2}{\sqrt{a^2m^2-b^2}}\right) = \left(\dfrac{-a^2m}{c}, \dfrac{b^2}{c}\right)$ and $\left(\dfrac{a^2m}{\sqrt{a^2m^2-b^2}}, \dfrac{-b^2}{\sqrt{a^2m^2-b^2}}\right) = \left(\dfrac{a^2m}{c}, \dfrac{-b^2}{c}\right)$ where $c^2 = a^2m^2 - b^2$

(iv) The condition for a straight line $lx + my + n = 0$ to be a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $n^2 = a^2l^2 - b^2m^2$

10.2.4 Theorem: The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on $\mathrm{S} = 0$ is $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

Proof: Let $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$ be two points on the hyperbola $\mathrm{S} \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$, then $\mathrm{S}_{11} = 0$ and $\mathrm{S}_{22} = 0$. Consider the second degree equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

$$i.e., \left[\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right] + \left[\frac{xx_2}{a^2} - \frac{yy_2}{b^2} - 1\right] = \left[\frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} - 1\right]$$

which represents a straight line.

Substituting $\mathrm{P}(x_1, y_1)$ it becomes $\mathrm{S}_{11} + \mathrm{S}_{12} = 0 + \mathrm{S}_{12} = \mathrm{S}_{12}$

$\therefore \mathrm{P}(x_1, y_1)$ satisfies the equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

Similarly $\mathrm{Q}(x_2, y_2)$ satisfies the equation $\mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$.

$\therefore \mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_{12}$ is a straight line passing through $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$

$\therefore$ The equation of the chord PQ is $S_1 + S_2 = S_{12}$.

10.2.5 Theorem: The equation of tangent at (x_1, y_1) to the hyperbola $S = 0$ is $S_1 = 0$.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, then $S_{11} = 0$ and $S_{22} = 0$. By Theorem 10.2.4, the equation of the chord PQ is $S_1 + S_2 = S_{12}$...(I)

The chord PQ becomes the tangent at P when Q approaches P.

i.e., (x_2, y_2) approaches to (x_1, y_1)

$\therefore$ The equation of the tangent at P is obtained by taking limits (x_2, y_2) tends to as (x_1, y_1) on either sides of (I)

So, the equation of the tangent at P is given by $\lim_{Q \to P} (S_1 + S_2) = \lim_{Q \to P} S_{12}$

i.e., $S_1 + S_1 = S_{11} \left[\because S_2 \to S_1, S_{12} \to S_{11} \text{ as } (x_2, y_2) \to (x_1, y_1) \right]$

$\therefore 2S_1 = 0 \Rightarrow S_1 = 0.$

$\therefore$ The equation of the tangent to the hyperbola $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, at (x_1, y_1) is $S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

10.2.6 Theorem: The equation of normal at (x_1, y_1) on the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2, (x_1 \neq 0, y_1 \neq 0).$

Proof: By Theorem 10.2.5, the equation of tangent to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ at (x_1, y_1) is $S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ is $= \frac{-x_1 / a^2}{-y_1 / b^2} = \frac{b^2 x_1}{a^2 y_1}.$

$\therefore$ Slope of the normal at $P(x_1, y_1)$ is $= -\frac{a^2 y_1}{b^2 x_1}.$

Hence the equation of normal at (x_1, y_1) is $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$.

10.2.7 Note:

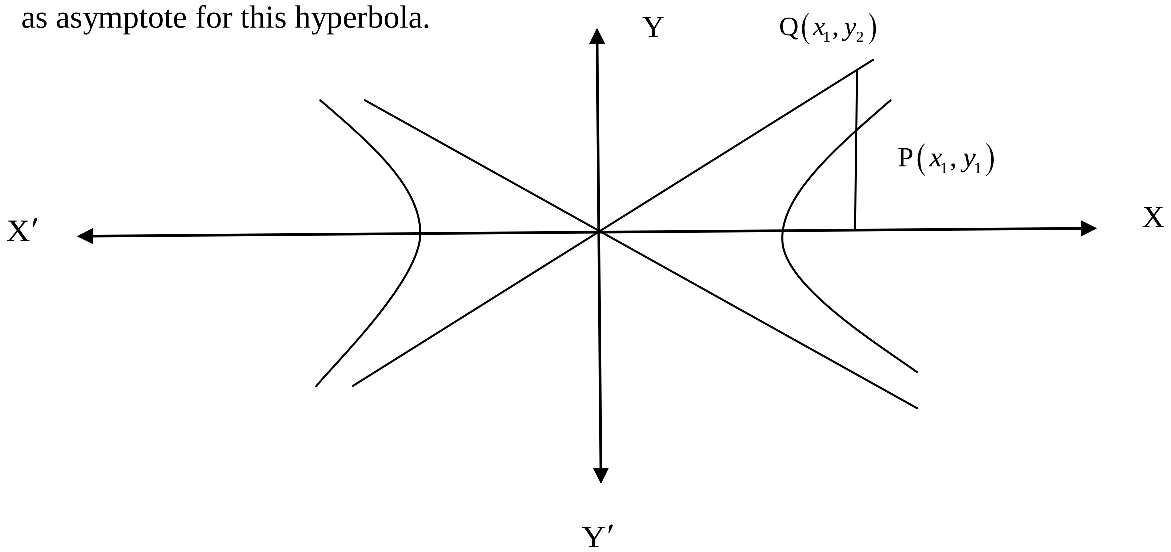
(i) If $x_1 = 0$ and $y_1 \neq 0$ then the equation of the normal at $\mathrm{P}(x_1, y_1) = (0, y_1) = (0, \pm b)$ is the y^- axis.

(ii) If $y_1 = 0$ and $x_1 \neq 0$ then the equation of the normal at $\mathrm{P}(x_1, y_1) = (x_1, 0) = (\pm a, 0)$ is the x^- axis.

10.2.8 Asymptotes of a curve: A non vertical line with equation $y = mx + c$ is called an asymptote of the graph of $y = f(x)$ if the difference $f(x) - (mx + c)$ is non zero and tend to 0 as $x \to \infty$ or $x \to -\infty$.

10.2.9 Asymptotes of the hyperbola $\mathrm{S} \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$**:** If $\mathrm{P}(x_1, y_1)$ is a point on the branch of the hyperbola and $\mathrm{Q}(x_1, y_2)$ is a point where ordinate through $\mathrm{P}(x_1, y_1)$ meets line $y = \dfrac{b}{a}x$, then $0 < (y_2 - y_1) = \dfrac{b}{a}\left(x_1 - \sqrt{x_1^2 - a^2}\right) = \dfrac{ab}{x_1 + \sqrt{x_1^2 - a^2}} \leq \dfrac{ab}{x_1}$ and $\dfrac{ab}{x_1} \to 0$ as $x_1 \to \infty$ therefore $y_2 - y_1 \to 0$ as $x_1 \to \infty$.

Therefore the line $y = \dfrac{b}{a}x$ is an asymptote of the hyperbola. By considering the portion of the curve $y = \dfrac{b}{a}\sqrt{x_1^2 - a^2}, x \leq a$ it can be similarly seen that $y = -\dfrac{b}{a}x$ is another as asymptote for this hyperbola.



10.2.10 Note:

(i) From the above equation of asymptotes, it is clear that they pass through the centre of the hyperbola and the axes of the hyperbola are the angle bisectors of the angle between the asymptotes.

(ii) Let $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be hyperbola. Then $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ is its conjugate hyperbola and asymptotes are $A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$

10.2.11 Solved Problems:

1. Problem: Find the centre, eccentricity, coordinates of foci, Length of latusrectum and equations of directrices of the following hyperbolas:

$(i)\ 9x^2 - 25y^2 = 225$ $(ii)\ 16x^2 - 9y^2 = 144$ $(iii) 9x^2 - 16y^2 + 36x + 32y - 124 = 0$

$(iv)\ x^2 - 3y^2 - 2x + 6y - 5 = 0$ $(v)\ x^2 - 2y^2 + 4x - 12y - 10 = 0$

$(vi)\ 4x^2 - y^2 + 8x - 2y + 7 = 0$ $(vii)\ 16x^2 - 9y^2 = -144$ $(viii)\ 4x^2 - y^2 = -4$

$(ix)\ 9(y+3)^2 - 4(x-2)^2 = 1$

Solution: (i) The given equation of hyperbola is $9x^2 - 25y^2 = 225 \Rightarrow \frac{x^2}{25} - \frac{y^2}{9} = 1$.......(I)

It is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.......(II)

Compare equations (I) and (II) we get $a^2 = 25, b^2 = 9 \Rightarrow a = 5, b = 3$

Centre $C = (0,0)$

eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{25}} = \sqrt{\frac{25+9}{25}} = \sqrt{\frac{34}{25}} = \frac{\sqrt{34}}{5}$

Focii $S = (\pm ae, 0) = \left(\pm 5\left(\frac{\sqrt{34}}{5}\right), 0 \right) = (\pm\sqrt{34}, 0)$

Length of latusrectum $= \frac{2b^2}{a} = \frac{2(4)^2}{5} = \frac{32}{5}$

Equation of directrices is $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{5}{\sqrt{34}/5} \Rightarrow x = \pm\frac{25}{\sqrt{34}} \Rightarrow \sqrt{34}x = \pm 25$

(ii) The given equation of hyperbola is $16x^2 - 9y^2 = 144 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$.......(I)

It is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.......(II)

Compare equations (I) and (II) we get $a^2=9, b^2=16 \Rightarrow a=3, b=4$

Centre $C=(0,0)$

eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{9+16}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$

Focii $S=(\pm ae,0)=\left(\pm 3\left(\frac{5}{3}\right),0\right)=(\pm 5,0)$

Length of latusrectum $=\frac{2b^2}{a}=\frac{2(4)^2}{3}=\frac{32}{3}$

Equation of directrices is $x=\pm\frac{a}{e}\Rightarrow x=\pm\frac{3}{5/3}\Rightarrow x=\pm\frac{9}{5}\Rightarrow 5x=\pm 9$

(iii) The given equation of hyperbola is $9x^2-16y^2+36x+32y-124=0$

$$\Rightarrow 9(x^2+4x)-16(y^2-2y)=124$$

$$\Rightarrow 9(x^2+2.x.2+2^2)-16(y^2-2.y.1+1^2)=124+9.2^2-16.1^2$$

$$\Rightarrow 9(x+2)^2-16(y-1)^2=144$$

$$\Rightarrow \frac{(x+2)^2}{16}-\frac{(y-1)^2}{9}=1.......(I)$$

It is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.......(II)

Compare equations (I) and (II) we get $h=-2, k=1$

$a^2=16, b^2=9\Rightarrow a=4, b=3$

Centre $C=(h,k)=(-2,1)$

eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{16+9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$

Focii $S=(h\pm ae,k)=\left(-2\pm 4\left(\frac{5}{4}\right),1\right)=(-2\pm 5,1)=(-2+5,1),(-2-5,1)=(3,1),(-7,1)$

Length of latusrectum $= \frac{2b^2}{a} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices is $x = h \pm \frac{a}{e} \Rightarrow x = -2 \pm \frac{4}{5/4} \Rightarrow x + 2 = \pm \frac{16}{5} \Rightarrow 5x + 10 = \pm 16$

$$\Rightarrow 5x - 6 = 0, 5x + 26 = 0$$

(iv) The given equation of hyperbola is $x^2 - 3y^2 - 2x + 6y - 5 = 0$

$$\Rightarrow (x^2 - 2x) - 3(y^2 - 2y) = 5$$

$$\Rightarrow (x^2 - 2.x.1 + 1^2) - 3(y^2 - 2.y.1 + 1^2) = 5 + 1^2 - 3.1^2$$

$$\Rightarrow (x-1)^2 - 3(y-1)^2 = 3$$

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-1)^2}{1} = 1.......(\text{I})$$

It is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.......(\text{II})$

Compare equations (I) and (II) we get $h = 1, k = 1$

$a^2 = 3, b^2 = 1 \Rightarrow a = \sqrt{3}, b = 1$

Centre $\text{C} = (h, k) = (1,1)$

eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{3+1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

Focii $\text{S} = (h \pm ae, k) = \left(1 \pm \sqrt{3}\left(\frac{2}{\sqrt{3}}\right), 1\right) = (1 \pm 2, 1) = (1+2, 1), (1-2, 1) = (3,1), (-1,1)$

Length of latusrectum $= \frac{2b^2}{a} = \frac{2(1)^2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

Equation of directrices is $x = h \pm \frac{a}{e} \Rightarrow x = 1 \pm \frac{\sqrt{3}}{2/\sqrt{3}} \Rightarrow x - 1 = \pm \frac{3}{2} \Rightarrow 2x - 2 = \pm 3$

$$\Rightarrow 2x - 2 = 3, 2x - 2 = -3 \Rightarrow 2x - 5 = 0, 2x + 1 = 0$$

(v) The given equation of hyperbola is $x^2 - 2y^2 + 4x - 12y - 10 = 0$

$$\Rightarrow (x^2 + 4x) - 2(y^2 + 6y) - 10 = 0$$

$$\Rightarrow (x^2+2.x.2+2^2)-2(y^2+2.y.3+3^2)-10=2^2-2.3^2$$

$$\Rightarrow (x+1)^2+2(y+3)^2=-4$$

$$\Rightarrow \frac{(x+1)^2}{4}-\frac{(y+3)^2}{2}=-1.......(\text{I})$$

It is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1.......(\text{II})$

Compare equations (I) and (II) we get $h=-1, k=-3$

$a^2=4, b^2=2 \Rightarrow a=2, b=\sqrt{2}$

Centre $\text{C}=(h,k)=(-1,-3)$

eccentricity $e=\sqrt{1+\frac{a^2}{b^2}}=\sqrt{1+\frac{4}{2}}=\sqrt{1+2}=\sqrt{3}$

Focii $\text{S}=(h,k\pm be)=\left(-1,-3\pm\sqrt{2}.\sqrt{3}\right)=\left(-1,-3\pm\sqrt{6}\right)$

Length of latusrectum $=\frac{2a^2}{b}=\frac{2(2)^2}{\sqrt{2}}=\frac{8}{\sqrt{2}}$

Equation of directrices is $y=k\pm\frac{b}{e}\Rightarrow y=-3\pm\frac{\sqrt{2}}{\sqrt{3}}\Rightarrow y+3=\pm\sqrt{\frac{2}{3}}$

(vi) The given equation of hyperbola is $4x^2-y^2+8x-2y+7=0$

$$\Rightarrow 4(x^2+2x)-(y^2+2y)+7=0$$

$$\Rightarrow 4(x^2+2.x.1+1^2)-(y^2+2.y.1+1^2)+7=4.1^2-1^2$$

$$\Rightarrow 4(x+1)^2-(y+1)^2=-4$$

$$\Rightarrow \frac{(x+1)^2}{1}-\frac{(y+1)^2}{4}=-1.......(\text{I})$$

It is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1.......(\text{II})$

Compare equations (I) and (II) we get $h=-1, k=-1$

$a^2=1, b^2=4 \Rightarrow a=1, b=2$

Centre $\text{C}=(h,k)=(-1,-1)$

eccentricity $e=\sqrt{1+\frac{a^2}{b^2}}=\sqrt{1+\frac{1}{4}}=\sqrt{\frac{4+1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

Focii $\mathrm{S}=(h,k\pm be)=\left(-1,-1\pm 2\left(\frac{\sqrt{5}}{2}\right)\right)=\left(-1,-1\pm\sqrt{5}\right)$

Length of latusrectum $=\frac{2a^2}{b}=\frac{2(1)^2}{2}=1$

Equation of directrices is $y=k\pm\frac{b}{e}\Rightarrow y=-1\pm\frac{2}{\sqrt{5}/2}\Rightarrow y+1=\pm 4/\sqrt{5}$

(vii) The given equation of hyperbola is $16x^2-9y^2=-144$

$$\Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=-1.......(\mathrm{I})$$

It is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.......(\mathrm{II})$

Compare equations (I) and (II) we get $a^2=9,b^2=16\Rightarrow a=3,b=4$

Centre $\mathrm{C}=(0,0)$

eccentricity $e=\sqrt{1+\frac{a^2}{b^2}}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{16+9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$

Focii $\mathrm{S}=(0,\pm be)=\left(0,\pm 4\left(\frac{5}{4}\right)\right)=\left(0,5\right)$

Length of latusrectum $=\frac{2a^2}{b}=\frac{2(3)^2}{4}=\frac{3}{2}$

Equation of directrices is $y=\pm\frac{b}{e}\Rightarrow y=\pm\frac{4}{5/4}\Rightarrow y=\pm 16/5\Rightarrow 5y=\pm 16$

$(viii)$ The given equation of hyperbola is $4x^2-y^2=-1$

$$\Rightarrow \frac{x^2}{1}-\frac{y^2}{4}=-1.......(\mathrm{I})$$

It is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.......(\mathrm{II})$

Compare equations (I) and (II) we get $a^2=1,b^2=4\Rightarrow a=1,b=2$

Centre $\mathrm{C} = (0,0)$

eccentricity $e = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{1}{4}} = \sqrt{\frac{4+1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

Focii $\mathrm{S} = (0, \pm be) = \left(0, \pm 2\left(\frac{\sqrt{5}}{2}\right)\right) = \left(0, \pm\sqrt{5}\right)$

Length of latusrectum $= \frac{2a^2}{b} = \frac{2(1)^2}{2} = 1$

Equation of directrices is $y = \pm\frac{b}{e} \Rightarrow y = \pm\frac{2}{\sqrt{5}/2} \Rightarrow y = \pm 4/\sqrt{5} \Rightarrow \sqrt{5}y = \pm 4$

(ix) The given equation of hyperbola is $9(y+3)^2 - 4(x-2)^2 = 1$

$$\Rightarrow \frac{(x-2)^2}{1/4} - \frac{(y+3)^2}{1/9} = -1 ....(\mathrm{I})$$

It is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1 ....(\mathrm{II})$

Compare equations (I) and (II) we get $h = 2, k = -3$

$a^2 = 1/4, b^2 = 1/9 \Rightarrow a = 1/2, b = 1/3$

Centre $\mathrm{C} = (h,k) = (2,-3)$

eccentricity $e = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{1/4}{1/9}} = \sqrt{\frac{9+4}{4}} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

Focii $\mathrm{S} = (h, k \pm be) = \left(2, -3 \pm \frac{1}{3}\left(\frac{\sqrt{13}}{2}\right)\right) = \left(2, -3 \pm \frac{\sqrt{13}}{6}\right)$

Length of latusrectum $= \frac{2a^2}{b} = \frac{2(1/2)^2}{1/3} = \frac{3}{2}$

Equation of directrices is $y = k \pm \frac{b}{e} \Rightarrow y = -3 \pm \frac{1/3}{\sqrt{13}/2} \Rightarrow y + 3 = \pm 2/3\sqrt{13}$

2. Problem: Find the equation of the hyperbola referred to its transverse and conjugate axes $x-, y-$axes respectively with latusrectum of length 4 and distance between foci $2\sqrt{3}$.

Solution: Given length of latusrectum is 4 $i.e., \frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$.......(I)

distance between foci is $i.e., 2ae = 2\sqrt{3} \Rightarrow ae = \sqrt{3}$

We have eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow a^2e^2 = a^2 + b^2 \Rightarrow (\sqrt{3})^2 = a^2 + 2a$

$\Rightarrow a^2 + 2a - 3 = 0 \Rightarrow a = -3, a = 1$

Since a need not be negative, therefore $a = 1$.

From equation (I), $\Rightarrow b^2 = 2.1 \Rightarrow b^2 = 2$

The required equation of hyperbola is $\frac{x^2}{1} - \frac{y^2}{2} = 1$

3. Problem: If the length of the transverse axis of a hyperbola is equal to the length of its conjugate axis then find the eccentricity of the hyperbola.

Solution: Length of the transverse axis of a hyperbola in standard form is $2a$

Length of the conjugate axis of a hyperbola in standard form is $2b$.

Given length of the transverse axis of a hyperbola is equal to the length of its conjugate axis

$i.e., 2a = 2b \Rightarrow a = b$

We have eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow e = \sqrt{1+\frac{b^2}{b^2}} \Rightarrow e = \sqrt{1+1} \Rightarrow e = \sqrt{2}$

4. Problem: If the length of the transverse axis of a hyperbola is twice the length of its conjugate axis then find the eccentricity of the hyperbola.

Solution: Length of the transverse axis of a hyperbola in standard form is $2a$

Length of the conjugate axis of a hyperbola in standard form is $2b$.

Given length of the transverse axis of a hyperbola is twice the length of its conjugate axis

$i.e., 2a = 2(2b) \Rightarrow a = 2b$

We have eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow e = \sqrt{1+\frac{b^2}{(2b)^2}} \Rightarrow e = \sqrt{1+\frac{1}{4}} \Rightarrow e = \sqrt{\frac{4+1}{4}}$

$\Rightarrow e = \sqrt{\frac{5}{4}} \quad \Rightarrow e = \frac{\sqrt{5}}{2}$

5. Problem: Find the equation of hyperbola in the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$ given the following data

(i) centre $(-1,2),$ one end of transverse axis is $(5,2), e=2.$

(ii) centre $(1,3),$ $e=\frac{3}{2},$ semi- transverse axis is $4.$

(iii) centre $(2,1),$ $e=2,$ length of latusrectum is $4.$

(iv) centre $(-3,1),$ one end of major axis is $(-1,1)$ and passes through $(-3,0).$

Solution: Let the required equation of hyperbola be $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(i) Given centre of the hyperbola is $(h,k)=(-1,2) \Rightarrow h=-1, k=2.$

One end of transverse axis is $(h+a,k)=(5,2) \Rightarrow h+a=5 \Rightarrow -1+a=5 \Rightarrow a=6.$

Also given that eccentricity $e=2.$

We have eccentricity $e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow e^2=1+\frac{b^2}{a^2} \Rightarrow a^2e^2=a^2+b^2 \Rightarrow b^2=a^2e^2-a^2$

$\Rightarrow b^2=6^2(2)^2-6^2=144-36=108$

Hence the required equation of hyperbola is $\frac{(x+1)^2}{36}-\frac{(y-2)^2}{108}=1$

(ii) Given centre of the hyperbola is $(h,k)=(1,3) \Rightarrow h=1, k=3.$

Semi transverse axis is $a=4.$

Also given that eccentricity $e=\frac{3}{2}.$

We have eccentricity $e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow e^2=1+\frac{b^2}{a^2} \Rightarrow a^2e^2=a^2+b^2 \Rightarrow b^2=a^2e^2-a^2$

$\Rightarrow b^2=4^2\left(\frac{3}{2}\right)^2-4^2=16\left(\frac{9}{4}\right)-16=36-16=20$

Hence the required equation of hyperbola is $\frac{(x-1)^2}{16}-\frac{(y-3)^2}{20}=1$

(iii) Given centre of the hyperbola is $(h,k)=(2,1) \Rightarrow h=2, k=1.$

length of latusrectum is 4 *i.e.,* $\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$

Also given that eccentricity $e = 2.$

We have eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}} \Rightarrow e^2 = 1 + \frac{b^2}{a^2} \Rightarrow a^2 e^2 = a^2 + b^2 \Rightarrow b^2 = a^2 e^2 - a^2$

$$\Rightarrow b^2 = a^2(e^2 - 1) \Rightarrow b^2 = a^2(2^2 - 1) \Rightarrow b^2 = 3a^2 \Rightarrow 2a = 3a^2 \Rightarrow a = \frac{2}{3}$$

$$b^2 = 2a \Rightarrow b^2 = 2\left(\frac{2}{3}\right) \Rightarrow b^2 = \frac{4}{3}$$

Hence the required equation of hyperbola is $\frac{(x-2)^2}{4/9} - \frac{(y-1)^2}{4/3} = 1$

$\Rightarrow 9(x-2)^2 - 3(y-1)^2 = 4$

(iv) Given centre of the hyperbola is $(h, k) = (-3, 1) \Rightarrow h = -3, k = 1.$

One end of transverse axis is $(h + a, k) = (-1, 1) \Rightarrow h + a = -1 \Rightarrow -3 + a = -1 \Rightarrow a = 2.$

The equation of hyperbola is $\frac{(x+3)^2}{4} - \frac{(y-1)^2}{b^2} = 1$

Since it passes through $(-7, 0)$, we have

$$\frac{(-7+3)^2}{4} - \frac{(0-1)^2}{b^2} = 1 \Rightarrow \frac{16}{4} - \frac{1}{b^2} = 1 \Rightarrow 4 - 1 = \frac{1}{b^2} \Rightarrow b^2 = \frac{1}{3}$$

Hence the required equation of hyperbola is $\frac{(x+3)^2}{4} - \frac{(y-1)^2}{1/3} = 1$

$\Rightarrow (x+3)^2 - 12(y-1)^2 = 4$

6. Problem: Find the equation of hyperbola whose focus is $(2, 1)$, eccentricity $\frac{4}{3}$ and directrix $2x - y + 3 = 0.$

Solution: Let $\mathrm{P}\left(x, y\right)$ be any point on the locus.

Given focus $\mathrm{S} = (2, 1)$, eccentricity $e = \frac{4}{3}$ and the equation of directrix is $\mathrm{M} \equiv 2x - y + 3 = 0.$

The equation of hyperbola having focus (x_1, y_1), eccentricity e and the directrix $ax+by+c=0$ is $(a^2+b^2)\left[(x-x_1)^2+(y-y_1)^2\right]=e^2(ax+by+c)^2$

The equation of hyperbola having focus $S=(2,1)$, eccentricity $e=\dfrac{4}{3}$ and the directrix $M \equiv 2x-y+3=0$ is $\left(2^2+(-1)^2\right)\left[(x-2)^2+(y-1)^2\right]=\left(\dfrac{4}{3}\right)^2(2x-y+3)^2$

$\Rightarrow 45\left[x^2-4x+4+y^2-2y+1\right]=16\left[4x^2+y^2+9-4xy+12x-6y\right]$

$\Rightarrow 45x^2-180x+45y^2-90y+225=64x^2+16y^2+144-64xy+192x-96y$

$\Rightarrow 19x^2-29y^2-64xy+372x-6y-81=0$ which is the required equation of hyperbola

7. Problem: Find the equations of the tangent and normal to the hyperbola $3x^2-4y^2=17$ at $(1,2)$.

Solution: The given equation of the hyperbola is $S \equiv 3x^2-4y^2=17....(I)$

Let $P(1,2)$ be any point on the hyperbola

Equation of tangent at $P(x_1, y_1)$ to the hyperbola $S \equiv 0$ is $S_1 \equiv 0$

i.e., $3xx_1-4yy_1-17=0 \Rightarrow 3x.1-4y.2-17=0$

$\Rightarrow 3x-8y-17=0....(II)$

Since equation of normal is perpendicular to the tangent and passing through $P(1,2)$

equation of the normal is $8(x-1)+3(y-2)=0$

$\Rightarrow 8x+3y-14=0$

8. Problem: Find the equations of the tangent and normal to the hyperbola $x^2-4y^2-8x-6y+1=0$ at $(-1,1)$.

Solution: The given equation of the hyperbola is $S \equiv x^2-4y^2-8x-6y+1=0....(I)$

Let $P(-1,1)$ be any point on the hyperbola

Equation of tangent at $P(x_1, y_1)$ to the hyperbola $S \equiv 0$ is $S_1 \equiv 0$

i.e., $xx_1-4yy_1-4(x+x_1)-3(y+y_1)+1=0 \Rightarrow x(-1)-4y.1-4(x-1)-3(y+1)+1=0$

$\Rightarrow 5x+7y-2=0....(II)$

Since equation of normal is perpendicular to the tangent and passing through $P(-1,1)$

equation of the normal is $7(x+1)-5(y-1)=0$

$$\Rightarrow 7x-5y+12=0$$

9. Problem: Find the value of k if the line $2x-y+k=0$ is a tangent to the hyperbola $x^2-2y^2=5$.

Solution: The given equation of the hyperbola is $S \equiv x^2-2y^2=5 \Rightarrow \frac{x^2}{5}-\frac{y^2}{5/2}=1$....(I)

The given line equation is $2x-y+k=0$....(II)

The condition that the line $lx+my+n=0$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

is $n^2=a^2l^2-b^2m^2$.

We have $a^2=5, b^2=5/2, l=2, m=-1, n=k$

$$k^2=5.2^2-\frac{5}{2}(-1)^2 \Rightarrow k^2=20-\frac{5}{2} \Rightarrow k^2=\frac{35}{2} \Rightarrow k=\pm\sqrt{\frac{35}{2}}$$

10. Problem: Find the equations of the tangent to the hyperbola $x^2-3y^2=4$ which are (i) parallel to the line $2x+3y+5=0$ and (ii) perpendicular to the line $3x+2y+4=0$.

Solution: The given equation of the hyperbola is $S \equiv x^2-3y^2=4 \Rightarrow \frac{x^2}{4}-\frac{y^2}{4/3}=1$....(I)

(i) The given line equation is $2x+3y+5=0$....(II)

The equation of a line parallel to the line (II) is $x+3y+k=0$....(III)

Since the line (III) is a tangent to the hyperbola (I).

We have $k^2=4.2^2-\frac{4}{3}.3^2 \Rightarrow k^2=16-12 \Rightarrow k^2=4 \Rightarrow k=\pm 2$

$\therefore$ The equation of the tangent to the hyperbola $x^2-3y^2=4$ which is parallel to the line $2x+3y+5=0$ is $2x+3y\pm 2=0$

(ii) The equation of a line perpendicular to the line $3x+2y+4=0$ is $2x-3y+k=0$....(IV)

Since the line (IV) is a tangent to the hyperbola (I).

We have $k^2 = 4.2^2 - \frac{4}{3}.(-3)^2 \Rightarrow k^2 = 16-12 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$

$\therefore$ The equation of the tangent to the hyperbola $x^2 - 3y^2 = 4$ which is perpendicular to the line $3x+2y+4=0$ is $2x-3y\pm 2=0$

11. Problem: Show that the angle between the asymptotes of a standard hyperbola is $2\tan^{-1}\left(\frac{b}{a}\right)$ *or* $2\sec^{-1}(e)$.

Solution: The asymptotes of a standard hyperbola are $y = \pm\frac{b}{a}x$

$$i.e., y = \frac{b}{a}x, y = -\frac{b}{a}x....(\text{I})$$

If θ be the angle between the lines $i.e., y = m_1x, y = m_2x$ then $\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1+m_1m_2}\right)$

$$i.e., \theta = \tan^{-1}\left(\frac{\left(\frac{b}{a}\right)-\left(-\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}\right) = \tan^{-1}\left(\frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

We have $\theta = 2\tan^{-1}\left(\frac{b}{a}\right) = 2\sec^{-1}\left(\sqrt{1+\left(\frac{b}{a}\right)^2}\right) = 2\sec^{-1}\left(\sqrt{1+\frac{b^2}{a^2}}\right) = 2\sec^{-1}(e)$

Exercise 10(a)

1. Define rectangular hyperbola and find its eccentricity.
2. If e and e' are the eccentricities of a hyperbola and its conjugate hyperbola, then prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1.$
3. One focus of a hyperbola is located at the point $(1,-3)$ and the corresponding directrix is the line $y = 2.$ Find the equation of the hyperbola if its eccentricity is $\frac{3}{2}.$
4. Find the equations of the hyperbola whose foci are $(5,0),(-5,0)$ the transverse axis is of length $8.$
5. Find the equation of the hyperbola, whose asymptotes are the straight lines $x+2y+3=0,\ 3x+4y+5=0$ and passing through the point $(1,-1).$
6. If $3x-4y+k=0$ is a tangent to $x^2-4y^2=5,$ find the value of $k.$

7. If the eccentricity of the hyperbola is $\frac{5}{4}$, then find eccentricity of the conjugate hyperbola.

8. Find the equation of the hyperbola whose asymptotes are $3x = \pm 5y$ and the vertices are $(5,0), (-5,0)$.

9. If the angle between the asymptotes is 30^0 then find its eccentricity.

10. Find the centre, eccentricity, coordinates of foci, Length of latusrectum and equations of directrices of the following hyperbolas:

(i) $16y^2 - 9x^2 = 144$ (ii) $x^2 - 4y^2 = 4$ (iii) $9x^2 - 16y^2 = 144$

(iv) $5x^2 - 4y^2 + 20x + 8y = 4$ (v) $9x^2 - 16y^2 + 72x - 32y - 16 = 0$

(vi) $4x^2 - 9y^2 - 8x - 32 = 0$ (vii) $4(y+3)^2 - 9(x-2)^2 = 1$

11. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are

(i) parallel (ii) perpendicular to the line $x + 2y = 0$.

12. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are

(i) parallel (ii) perpendicular to the line $y = x - 7$.

13. Find the equations of the tangents to the hyperbola $2x^2 - 3y^2 = 6$ through $(-2,1)$.

Key concepts

1. A conic with eccentricity greater than unity is called a *hyperbola*. Hence hyperbola is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio e, which is greater than unity. The fixed point is called the *focus* and the fixed line is called the *directrix* of the hyperbola.

2. Let $S(x_1, y_1)$ be the focus, e be the eccentricity and the directrix be $ax + by + c = 0$. The equation of the hyperbola is $(a^2 + b^2)\left[(x - x_1)^2 + (y - y_1)^2\right] = e^2(ax + by + c)^2$ and the equation of the axis of the above hyperbola is $b(x - x_1) - a(y - y_1) = 0$.

3. The equation of hyperbola in standard form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. For the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the focus is $S = (ae, 0)$, directrix is $x = \frac{a}{e}$ and axis is $y = 0$.

The point $A = (0,0)$ is called the vertex of the hyperbola.

4. The line joining two points of a hyperbola is called a *chord* of hyperbola.

5. A chord passing through focus is called a *focal chord*.

6. A chord through a point P on the hyperbola, which is perpendicular to the axis of the hyperbola, is called the *double ordinate* of the point P.

7. The double ordinate passing though the focus is called the focus is called the *latusrectum* of the hyperbola.

8. The length of the latusrectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$.

9. (i) If the equation of hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$ then Eccentricity is $e=\sqrt{1+\frac{b^2}{a^2}},$ Centre $C=(0,0),$ vertices are $A=(a,0), A'=(-a,0)$ Focii are $S=(ae,0), S'=(-ae,0)$ Equation of directrices are $x=\pm\frac{a}{e},$ Length of transverse axis is $AA'=2a$ and length of conjugate axis is $BB'=2b$

(ii) If the equation of hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$ then Eccentricity is $e=\sqrt{1+\frac{a^2}{b^2}},$ Centre $C=(0,0),$ vertices are $B=(0,b), B'=(0,-b)$ Focii are $S=(0,be), S'=(0,-be)$ Equation of directrices are $y=\pm\frac{b}{e},$ Length of transverse axis is $BB'=2b$ and length of conjugate axis is $AA'=2a$

(iii) If the equation of hyperbola is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$ then Eccentricity is $e=\sqrt{1+\frac{b^2}{a^2}},$ Centre $C=(h,k),$ vertices are $A=(h+a,k), A'=(h-a,k)$ Focii are $S=(h+ae,k), S'=(h-ae,k)$ Equation of directrices are $x=h\pm\frac{a}{e},$ Length of transverse axis is $AA'=2a$ and length of conjugate axis is $BB'=2b$

(iv) If the equation of hyperbola is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1,$ then Eccentricity is $e=\sqrt{1+\frac{a^2}{b^2}},$ Centre $C=(h,k),$ vertices are $B=(h,k+b), B'=(h,k-b)$ Focii are $S=(h,k+be), S'=(h,k-be)$ Equation of directrices are $y=k\pm\frac{b}{e},$ Length of transverse axis is $BB'=2b$ and length of conjugate axis is $AA'=2a$

10. The circle described on the major axis of a hyperbola as diameter is called *auxiliary circle* of the hyperbola. The equation of the auxiliary circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$.

11. The distance of a point on the hyperbola from its focus is called the *focal distance* of the point.

12. The following notation will be adapted.

(i) $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$ (ii) $S_1\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$ (iii) $S_{12}\equiv\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}-1$ (iv) $S_{11}\equiv\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

where (x_1,y_1) and (x_2,y_2) are the points in the plane of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

13. The condition that the point P lies outside, on or inside the hyperbola $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ according as $S_{11}>=<0$.

14. The condition for a straight line $y=mx+c$ to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$

15. The condition for a straight line $lx+my+n=0$ to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $n^2=a^2l^2-b^2m^2$

16. Two tangents can be drawn from an external point (x_1,y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

17. The equation of the chord joining two points (x_1,y_1) and (x_2,y_2) on $S=0$ is $S_1+S_2=S_{12}$.

18. The equation of tangent at (x_1,y_1) to the hyperbola $S=0$ is $S_1=0$ or the equation of the tangent to the hyperbola $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0,$ at (x_1,y_1) is $S_1\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$

19. The equation of normal at (x_1,y_1) on the hyperbola $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0,$ is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2.$$

20. If in a hyperbola the length of transverse axis $(2a)$ is equal to the length of conjugate axis $(2b)$, the hyperbola is called the rectangular hyperbola.

21. The equation of the rectangular hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2-y^2=a^2$.

21. The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

Answers
Exercise 10(a)

1. $\sqrt{2}$ 3. $4x^2-5y^2-8x+60y+4=0$ 4. $9x^2-16y^2=144$

5. $3x^2+10xy+8y^2+14x+22y+7=0$ 6. $k=\pm 5$ 7. $\frac{5}{3}$ 8. $9x^2-25y^2=225$

9. $\sqrt{6}-\sqrt{2}$

10. (i) centre:$(0,0)$, eccentricity:$5/3$, focii:$(0,\pm 5)$, length of latusrectum: 32/3, eq.of directrices:$5y\pm 9=0$

(ii) centre:$(0,0)$, eccentricity:$\sqrt{5}/2$, focii:$(\pm\sqrt{5},0)$, length of latusrectum: 1, eq.of directrices:$\sqrt{5}x\pm 4=0$

(iii) centre:$(0,0)$, eccentricity:$5/4$, focii:$(\pm 5,0)$, length of latusrectum: $9/2$, eq.of directrices:$5x\pm 16=0$

(iv) centre:$(-2,1)$, eccentricity:$3/2$, focii:$(1,1),(-5,1)$ length of latusrectum:5, eq.of directrices:$3x+2=0, 3x+10=0$

(v) centre:$(-4,-1)$, eccentricity:$5/4$, focii:$(1,-1),(-9,-1)$ length of latusrectum:$9/2$, eq.of directrices:$5x+4=0, 5x+36=0$

(vi) centre:$(1,0)$, eccentricity:$\sqrt{13}/3$, focii:$(1\pm\sqrt{13},0)$, length of latusrectum: $8/3$, eq.of directrices:$x=1\pm(9/\sqrt{13})$

(vii) centre:$(2,-3)$, eccentricity:$\sqrt{13}/3$, focii:$\left(2,-3\pm(\sqrt{13}/6)\right)$, length of latusrectum: $4/9$, eq.of directrices: $y=-3\pm(3/2\sqrt{13})$

11. (i) no parallel tangents (ii) $2x-y\pm\sqrt{15}=0.$

12. (i) $y=x\pm 1$ (ii) $x+y=\pm 1$ 13. $x+y+1=0, 3x+y+5=0$

11. INDEFINITE INTEGRATION

Introduction:

We have learnt the concept of differentiation in the first year of the intermediate course. If a function f is differentiable in an interval I, *i.e.,* the derivative of f namely f' exists at each point of I, then the following question arises naturally: "given f' on I, can we determine f ? ". In this chapter, we answer this question by introducing the concept of integration as the inverse process of differentiation. Also we discuss standard forms and properties of integrals.

Throughout this chapter, R denotes the set of all real numbers and I, an interval in R. Unless otherwise stated, all the functions consider here are real valued functions defined over subsets of R.

11.1 Definitionof Integration:

We begin with the definition of an indefinite integral of a function and then state the standard forms of integrals for certain functions.

11.1.1 Definition: Let E be a subset of R such that E contains a right or left neighbourhood of each of its points and let $f : E \to R$ be a function. If there is a function F on E such that $F'(x) = f(x)$ for all $x \in E,$ then we call F an *anti-derivative* of f or a *primitive* of f

For example, we know that $\frac{d}{dx}(\sin x) = \cos x, x \in R.$ Hence, if f is a function given by $f(x) = \cos x, x \in R,$ then the function F given by $F(x) = \sin x, x \in R,$ is an *anti-derivative* of f or a *primitive* of f

11.1.2 Definition: Let $f : I \to R.$ Suppose that f has an anti-derivative F on $I.$ Then we say that f has an integral on I for any real number $c,$ we call $F + c$ is an *indefinite integral* of f over $I,$ denote it by $\int f(x)dx$ and read it as integral $f(x)dx.$ Thus we have $\int f(x)dx = F(x) + c.$ Here c is called a '*constant of integration'*.

In the indefinite integral $\int f(x)dx, f$ is called the *'integrand'* and x is called the *'variable of integration'.*

11.2 Integration of different types of functions:

In this section we discuss the integration for different functions

11.2.1 Standard forms: In the I year intermediate course, we have studied the derivatives of some functions. With this background, let us obtain indefinite integrals of some functions.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

2. $\int \frac{1}{x} dx = \log|x| + c$

3. $\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + c$

4. $\int e^x dx = e^x + c$

5. $\int a^x dx = a^x \log a + c$

6. $\int \sin x dx = -\cos x + c$

7. $\int \cos x dx = \sin x + c$

8. $\int \sec^2 (ax+b) dx = \frac{1}{a} \tan(ax+b) + c$

9. $\int \text{cosec}^2 (ax+b) dx = -\frac{1}{a} \cot(ax+b) + c$

10. $\int \sec(ax+b) \tan(ax+b) dx = \frac{1}{a} \sec(ax+b) + c$

11. $\int \text{cosec}(ax+b) \cot(ax+b) dx = -\frac{1}{a} \text{cosec}(ax+b) + c$

12. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c \ (or) \ -\cos^{-1} x + c$

13. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \ (or) \ -\cot^{-1} x + c$

14. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c \ (or) \ -\text{cosec}^{-1} x + c$

15. $\int \sinh x dx = \cosh x + c$

16. $\int \cosh x dx = \sinh x + c$

17. $\int \text{sech}^2 x dx = \tanh x + c$

18. $\int \operatorname{cosec} h^2 x dx = -\coth x + c$

19. $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + c$

20. $\int \operatorname{cosec} hx \coth x dx = -\operatorname{cosec} hx + c$

21. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c$

22. $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c$

23. $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + c \; (or) \cot^{-1} x + c$

We shall now write the algebraic properties of the definite integrals. We state with out proof, Theorem 11.2.2 and 11.2.3 which will be discussed in the subsequent development of theory and in solving problems.

11.2.2 Theorem: If the functions f and g have integrals on I, then $f+g$ has an integral on I and $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx + c$, where c is a constant.

11.2.3 Theorem: If f has an integral on I and a is a real number then, af also has an integral on I and $\int (af)(x)dx = a\int f(x)dx + c$, where c is a constant.

11.2.4 Remark: From Theorems 11.2.2 and 11.2.3, we can easily write the following statements in which c is a constant.

(i) If f and g have integrals on I, then $f-g$ has an integral on I and $\int (f-g)(x)dx = \int f(x)dx - \int g(x)dx + c$.

(ii) If $f_1, f_2, f_3, \ldots f_n$ have integrals on I, then $f_1 + f_2 + f_3 + \ldots + f_n$ has an integral on I and $\int (f_1 + f_2 + f_3 + \ldots + f_n)(x)dx = \int f_1(x)dx + \int f_2(x)dx + \int f_3(x)dx + \ldots + \int f_n(x)dx + c$.

(ii) If $f_1, f_2, f_3, \ldots f_n$ have integrals on I and $k_1, k_2, k_3, \ldots k_n$ are constants, then $k_1 f_1 + k_2 f_2 + k_3 f_3 + \ldots + k_n f_n$ has an integral on I and

$$\int (k_1 f_1 + k_2 f_2 + k_3 f_3 + \ldots + k_n f_n)(x)dx$$
$$= k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + k_3 \int f_3(x)dx + \ldots + k_n \int f_n(x)dx + c.$$

11.2.5 Solved Problems:

1. Problem: Evaluate $\int (e^x - \sin x + x^4)dx$

Solution: Let $\text{I} = \int (e^x - \sin x + x^4)dx$

$$= \int e^x dx - \int \sin x dx + \int x^4 dx$$

$$= e^x - (-\cos x) + \frac{x^{4+1}}{4+1}$$

$$= e^x + \cos x + \frac{x^5}{5} + c$$

2. Problem: Evaluate $\int (\frac{1}{x} - e^x + x^4)dx$

Solution: Let $\text{I} = \int (\frac{1}{x} - e^x + x^4)dx$

$$= \int \frac{1}{x} dx - \int e^x dx + \int x^4 dx$$

$$= \log x - e^x + \frac{x^{4+1}}{4+1}$$

$$= \log x - e^x + \frac{x^5}{5} + c$$

3. Problem: Evaluate $\int (\frac{1}{2\sqrt{x}} + 7\sec^2 x + \frac{1}{x})dx$

Solution: Let $\text{I} = \int (\frac{1}{2\sqrt{x}} + 7\sec^2 x + \frac{1}{x})dx$

$$= \int \frac{1}{2\sqrt{x}} dx + 7\int \sec^2 x\, dx + \int \frac{1}{x} dx$$

$$= \sqrt{x} + 7\tan x + \log x + c$$

4. Problem: Evaluate $\int (e^x + 2\sin x + \frac{6}{\sqrt{1-x^2}})dx$

Solution: Let $\text{I} = \int (e^x + 2\sin x + \frac{6}{\sqrt{1-x^2}})dx$

$$= \int e^x\, dx + 2\int \sin x dx + 6\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= e^x + 2(-\cos x) + 6\sin^{-1} x = e^x - 2\cos x + 6\sin^{-1} x + c$$

5. Problem: Evaluate $\int (x^5 + 5^x + 5x)dx$

Solution: Let $\mathrm{I} = \int (x^5 + 5^x + 5x)dx$

$$= \int x^5 dx + \int 5^x\, dx + 5\int x dx$$

$$= \frac{x^{5+1}}{5+1} + \frac{5^x}{\log 5} + 5\frac{x^{1+1}}{1+1} = \frac{x^6}{6} + \frac{5^x}{\log 5} + \frac{5x^2}{2} + c$$

6. Problem: Evaluate $\int (x^2 + \frac{1}{x^2})dx$

Solution: Let $\mathrm{I} = \int (x^2 + \frac{1}{x^2})dx$

$$= \int x^2 dx + \int x^{-2} dx$$

$$= \frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} = \frac{x^3}{3} - \frac{1}{x} + c$$

7. Problem: Evaluate $\int \left(x + \frac{1}{x} \right)^2 dx$

Solution: Let $\mathrm{I} = \int \left(x + \frac{1}{x} \right)^2 dx \; = \int (x^2 + \frac{1}{x^2} + 2)dx$

$$= \int x^2 dx + \int x^{-2} dx + 2\int dx$$

$$= \frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} + 2x = \frac{x^3}{3} - \frac{1}{x} + 2x + c$$

8. Problem: Evaluate $\int \sqrt{1 + \sin 2x} dx$

Solution: Let $\mathrm{I} = \int \sqrt{1 + \sin 2x} dx \; = \int \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x} dx$

$$= \int \sqrt{(\cos x + \sin x)^2} dx = \int (\cos x + \sin x) dx = \int \cos x dx + \int \sin x dx$$

$$= \sin x - \cos x + c$$

9. Problem: Evaluate $\int \sqrt{1 + \cos 2x} dx$

Solution: Let $\mathrm{I} = \int \sqrt{1 + \cos 2x} dx \; = \int \sqrt{1 + 2\cos^2 x - 1} dx$

$$= \int \sqrt{2\cos^2 x} dx = \sqrt{2} \int \cos x dx = \sqrt{2} \sin x + c$$

10. Problem: Evaluate $\int \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}dx$

Solution: Let $I = \int \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}dx = \int \sqrt{\frac{1-(1-2\sin^2 x)}{1+2\cos^2 x-1}}dx = \int \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}dx = \int \sqrt{\frac{\sin^2 x}{\cos^2 x}}dx$

$$= \int \sqrt{\tan^2 x}dx = \int \tan x\, dx = -\log(\cos x) + c$$

11. Problem: Evaluate $\int \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}dx$

Solution: Let $I = \int \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}dx = \int \frac{\cos x - \sin x}{\sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}}dx$

$$= \int \frac{\cos x - \sin x}{\sqrt{(\cos x - \sin x)^2}}dx = \int \frac{\cos x - \sin x}{\cos x - \sin x}dx$$

$$= \int 1 dx = x + c$$

12. Problem: Evaluate $\int \frac{1}{1+\sin x}dx$

Solution: Let $I = \int \frac{1}{1+\sin x}dx = \int \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right)dx = \int \frac{1-\sin x}{1-\sin^2 x}dx$

$$= \int \frac{1-\sin x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx = \int \sec^2 x dx - \int \sec x \tan x dx$$

$$= \tan x - \sec x + c$$

13. Problem: Evaluate $\int \frac{1}{1-\cos x}dx$

Solution: Let $I = \int \frac{1}{1-\cos x}dx = \int \left(\frac{1}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)dx = \int \frac{1+\cos x}{1-\cos^2 x}dx$

$$= \int \frac{1+\cos x}{\sin^2 x}dx = \int \frac{1}{\sin^2 x}dx + \int \frac{\cos x}{\sin^2 x}dx = \int \operatorname{cosec}^2 x dx + \int \operatorname{cosec} x \cot x dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

14. Problem: Evaluate $\int \frac{1-\cos 2x}{1+\cos 2x}dx$

Solution: Let $I = \int \frac{1-\cos 2x}{1+\cos 2x}dx = \int \frac{1-(1-2\sin^2 x)}{1+2\cos^2 x-1}dx = \int \frac{2\sin^2 x}{2\cos^2 x}dx = \int \frac{\sin^2 x}{\cos^2 x}dx$

$$= \int \tan^2 x dx = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int 1 dx = \tan x - x + c$$

15. Problem: Evaluate $\int \frac{1}{\sin^2 x\cos^2 x}dx$

Solution: Let $I=\int \frac{1}{\sin^2 x\cos^2 x}dx \;=\int \frac{\cos^2 x+\sin^2 x}{\sin^2 x\cos^2 x}dx$

$$=\int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx+\int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx=\int \frac{1}{\sin^2 x}dx+\int \frac{1}{\cos^2 x}dx$$

$$=\int \operatorname{cosec}^2 x dx+\int \sec^2 x dx=-\cot x+\tan x+c$$

16. Problem: Evaluate $\int \frac{1}{\sin x\cos^2 x}dx$

Solution: Let $I=\int \frac{1}{\sin x\cos^2 x}dx \;=\int \frac{\cos^2 x+\sin^2 x}{\sin x\cos^2 x}dx$

$$=\int \frac{\cos^2 x}{\sin x\cos^2 x}dx+\int \frac{\sin^2 x}{\sin x\cos^2 x}dx=\int \frac{1}{\sin x}dx+\int \frac{\sin x}{\cos^2 x}dx$$

$$=\int \operatorname{cosec} x dx+\int \sec x\tan x dx=\log(\operatorname{cosec} x-\cot x)+\sec x+c$$

Exercise 11(a)

I Evaluate the following integrals:

$(i)\int \frac{x^2+2x+2}{x^4}dx$ $(ii)\int (x^3+3^x+2)dx$ $(iii)\int (x^5+5^x+5x)dx$ $(iv)\int (x^3+7^x+11x)dx$

$(v)\int (\cos x-\sin x)dx$ $(vi)\int (\sec^2 x-e^x+\sin x)dx$ $(vii)\int (e^{-x}+a^x+\frac{1}{x}+3)dx$

$(viii)\int (4\sec^2 x-2e^x+3\sin x)dx$ $(ix)\int (x^3+4^x+4x)dx$ $(x)\int (x^9+9^x+9x)dx$

$(xi)\int (x^2+2x+3)dx$ $(xii)\int (x^5+3\cos x-\frac{4}{x})dx$ $(xiii)\int (x^7-\frac{3}{x}+\sin x)dx$

$(xiv)\int \sec^2 x\operatorname{cosec}^2 x\,dx$ $(xv)\int \frac{1}{\sin x\cos^2 x}dx$ $(xvi)\int \frac{1}{\sin^2 x\cos x}dx$ $(xvii)\int (x^2+2x+3)dx$

$(xviii)\int \frac{1}{1+\cos x}dx$ $(xix)\int \frac{1}{1-\sin x}dx$ $(xx)\int (x+\frac{1}{x})^3dx$ $(xxi)\int \frac{\cos x+\sin x}{\sqrt{1+\sin 2x}}dx$

$(xxii)\int \sqrt{1-\sin 2x}dx$ $(xxiii)\int \sqrt{1-\cos 2x}dx$ $(xxiv)\int \sqrt{\frac{1+\cos 2x}{1-\cos 2x}}dx$ $(xxv)\int \frac{1+\cos 2x}{1-\cos 2x}dx$

11.3 Integration of methods of substitution:

In this section, we reduce certain integrals to some standard forms by using a suitable substitution. Here we discuss mainly the integration of algebraic, trigonometric functions and simple forms of exponential functions and some functions which are combination of these forms.

We state without proof, Theorem 11.3.1, 11.3.2 and 11.3.4 which will be discussed in the subsequent development of theory and in solving problems.

11.3.1 Theorem: Let $f : I \to R$ have an interval on I and F be a primitive of f on I. Let J be an interval in R and $g : J \to I$ be a differentiable function. Then $(f \circ g)g'$ has an integral on J and $\int f(g(x))g'(x)dx = F(g(x)) + c$

$$i.e., \quad \int f(g(x))g'(x)dx = \left[\int f(t)\,dt\right]_{t=g(x)} = F(g(x)) + c$$

11.3.2 Corollary: Let $f : I \to R$ have an interval on I and F be a primitive of f on I. Let $a, b \in R$ with $a \neq 0$. Then $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c$ for all $x \in J$, where

$J = \{x \in R : ax + b \in I\}$ and c is an integral constant.

11.3.3 Some important formulae:

The following formulae can be obtained by using some of the standard integrals given in 11...... and corollary 11.3.2.

1. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c, n \neq -1$

2. $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c$

3. $\int \frac{1}{2\sqrt{ax+b}}dx = \frac{1}{a}\sqrt{ax+b} + c$

4. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

5. $\int a^{mx+n}dx = \frac{1}{m}a^{mx+n}\log a + c$

6. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

7. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

8. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$

9. $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

10. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c$

11. $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

12. $\int \frac{1}{\sqrt{1-(ax+b)^2}}dx = \frac{1}{a}\sin^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\cos^{-1}(ax+b)+c$

13. $\int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}\tan^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\cot^{-1}(ax+b)+c$

14. $\int \frac{1}{|ax+b|\sqrt{(ax+b)^2-1}}dx = \frac{1}{a}\sec^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\operatorname{cosec}^{-1}(ax+b)+c$

15. $\int \sinh(ax+b)dx = \frac{1}{a}\cosh(ax+b)+c$

16. $\int \cosh(ax+b)dx = \frac{1}{a}\sinh(ax+b)+c$

17. $\int \operatorname{sech}^2(ax+b)dx = \frac{1}{a}\tanh(ax+b)+c$

18. $\int \operatorname{cosec} h^2(ax+b)dx = -\frac{1}{a}\coth(ax+b)+c$

19. $\int \operatorname{sech}(ax+b)\tanh(ax+b)dx = -\frac{1}{a}\operatorname{sech}(ax+b)+c$

20. $\int \operatorname{cosec}\mathrm{h}(ax+b)\coth(ax+b)dx = -\frac{1}{a}\operatorname{cosec}\mathrm{h}(ax+b)+c$

21. $\int \frac{1}{\sqrt{1+(ax+b)^2}}dx = \sinh^{-1}(ax+b)+c$

22. $\int \frac{1}{\sqrt{(ax+b)^2-1}}dx = \cosh^{-1}(ax+b)+c$

23. $\int \frac{1}{1-(ax+b)^2}dx = \frac{1}{a}\tanh^{-1}(ax+b)+c \ (or) \frac{1}{a}\cot^{-1}(ax+b)+c$

Let us now write integrals of functions of particular form by using the method of substitution.

11.3.4 Theorem: Let $f: I \to R$ be a differentiable function. Then the following statements are true.

(i) If f is never zero on I then $\frac{f'}{f}$ has an integral on I and $\int \frac{f'}{f}dx = \log|f(x)|+c$ on I

(ii) $\int (f(x))^{\alpha} f'(x)dx = \frac{(f(x))^{\alpha+1}}{\alpha+1}+c, \alpha \neq -1$

(iii) $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$

(iv) $\int f'(ax+b)dx = \frac{1}{a}f(ax+b)+c, a \neq 0$

11.3.5 Solved Problems:

1. Problem: Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Solution: Let $\text{I} = \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

$\therefore \text{I} = \int \sin t.2dt = 2\int \sin t dt = 2(-\cos t) = -2\cos t = -2\cos\sqrt{x}+c$

2. Problem: Evaluate $\int \frac{\cos(\log x)}{x}dx$

Solution: Let $\text{I} = \int \frac{\cos(\log x)}{x}dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \text{I} = \int \cos t\, dt = \sin t = \sin(\log x)+c$

3. Problem: Evaluate $\int \frac{1}{x(\log x)^4} dx$

Solution: Let $I = \int \frac{1}{x(\log x)^4} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore I = \int \frac{1}{t^4} dt = \int t^{-4} dt = \frac{t^{-4+1}}{-4+1} = \frac{t^{-3}}{-3} = \frac{-1}{3t^3} = \frac{-1}{3(\log x)^3} + c$$

4. Problem: Evaluate $\int \frac{(1+\log x)^5}{x} dx$

Solution: Let $I = \int \frac{(1+\log x)^5}{x} dx$

Put $1+\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore I = \int t^5 dt = \frac{t^{5+1}}{5+1} = \frac{t^6}{6} = \frac{(1+\log x)^6}{6} + c$$

5. Problem: Evaluate $\int \frac{(\sin^{-1} x)^4}{\sqrt{1-x^2}} dx$

Solution: Let $I = \int \frac{(\sin^{-1} x)^4}{\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$$\therefore I = \int t^4 dt = \frac{t^{4+1}}{4+1} = \frac{t^5}{5} = \frac{(\sin^{-1} x)^5}{5} + c$$

6. Problem: Evaluate $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

Solution: Let $I = \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\therefore I = \int e^{mt} dt = \frac{e^{mt}}{m} = \frac{e^{m \tan^{-1} x}}{m} + c$$

7. Problem: Evaluate $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

Solution: Let $I = \int \frac{\sec^2 x}{(1+\tan x)^3} dx$

Put $1+\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int \frac{1}{t^3} dt = \int t^{-3} dt = \frac{t^{-3+1}}{-3+1} = \frac{t^{-2}}{-2} = \frac{-1}{2t^2} = \frac{-1}{2(1+\tan x)^2} + c$$

8. Problem: Evaluate $\int 2x \sin(x^2+1)\, dx$

Solution: Let $I = \int 2x \sin(x^2+1)\, dx$

Put $x^2+1 = t \Rightarrow 2x dx = dt$

$$\therefore I = \int \sin t dt = -\cos t = -\cos(x^2+1) + c$$

9. Problem: Evaluate $\int \frac{2x+1}{x^2+x+1} dx$

Solution: Let $I = \int \frac{2x+1}{x^2+x+1} dx$

Put $x^2+x+1 = t \Rightarrow (2x+1)dx = dt$

$$\therefore I = \int \frac{1}{t} dt = \log t = \log(x^2+x+1) + c$$

10. Problem: Evaluate $\int \frac{3x^2}{x^6+1} dx$

Solution: Let $I = \int \frac{3x^2}{x^6+1} dx = \int \frac{3x^2}{(x^3)^2+1} dx$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore I = \int \frac{1}{1+t^2} dt = \tan^{-1} t = \tan^{-1}(x^3) + c$$

11. Problem: Evaluate $\int \frac{x}{\sqrt{x^4-1}} dx$

Solution: Let $I = \int \frac{x}{\sqrt{x^4-1}} dx = \int \frac{x}{\sqrt{(x^2)^2-1}} dx$

Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$

$$\therefore \text{I} = \int \frac{1}{\sqrt{t^2-1}} \frac{1}{2} dt = \frac{1}{2}\int \frac{1}{\sqrt{t^2-1}} dt = \frac{1}{2}\cosh^{-1} t = \frac{1}{2}\cosh^{-1}(x^2) + c$$

12. Problem: Evaluate $\int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$

Solution: Let $\text{I} = \int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$

Put $xe^x = t \Rightarrow (x+1)e^x dx = dt$

$$\therefore \text{I} = \int \frac{1}{\cos^2 t} dt = \int \sec^2 t dt = \tan t = \tan(xe^x) + c$$

13. Problem: Evaluate $\int \frac{12x+22}{6x^2+22x+1} dx$

Solution: Let $\text{I} = \int \frac{12x+22}{6x^2+22x+1} dx$

Put $6x^2+22x+1 = t \Rightarrow (12x+22)dx = dt$

$$\therefore \text{I} = \int \frac{1}{t} dt = \log t = \log(6x^2+22x+1) + c$$

14. Problem: Evaluate $\int \sin^2 x \, dx$

Solution: Let $\text{I} = \int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$$= \frac{1}{2}\left(\int 1 dx - \int \cos 2x \, dx\right) = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

15. Problem: Evaluate $\int \sin^3 x \, dx$

Solution: Let $\text{I} = \int \sin^3 x \, dx = \int \frac{3\sin x + \sin 3x}{4} dx \left[\because \sin^3\theta = \frac{3\sin\theta + \sin 3\theta}{4}\right]$

$$= \frac{1}{4}\left(3\int \sin x \, dx + \int \sin 3x \, dx\right) = \frac{1}{4}\left[3(-\cos x) + \left(-\frac{\cos 3x}{3}\right)\right] + c$$

$$= \frac{1}{4}\left[-3\cos x - \frac{\cos 3x}{3}\right] + c$$

16. Problem: Evaluate $\int \sin^4 x \, dx$

Solution: Let $\text{I} = \int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 2(2x)}{2}\right)dx \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{8}\int \left(3 - 4\cos 2x + \cos 4x\right)dx$$

$$= \frac{1}{8}\left[3x - 4\left(\frac{\sin 2x}{2}\right) + \left(\frac{\sin 4x}{4}\right)\right]$$

$$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{\sin 4x}{4}\right] + c$$

17. Problem: Evaluate $\int \sin 8x \cos 3x \, dx$

Solution: Let $\text{I} = \int \sin 8x \cos 3x \, dx$

We have $2\sin \text{A} \cos \text{B} = \sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})$

$$\Rightarrow \sin \text{A} \cos \text{B} = \frac{1}{2}\left(\sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})\right)$$

Now $\text{I} = \int \sin 8x \cos 3x \, dx = \frac{1}{2}\int \left(\sin(8x+3x) + \sin(8x-3x)\right)dx$

$$= \frac{1}{2}\int \left(\sin 11x + \sin 5x\right)dx = \frac{1}{2}\left(\int \sin 11x \, dx + \int \sin 5x \, dx\right)$$

$$= \frac{1}{2}\left[\left(-\frac{\cos 11x}{11}\right) + \left(-\frac{\cos 5x}{5}\right)\right] = -\frac{1}{2}\left[\frac{\cos 11x}{11} + \frac{\cos 5x}{5}\right] + c$$

18. Problem: Evaluate $\int \cos 5x \sin 3x \, dx$

Solution: Let $\text{I} = \int \cos 5x \sin 3x \, dx$

We have $2\cos \text{A} \sin \text{B} = \sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B})$

$$\Rightarrow \cos \text{A} \sin \text{B} = \frac{1}{2}\left(\sin(\text{A}+\text{B}) - \sin(\text{A}-\text{B})\right)$$

Now $\text{I} = \int \cos 5x \sin 3x \, dx = \frac{1}{2}\int \left(\sin(5x+3x) - \sin(5x-3x)\right)dx$

$$= \frac{1}{2}\int \left(\sin 8x - \sin 2x\right)dx = \frac{1}{2}\left(\int \sin 8x \, dx - \int \sin 2x \, dx\right)$$

$$= \frac{1}{2}\left[\left(-\frac{\cos 8x}{8}\right) - \left(-\frac{\cos 2x}{2}\right)\right] = -\frac{1}{4}\left[\frac{\cos 8x}{4} - \cos 2x\right] + c$$

19. Problem: Evaluate $\int \cos 7x \cos 2x \, dx$

Solution: Let $\mathrm{I} = \int \cos 7x \cos 2x \, dx$

We have $2\cos \mathrm{A} \cos \mathrm{B} = \cos(\mathrm{A}+\mathrm{B}) + \cos(\mathrm{A}-\mathrm{B})$

$$\Rightarrow \cos \mathrm{A} \cos \mathrm{B} = \frac{1}{2}\left(\cos(\mathrm{A}+\mathrm{B}) + \cos(\mathrm{A}-\mathrm{B})\right)$$

Now $\mathrm{I} = \int \cos 7x \cos 2x \, dx = \frac{1}{2}\int \left(\cos(7x+2x) + \cos(7x-2x)\right) dx$

$$= \frac{1}{2}\int \left(\cos 9x + \cos 5x\right) dx = \frac{1}{2}\left(\int \cos 9x \, dx + \int \cos 5x \, dx\right)$$

$$= \frac{1}{2}\left[\frac{\sin 9x}{9} + \frac{\sin 5x}{5}\right] + c$$

20. Problem: Evaluate $\int \sin 7x \sin 3x \, dx$

Solution: Let $\mathrm{I} = \int \sin 7x \sin 3x \, dx$

We have $2\sin \mathrm{A} \sin \mathrm{B} = \cos(\mathrm{A}-\mathrm{B}) - \cos(\mathrm{A}+\mathrm{B})$

$$\Rightarrow \sin \mathrm{A} \sin \mathrm{B} = \frac{1}{2}\left(\cos(\mathrm{A}-\mathrm{B}) - \cos(\mathrm{A}+\mathrm{B})\right)$$

Now $\mathrm{I} = \int \sin 7x \sin 3x \, dx = \frac{1}{2}\int \left(\cos(7x-3x) - \cos(7x+3x)\right) dx$

$$= \frac{1}{2}\int \left(\cos 4x - \cos 10x\right) dx = \frac{1}{2}\left(\int \cos 4x \, dx - \int \cos 10x \, dx\right)$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 10x}{10}\right] + c$$

21. Problem: Evaluate $\int \sin^4 x \cos^3 x \, dx$

Solution: Let $\mathrm{I} = \int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx = \int \sin^4 x (1-\sin^2 x) \cos x \, dx$

$$= \int (\sin^4 x - \sin^6 x) \cos x \, dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

Now $I=\int (t^4-t^6)\,dt=\frac{t^5}{5}-\frac{t^7}{7}=\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c$

22. Problem: Evaluate $\int \sin^3 x\cos^4 x\,dx$

Solution: Let $I=\int \sin^3 x\cos^4 x\,dx\ =\int \sin^2 x\cos^4 x\sin x\,dx\ =\int \cos^4 x(1-\cos^2 x)\sin x\,dx$

$$=\int (\cos^4 x-\cos^6 x)\sin x\,dx$$

Put $\cos x=t\Rightarrow -\sin xdx=dt\Rightarrow \sin xdx=-dt$

Now $I=\int (t^4-t^6)(-dt)=\int (t^6-t^4)\,dt=\frac{t^7}{7}-\frac{t^5}{5}=\frac{\cos^7 x}{7}-\frac{\cos^5 x}{5}+c$

23. Problem: Evaluate $\int \sin^2 x\cos^2 x\,dx$

Solution: Let $I=\int \sin^2 x\cos^2 x\,dx\ =\frac{1}{4}\int 4\sin^2 x\cos^2 x\,dx\ =\frac{1}{4}\int (2\sin x\cos x)^2\,dx$

$$=\frac{1}{4}\int (\sin 2x)^2\,dx\ =\frac{1}{4}\int \sin^2 2xdx\ =\frac{1}{4}\int \frac{1-\cos 2(2x)}{2}\,dx\left[\because \sin^2\theta=\frac{1-\cos 2\theta}{2}\right]$$

$$=\frac{1}{8}\left(\int 1\,dx-\int \cos 4x\,dx\right)=\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+c$$

Now $I=\int (t^4-t^6)(-dt)=\int (t^6-t^4)\,dt=\frac{t^7}{7}-\frac{t^5}{5}=\frac{\cos^7 x}{7}-\frac{\cos^5 x}{5}+c$

Exercise 11(b)

I Evaluate the following integrals:

$(i)\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ $\quad (ii)\int \frac{\sin(\log x)}{x}dx$ $\quad (iii)\int \frac{1}{x\log x}dx$ $\quad (iv)\int \frac{(\log x)}{x}dx$

$(v)\int \frac{1}{x(\log x)^2}dx$ $\quad (vi)\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ $\quad (vii)\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$ $\quad (viii)\int \frac{\tan(\sin^{-1}x)}{\sqrt{1-x^2}}dx$

$(ix)\int \frac{(\cos^{-1}x)^5}{\sqrt{1-x^2}}dx$ $\quad (x)\int \frac{1}{x\log x\log(\log x)}dx$ $\quad (xi)\int \frac{(\tan^{-1}x)^3}{1+x^2}dx$ $\quad (xii)\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

$(xiii)\int \frac{\tan^{-1}x}{1+x^2}dx$ $\quad (xiv)\int 2xe^{x^2}\,dx$ $\quad (xv)\int \frac{x^8}{1+x^{18}}dx$ $\quad (xvi)\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$

II Evaluate the following integrals:

$(i)\int \cos^2 x\,dx$ $(ii)\int \cos^3 x\,dx$ $(iii)\int \cos^4 x\,dx$ $(iv)\int \frac{\sin^2(\log x)}{x}dx$

III Evaluate the following integrals:

$(i)\int \sin 6x\cos 2x\,dx$ $(ii)\int \sin 7x\cos 5x\,dx$ $(iii)\int \sin 7x\cos 3x\,dx$ $(iv)\int \cos 7x\sin 2x\,dx$
$(v)\int \sin 2x\cos 3x\,dx$ $(vi)\int \cos 3x\cos 2x\,dx$ $(vii)\int \sin 5x\cos 2x\,dx$ $(viii)\int \sin 3x\cos x\,dx$
$(ix)\int \cos 2x\cos x\,dx$ $(x)\int \sin 7x\cos 2x\,dx$

IV Evaluate the following integrals:

$(i)\int \sin^6 x\cos^3 x\,dx$ $(ii)\int \sin^3 x\cos^{10} x\,dx$ $(iii)\int \sin^3 x\cos^3 x\,dx$ $(iv)\int \sin^3 x\cos^6 x\,dx$
$(v)\int \sin^5 x\cos^3 x\,dx$ $(vi)\int \sin^2 x\cos^3 x\,dx$

11.3.6 Evaluation of integrals of algebraic functions of special forms:

In the following integrals, a is a positive real number. Let us know that

1. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

2. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

3. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

4. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\frac{x}{a}+c$

5. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\frac{x}{a}+c$

6. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+c$

7. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$

8. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$

9. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

11.3.7 Solved Problems:

1. Problem: Evaluate the following integrals

$(a)\int \frac{1}{x^2+4}dx$ $(b)\int \frac{1}{x^2-4}dx$ $(c)\int \frac{1}{4-x^2}dx$

$(d)\int \frac{1}{\sqrt{x^2+4}}dx$ $(e)\int \frac{1}{\sqrt{x^2-4}}dx$ $(f)\int \frac{1}{\sqrt{4-x^2}}dx$

$(g)\int \sqrt{x^2+4}\,dx$ $(h)\int \sqrt{x^2-4}\,dx$ $(i)\int \sqrt{4-x^2}\,dx$

Solution:

(a) Let $I=\int \frac{1}{x^2+4}dx=\int \frac{1}{x^2+2^2}dx=\frac{1}{2}\tan^{-1}\frac{x}{2}+c \quad \left[\because \int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

(b) Let $I=\int \frac{1}{x^2-4}dx=\int \frac{1}{x^2-2^2}dx$

$$=\frac{1}{2.2}\log\left(\frac{x-2}{x+2}\right)+c \quad \left[\because \int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c\right]$$

$$=\frac{1}{4}\log\left(\frac{x-2}{x+2}\right)+c$$

(c) Let $I=\int \frac{1}{4-x^2}dx=\int \frac{1}{2^2-x^2}dx$

$$=\frac{1}{2.2}\log\left(\frac{2+x}{2-x}\right)+c \quad \left[\because \int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c\right]$$

$$=\frac{1}{4}\log\left(\frac{2+x}{2-x}\right)+c$$

(d) Let $I=\int \frac{1}{\sqrt{x^2+4}}dx=\int \frac{1}{\sqrt{x^2+2^2}}dx=\sinh^{-1}\frac{x}{2}+c \quad \left[\because \int \frac{1}{\sqrt{x^2+a^2}}dx=\sinh^{-1}\frac{x}{a}+c\right]$

(e) Let $I=\int \frac{1}{\sqrt{x^2-4}}dx=\int \frac{1}{\sqrt{x^2-2^2}}dx$

$$=\cosh^{-1}\left(\frac{x}{2}\right)+c \quad \left[\because \int \frac{1}{\sqrt{x^2-a^2}}dx=\cosh^{-1}\left(\frac{x}{a}\right)+c\right]$$

(f) Let $I=\int \frac{1}{\sqrt{4-x^2}}dx=\int \frac{1}{\sqrt{2^2-x^2}}dx$

$$= \sin^{-1}\left(\frac{x}{2}\right) + c \quad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

(g) Let $\mathrm{I} = \int \sqrt{x^2+4}\, dx = \int \sqrt{x^2+2^2}\, dx = \frac{x}{2}\sqrt{x^2+2^2} + \frac{2^2}{2}\sinh^{-1}\frac{x}{2} + c$

$$\left[\because \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x}{2}\sqrt{x^2+4} + 2\sinh^{-1}\frac{x}{2} + c$$

(h) Let $\mathrm{I} = \int \sqrt{x^2-4}\, dx = \int \sqrt{x^2-2^2}\, dx = \frac{x}{2}\sqrt{x^2-2^2} - \frac{2^2}{2}\cosh^{-1}\frac{x}{2} + c$

$$\left[\because \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x}{2}\sqrt{x^2-4} - 2\cosh^{-1}\frac{x}{2} + c$$

(i) Let $\mathrm{I} = \int \sqrt{4-x^2}\, dx = \int \sqrt{2^2-x^2}\, dx = \frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + c$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + c$$

2. Problem: Evaluate the following integrals

$(a)\int \frac{1}{2x^2+6}\, dx$ $(b)\int \frac{1}{2x^2-6}\, dx$ $(c)\int \frac{1}{6-2x^2}\, dx$

$(d)\int \frac{1}{\sqrt{2x^2+6}}\, dx$ $(e)\int \frac{1}{\sqrt{2x^2-6}}\, dx$ $(f)\int \frac{1}{\sqrt{6-2x^2}}\, dx$

$(g)\int \sqrt{2x^2+6}\, dx$ $(h)\int \sqrt{2x^2-6}\, dx$ $(i)\int \sqrt{6-2x^2}\, dx$

Solution:

(a) Let $\mathrm{I} = \int \frac{1}{2x^2+6}\, dx = \frac{1}{2}\int \frac{1}{x^2+(\sqrt{3})^2}\, dx$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) + c \quad \left[\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + c$$

(b) Let $\mathrm{I} = \int \frac{1}{2x^2 - 6} dx = \frac{1}{2} \int \frac{1}{x^2 - (\sqrt{3})^2} dx$

$$= \frac{1}{2} \left(\frac{1}{2\sqrt{3}} \log \left(\frac{x - \sqrt{3}}{x + \sqrt{3}} \right) \right) + c \quad \left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x - a}{x + a} \right) + c \right]$$

$$= \frac{1}{4\sqrt{3}} \log \left(\frac{x - \sqrt{3}}{x + \sqrt{3}} \right) + c$$

(c) Let $\mathrm{I} = \int \frac{1}{6 - 2x^2} dx = \frac{1}{2} \int \frac{1}{(\sqrt{3})^2 - x^2} dx$

$$= \frac{1}{2} \left(\frac{1}{2.\sqrt{3}} \log \left(\frac{\sqrt{3} + x}{\sqrt{3} - x} \right) \right) + c \quad \left[\because \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a + x}{a - x} \right) + c \right]$$

$$= \frac{1}{4\sqrt{3}} \log \left(\frac{\sqrt{3} + x}{\sqrt{3} - x} \right) + c$$

(d) Let $\mathrm{I} = \int \frac{1}{\sqrt{2x^2 + 6}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + (\sqrt{3})^2}} dx$

$$= \frac{1}{\sqrt{2}} \sinh^{-1} \frac{x}{\sqrt{3}} + c \quad \left[\because \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \frac{x}{a} + c \right]$$

(e) Let $\mathrm{I} = \int \frac{1}{\sqrt{2x^2 - 6}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 - (\sqrt{3})^2}} dx$

$$= \frac{1}{\sqrt{2}} \cosh^{-1} \left(\frac{x}{\sqrt{3}} \right) + c \quad \left[\because \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c \right]$$

(f) Let $\mathrm{I} = \int \frac{1}{\sqrt{6 - 2x^2}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(\sqrt{3})^2 - x^2}} dx$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x}{\sqrt{3}} \right) + c \quad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \right]$$

(g) Let $\mathrm{I} = \int \sqrt{2x^2 + 6}\, dx = \sqrt{2} \int \sqrt{x^2 + (\sqrt{3})^2}\, dx$

$$=\sqrt{2}\left(\frac{x}{2}\sqrt{x^2+(\sqrt{3})^2}+\frac{(\sqrt{3})^2}{2}\sinh^{-1}\frac{x}{\sqrt{3}}\right)+c$$

$$\left[\because \int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c\right]$$

$$=\frac{x}{\sqrt{2}}\sqrt{x^2+3}+\frac{3}{\sqrt{2}}\sinh^{-1}\frac{x}{\sqrt{3}}+c$$

(h) Let $\mathrm{I}=\int\sqrt{2x^2-6}\,dx=\sqrt{2}\int\sqrt{x^2-(\sqrt{3})^2}\,dx$

$$=\sqrt{2}\left(\frac{x}{2}\sqrt{x^2-(\sqrt{3})^2}-\frac{(\sqrt{3})^2}{2}\cosh^{-1}\frac{x}{\sqrt{3}}\right)+c=\frac{x}{\sqrt{2}}\sqrt{x^2-3}-\frac{3}{\sqrt{2}}\cosh^{-1}\frac{x}{\sqrt{3}}+c$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c\right]$$

(i) Let $\mathrm{I}=\int\sqrt{6-2x^2}\,dx=\sqrt{2}\int\sqrt{(\sqrt{3})^2-x^2}\,dx$

$$=\sqrt{2}\left(\frac{x}{2}\sqrt{(\sqrt{3})^2-x^2}+\frac{(\sqrt{3})^2}{2}\sin^{-1}\frac{x}{\sqrt{3}}\right)+c$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c\right]$$

$$=\frac{x}{\sqrt{2}}\sqrt{3-x^2}+\frac{3}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{3}}+c$$

Exercise 11(c)

I Evaluate the following integrals:

(1) $(a)\int\frac{1}{x^2+9}dx$ $(b)\int\frac{1}{x^2-9}dx$ $(c)\int\frac{1}{9-x^2}dx$

$(d)\int\frac{1}{\sqrt{x^2+9}}dx$ $(e)\int\frac{1}{\sqrt{x^2-9}}dx$ $(f)\int\frac{1}{\sqrt{9-x^2}}dx$

$(g)\int\sqrt{x^2+9}\,dx$ $(h)\int\sqrt{x^2-9}\,dx$ $(i)\int\sqrt{9-x^2}\,dx$

(2) $(a)\int \frac{1}{x^2+16}dx$ $(b)\int \frac{1}{x^2-16}dx$ $(c)\int \frac{1}{16-x^2}dx$

$(d)\int \frac{1}{\sqrt{x^2+16}}dx$ $(e)\int \frac{1}{\sqrt{x^2-16}}dx$ $(f)\int \frac{1}{\sqrt{16-x^2}}dx$

$(g)\int \sqrt{x^2+16}\,dx$ $(h)\int \sqrt{x^2-16}\,dx$ $(i)\int \sqrt{16-x^2}\,dx$

(3) $(a)\int \frac{1}{x^2+25}dx$ $(b)\int \frac{1}{x^2-25}dx$ $(c)\int \frac{1}{25-x^2}dx$

$(d)\int \frac{1}{\sqrt{x^2+25}}dx$ $(e)\int \frac{1}{\sqrt{x^2-25}}dx$ $(f)\int \frac{1}{\sqrt{25-x^2}}dx$

$(g)\int \sqrt{x^2+25}\,dx$ $(h)\int \sqrt{x^2-25}\,dx$ $(i)\int \sqrt{25-x^2}\,dx$

(4) $(a)\int \frac{1}{x^2+36}dx$ $(b)\int \frac{1}{x^2-36}dx$ $(c)\int \frac{1}{36-x^2}dx$

$(d)\int \frac{1}{\sqrt{x^2+36}}dx$ $(e)\int \frac{1}{\sqrt{x^2-36}}dx$ $(f)\int \frac{1}{\sqrt{36-x^2}}dx$

$(g)\int \sqrt{x^2+36}\,dx$ $(h)\int \sqrt{x^2-36}\,dx$ $(i)\int \sqrt{36-x^2}\,dx$

(5) $(a)\int \frac{1}{x^2+8}dx$ $(b)\int \frac{1}{x^2-8}dx$ $(c)\int \frac{1}{8-x^2}dx$

$(d)\int \frac{1}{\sqrt{x^2+8}}dx$ $(e)\int \frac{1}{\sqrt{x^2-8}}dx$ $(f)\int \frac{1}{\sqrt{8-x^2}}dx$

$(g)\int \sqrt{x^2+}8\,dx$ $(h)\int \sqrt{x^2-8}\,dx$ $(i)\int \sqrt{8-x^2}\,dx$

(6) $(a)\int \frac{1}{7(x+1)^2+1}dx$ $(b)\int \frac{1}{7(x+1)^2-1}dx$ $(c)\int \frac{1}{1-7(x+1)^2}dx$

$(d)\int \frac{1}{\sqrt{7(x+1)^2+1}}dx$ $(e)\int \frac{1}{\sqrt{7(x+1)^2-1}}dx$ $(f)\int \frac{1}{\sqrt{1-7(x+1)^2}}dx$

$(g)\int \sqrt{7(x+1)^2+1}\,dx$ $(h)\int \sqrt{7(x+1)^2-1}\,dx$ $(i)\int \sqrt{1-7(x+1)^2}\,dx$

11.3.8 Integration by parts:

In this section we discuss the integration of exponential, logarithmic and inverse trigonometric functions and some functions obtained as combination of these.

11.3.9 Theorem: Let u and v be real valued differentiable functions on I such that $u'v$ has an integrable on I then uv' has integrable on I and $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx + c$ where c is a constant.

11.3.10 Note: For a given differentiable function f on I $\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x) + c$ where c is a constant.

11.3.11 Solved Problems:

1. Problem: Evaluate the following integrals:

$$(a)\int xe^x\,dx\,(b)\int x^2e^x\,dx\,(c)\int x^3e^x\,dx$$

Solution: (a) Let $\mathrm{I}=\int xe^x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\mathrm{I}=\int xe^x\,dx = x\int e^x dx - \int\left(\frac{d}{dx}(x)\int e^x dx\right)dx$$

$$= xe^x - \int 1.e^x dx = xe^x - e^x = e^x(x-1)+c$$

(b) Let $\mathrm{I}=\int x^2e^x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\mathrm{I}=\int x^2e^x\,dx = x^2\int e^x dx - \int\left(\frac{d}{dx}(x^2)\int e^x dx\right)dx$$

$$= x^2e^x - \int 2xe^x dx = x^2e^x - 2\int xe^x dx = x^2e^x - 2e^x(x-1)\left[\because \text{from}\,(a)\right]$$

$$= e^x(x^2-2x+2)+c$$

(c) Let $\mathrm{I}=\int x^3e^x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\mathrm{I}=\int x^3e^x\,dx = x^3\int e^x dx - \int\left(\frac{d}{dx}(x^3)\int e^x dx\right)dx$$

$$= x^3e^x - \int 3x^2e^x dx = x^3e^x - 3\int x^2e^x dx = x^3e^x - 3e^x(x^2-2x+2)\left[\because \text{from}\,(b)\right]$$

$$= e^x(x^3-3x^2+6x-6)+c$$

2. Problem: Evaluate $\int x^2e^{3x}\,dx$

Solution: Let $\mathrm{I}=\int x^2e^{3x}\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\text{I} = \int x^2 e^{3x}\,dx = x^2\int e^{3x}dx - \int\left(\frac{d}{dx}(x^2)\int e^{3x}dx\right)dx = x^2\,\frac{e^{3x}}{3} - \int 2x.\frac{e^{3x}}{3}dx$$

$$= x^2\,\frac{e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx = x^2\,\frac{e^{3x}}{3} - \frac{2}{3}\left[x\int e^{3x}dx - \int\left(\frac{d}{dx}(x)\int e^{3x}dx\right)dx\right]$$

$$= x^2\,\frac{e^{3x}}{3} - \frac{2}{3}\left[x.\frac{e^{3x}}{3} - \int\left(1.\frac{e^{3x}}{3}\right)dx\right] = x^2\,\frac{e^{3x}}{3} - \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{1}{3}.\frac{e^{3x}}{3}\right]$$

$$= \frac{e^{3x}}{3}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) + c$$

3. Problem: Evaluate the following integrals:

$(a)\int \log x\,dx\,(b)\int x\log x\,dx\,(c)\int x^n \log x\,dx$

Solution: (a) Let $\text{I} = \int \log x\,dx = \int \log x.1dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\text{I} = \int \log x\,dx = \log x\int 1dx - \int\left(\frac{d}{dx}(\log x)\int 1dx\right)dx$$

$$= \log x.x - \int \frac{1}{x}.xdx = x\log x - \int 1dx = x\log x - x = x(\log x - 1) + c$$

(b) Let $\text{I} = \int x\log x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\text{I} = \int x\log x\,dx = \log x\int xdx - \int\left(\frac{d}{dx}(\log x)\int xdx\right)dx$$

$$= \log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx = \frac{x^2}{2}\log x - \frac{1}{2}\int xdx = \frac{x^2}{2}\log x - \frac{1}{2}.\frac{x^2}{2}$$

$$= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + c$$

(c) Let $\text{I} = \int x^n \log x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$I = \int x^n \log x\,dx = \log x\int x^n dx - \int\left(\frac{d}{dx}(\log x)\int x^n dx\right)dx$$

$$= \log x.\frac{x^{n+1}}{n+1} - \int \frac{1}{x}.\frac{x^{n+1}}{n+1}dx = \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n dx = \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}.\frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + c = \frac{1}{2}\left(\frac{4}{\sqrt{7}}\tan^{-1}\frac{4t-3}{\sqrt{7}}\right) = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4\tan\frac{x}{2}-3}{\sqrt{7}}\right) + c$$

4. Problem: Evaluate $\int x\sin 2x\,dx$

Solution: Let $I = \int x\sin 2x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$I = \int x\sin 2x\,dx = x\int \sin 2x dx - \int\left(\frac{d}{dx}(x)\int \sin 2x dx\right)dx$$

$$= x\left(\frac{-\cos 2x}{2}\right) - \int 1\left(\frac{-\cos 2x}{2}\right)dx = \frac{-x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}.\frac{\sin 2x}{2} = \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

5. Problem: Evaluate the following integrals:

$(a)\int \sin^{-1} x dx$ $(b)\int \tan^{-1} x dx$ $(c)\int x\cos^{-1} x dx$ $(d)\int x\cot^{-1} x dx$

Solution: (a) Let $I = \int \sin^{-1} x\ dx = \int \sin^{-1} x.1dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$I = \int \sin^{-1} x\,dx = \sin^{-1} x\int 1dx - \int\left(\frac{d}{dx}(\sin^{-1} x)\int 1dx\right)dx$$

$$= \sin^{-1} x.x - \int \frac{1}{\sqrt{1-x^2}}.xdx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1}x - (-\sqrt{1-x^2})\left[\begin{array}{l}\because \text{Put}\, 1-x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt \\ \int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{-tdt}{\sqrt{t^2}} = -\int \frac{tdt}{t} = -\int dt = -t = -\sqrt{1-x^2}\end{array}\right]$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

(b) Let $\text{I} = \int \tan^{-1}x\ dx = \int \tan^{-1}x.1dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\text{I} = \int \tan^{-1}x\,dx = \tan^{-1}x\int 1dx - \int\left(\frac{d}{dx}(\tan^{-1}x)\int 1dx\right)dx$$

$$= \tan^{-1}x.x - \int \frac{1}{1+x^2}.xdx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\left[\begin{array}{l}\because \text{Put}\, 1+x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}tdt \\ \int \frac{x}{1+x^2}dx = \int \frac{1}{t}\frac{1}{2}dt = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log(1+x^2)\end{array}\right]$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + c$$

(c) Let $\text{I} = \int x\cos^{-1}x\ dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\text{I} = \int x\cos^{-1}x\,dx = \cos^{-1}x\int xdx - \int\left(\frac{d}{dx}(\cos^{-1}x)\int xdx\right)dx$$

$$= \cos^{-1}x.\frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx = \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx = \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \left[\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right] - \frac{1}{2}\left[\cos^{-1}x\right]$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{x}{2}\sqrt{1-x^2} + c$$

(d) Let $\mathrm{I}=\int x\cot^{-1}x\,dx$

We have $\int f(x)g(x)dx = f(x)\int g(x)dx-\int\left(\frac{d}{dx}(f(x))\int g(x)\right)dx$

$$\mathrm{I}=\int x\cot^{-1}x\,dx=\cot^{-1}x\int xdx-\int\left(\frac{d}{dx}(\cot^{-1}x)\int xdx\right)dx$$

$$=\cot^{-1}x.\frac{x^2}{2}-\int\frac{-1}{1+x^2}.\frac{x^2}{2}dx=\frac{x^2}{2}\cot^{-1}x+\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2}{2}\cot^{-1}x+\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx\quad=\frac{x^2}{2}\cot^{-1}x+\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{-1}{1+x^2}dx$$

$$=\frac{x^2}{2}\cot^{-1}x+\frac{x}{2}+\frac{1}{2}\cot^{-1}x=\left(\frac{x^2+1}{2}\right)\cot^{-1}x+\frac{x}{2}+c$$

6. Problem: Evaluate the following integrals:

$$(a)\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx\quad(b)\int\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx\quad(c)\int\sin^{-1}\left(3x-4x^3\right)dx$$

Solution: (a) Let $\mathrm{I}=\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x=\tan\theta\Rightarrow\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}\left(\sin 2\theta\right)=2\theta=2\tan^{-1}x.$

$$\mathrm{I}=\int 2\tan^{-1}x\,dx=2\int\tan^{-1}xdx=2\left(x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\right)$$

$$=2x\tan^{-1}x-\log(1+x^2)+c$$

(b) Let $\mathrm{I}=\int\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Put $x=\tan\theta\Rightarrow\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}\left(\tan 2\theta\right)=2\theta=2\tan^{-1}x.$

$$\mathrm{I}=\int 2\tan^{-1}x\,dx=2\int\tan^{-1}xdx=2\left(x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\right)$$

$$=2x\tan^{-1}x-\log(1+x^2)+c$$

(c) Let $\mathrm{I}=\int\sin^{-1}\left(3x-4x^3\right)dx$

Put $x=\sin\theta \Rightarrow \sin^{-1}(3x-4x^3)=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$

$$=\sin^{-1}\left(\sin 3\theta\right)=3\theta=3\sin^{-1}x.$$

$$\text{I}=\int 3\sin^{-1}x\,dx=3\int \sin^{-1}x dx=3\left(x\sin^{-1}x+\sqrt{1-x^2}\right)+c$$

7. Problem: Evaluate the following integrals:

$(a)\int e^x\left(\sin x+\cos x\right)dx$ $\quad (b)\int e^x \sec x\left(1+\tan x\right)dx$

$(c)\int e^x\left(\tan^{-1}x+\dfrac{1}{1+x^2}\right)dx$ $\quad (d)\int \dfrac{x}{\left(x+1\right)^2}e^x\,dx$

Solution: (a) Let $\text{I}=\int e^x\left(\sin x+\cos x\right)dx$

We have $\int e^x(f(x)+f'(x))dx=e^x f(x)+c$

Put $f(x)=\sin x \Rightarrow f'(x)=\cos x.$

$$\text{I}=\int e^x\left(\sin x+\cos x\right)dx=e^x\sin x+c$$

(b) Let $\text{I}=\int e^x\sec x\left(1+\tan x\right)dx=\int e^x\left(\sec x+\sec x\tan x\right)dx$

We have $\int e^x(f(x)+f'(x))dx=e^x f(x)+c$

Put $f(x)=\sec x \Rightarrow f'(x)=\sec x\tan x.$

$$\text{I}=\int e^x\sec x\left(1+\tan x\right)dx=e^x\sec x+c$$

(c) Let $\text{I}=\int e^x\left(\tan^{-1}x+\dfrac{1}{1+x^2}\right)dx$

We have $\int e^x(f(x)+f'(x))dx=e^x f(x)+c$

Put $f(x)=\tan^{-1}x \Rightarrow f'(x)=\dfrac{1}{1+x^2}$

$$\text{I}=\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx=e^x\tan^{-1}x+c$$

(d) Let $\text{I}=\int \dfrac{x}{\left(x+1\right)^2}e^x\,dx=\int e^x\left(\dfrac{x+1-1}{\left(x+1\right)^2}\right)dx=\int e^x\left(\dfrac{1}{x+1}-\dfrac{1}{(x+1)^2}\right)dx$

We have $\int e^x(f(x)+f'(x))dx=e^x f(x)+c$

Put $f(x)=\frac{1}{x+1} \Rightarrow f'(x)=-\frac{1}{(x+1)^2}$

$$\text{I}=\int e^x\left(\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx=e^x.\frac{1}{1+x}+c=\frac{e^x}{1+x}+c$$

Exercise 11(d)

I Evaluate the following integrals:

(1) $(a)\int x^2e^{3x}\,dx$ $(b)\int xe^{-4x}\,dx$ $(c)\int x^2e^{5x}\,dx$ $(d)\int x^2e^{7x}\,dx$

$(e)\int x^2e^{-5x}\,dx$ $(f)\int x^2\log x\,dx$ $(g)\int x^3\log x\,dx$ $(h)\int x\sin x\,dx$

$(i)\int x\cos x\,dx$ $(j)\int x^2\sin 3x\,dx$ $(k)\int x^2\cos 2x\,dx$ $(l)\int x^3\cos 3x\,dx$

(2) $(a)\int \cos^{-1}x\,dx$ $(b)\int \cot^{-1}x\,dx$ $(c)\int x\sin^{-1}x\,dx$ $(d)\int x\tan^{-1}x\,dx$

$(e)\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ $(f)\int \cos^{-1}\left(4x^3-3x\right)dx$ $(g)\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

(3) $(a)\int e^x\left(\sin x-\cos x\right)dx$ $(b)\int e^x\operatorname{cosec}x\left(1-\cot x\right)dx$ $(c)\int e^x\operatorname{cosec}x\left(1-\cot x\right)dx$

$(d)\int e^x\left(\sec^2 x+\tan x\right)dx$ $(e)\int e^x\left(\operatorname{cosec}^2x-\cot x\right)dx$ $(f)\int e^x\left(\tan^2 x+\tan x+1\right)dx$

$(g)\int e^x\left(\cot^2 x-\cot x+1\right)dx$ $(h)\int e^x\left(x^2+2x\right)dx$ $(i)\int e^x\left(\frac{1}{x}+\log x\right)dx$

$(j)\int e^x\left(\sin^{-1}x+\frac{1}{\sqrt{1-x^2}}\right)dx$ $(k)\int \frac{x+1}{(x+2)^2}e^x\,dx$

11.4 Reduction formulae:

There are many functions whose integrals cannot be not be reduced to one or the other of the well known standard forms of integration. However, in some cases these integrals can be connected algebraically with integrals of other expressions in the form of a recurrence relation ration which are directly integrable or which may be easier to integrate than the original functions. Such connecting algebraic relations are called 'reduction formulae'. These formulae connect an integral with another one which is of the same type, with a lower integer parameter which is relatively easier to integrate. In this section, we illustrate the method of integration by successive reduction.

11.4.1 Reduction formula for $\int x^n e^{ax}dx, n$ being a positive integer:

If $\text{I}_n=\int x^ne^{ax}dx$ then $\text{I}_n=\frac{x^ne^{ax}}{a}-\frac{n}{a}\text{I}_{n-1}$.

This is called a 'reduction formula' for $\int x^n e^{ax} dx.$ Now I_{n-1} in turn can be connected to I_{n-2}. By successive reduction of $n,$ the original integral I_n finally depends on I_0, where $\mathrm{I}_0 = \frac{e^{ax}}{a}.$

11.4.2 Solved Problems:

1. Problem: Evaluate $\int x^3 e^{5x} dx$

Solution: We take $a = 5$ and use the reduction formula 11.4.1 for $n = 3, 2, 1$ in that order.

Then we have $\mathrm{I}_3 = \int x^3 e^{5x} dx = \frac{x^3 e^{5x}}{5} - \frac{3}{5}\mathrm{I}_2.$

$$\mathrm{I}_2 = \int x^2 e^{5x} dx = \frac{x^2 e^{5x}}{5} - \frac{2}{5}\mathrm{I}_1.$$

$$\mathrm{I}_1 = \int x e^{5x} dx = \frac{x e^{5x}}{5} - \frac{1}{5}\mathrm{I}_0 \text{ and } \mathrm{I}_0 = \frac{e^{5x}}{5} + c$$

Hence $\mathrm{I}_3 = \frac{x^3 e^{5x}}{5} - \frac{3}{5^2} x^2 e^{5x} + \frac{6}{5^3} x e^{5x} - \frac{6}{5^4} e^{5x} + c.$

2. Problem: Evaluate $\int x^4 e^{-2x} dx$

Solution: We take $a = -2$ and use the reduction formula 11.4.1 for $n = 4, 3, 2, 1$ in that order. Then we have $\mathrm{I}_4 = \int x^4 e^{-2x} dx = \frac{x^4 e^{-2x}}{-2} - \left(\frac{4}{-2}\right)\mathrm{I}_3 \Rightarrow \mathrm{I}_4 = \frac{x^4 e^{-2x}}{-2} + 2\mathrm{I}_3$

$$\mathrm{I}_3 = \int x^3 e^{-2x} dx = \frac{x^3 e^{-2x}}{-2} - \left(\frac{3}{-2}\right)\mathrm{I}_2 \Rightarrow \mathrm{I}_3 = \frac{x^3 e^{-2x}}{-2} + \frac{3}{2}\mathrm{I}_2$$

$$\mathrm{I}_2 = \int x^2 e^{-2x} dx = \frac{x^2 e^{-2x}}{-2} - \left(\frac{2}{-2}\right)\mathrm{I}_1 \Rightarrow \mathrm{I}_2 = \frac{x^2 e^{-2x}}{-2} + \mathrm{I}_1$$

$$\mathrm{I}_1 = \int x e^{-2x} dx = \frac{x e^{-2x}}{-2} - \left(\frac{1}{-2}\right)\mathrm{I}_0 \Rightarrow \mathrm{I}_1 = \frac{x e^{-2x}}{-2} + \frac{1}{2}\mathrm{I}_0 \text{ and } \mathrm{I}_0 = \frac{e^{-2x}}{-2} + c$$

Hence $\mathrm{I}_4 = \frac{x^4 e^{-2x}}{-2} - x^3 e^{-2x} - \frac{3}{2} x^2 e^{-2x} - \frac{3}{2} x e^{-2x} - \frac{3}{4} e^{-2x} + c$

11.4.3 Reduction formula for $\int \sin^n x dx$ for an integer $n \geq 2.$:

If $\mathrm{I}_n = \int \sin^n x \, dx$ then $\mathrm{I}_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\mathrm{I}_{n-2}.$

This is called a 'reduction formula' for $\int \sin^n x dx.$

If n is even, after successive reduction we get $I_0 = \int (\sin x)^0 \, dx = \int 1 dx = x + c$

If is n odd, after successive reduction we get $I_1 = \int (\sin x)^1 \, dx = \int \sin x \, dx = -\cos x + c$

11.4.4 Solved Problems:

1. Problem: Evaluate $\int \sin^4 x \, dx$

Solution: We take $n = 4$ and use the reduction formula 11.4.3 for $n = 4, 2$ in that order.

Then we have $I_4 = \int \sin^4 x dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2.$

$$I_2 = \int \sin^2 x dx = \frac{-\sin x \cos x}{2} + \frac{1}{2} I_0$$

and $I_0 = \int \sin^0 x dx = \int 1 dx = x + c$

Hence $\quad I_4 = \int \sin^4 x dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left[\frac{-\sin x \cos x}{2} + \frac{1}{2} I_0\right]$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3\sin x \cos x}{8} + \frac{3}{8} I_0 = \frac{-\sin^3 x \cos x}{4} - \frac{3\sin x \cos x}{8} + \frac{3}{8} x + c$$

2. Problem: Evaluate $\int \sin^5 x \, dx$

Solution: We take $n = 5$ and use the reduction formula 11.4.3 for $n = 5, 3$ in that order.

Then we have $I_5 = \int \sin^5 x dx = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5} I_3.$

$$I_3 = \int \sin^3 x dx = \frac{-\sin^2 x \cos x}{3} + \frac{2}{3} I_1$$

and $I_1 = \int \sin^1 x dx = \int \sin x dx = -\cos x + c$

Hence $\quad I_5 = \int \sin^5 x dx = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5}\left[\frac{-\sin^2 x \cos x}{3} + \frac{2}{3} I_1\right]$

$$I_5 = \int \sin^5 x dx = \frac{-\sin^4 x \cos x}{5} - \frac{4\sin^2 x \cos x}{15} + \frac{8}{15} I_1$$

$$= \frac{-\sin^4 x \cos x}{5} - \frac{4\sin^2 x \cos x}{15} - \frac{8}{15} \cos x + c$$

11.4.5 Reduction formula for $\int \cos^n x dx$ for an integer $n \geq 2.$:

If $\mathrm{I}_n = \int \cos^n x\, dx$ then $\mathrm{I}_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \mathrm{I}_{n-2}.$

This is called a 'reduction formula' for $\int \cos^n x dx$

If n is even, after successive reduction we get $\mathrm{I}_0 = \int (\cos x)^0\, dx = \int 1 dx = x + c$

If n is odd, after successive reduction we get $\mathrm{I}_1 = \int (\cos x)^1\, dx = \int \cos x\, dx = \sin x + c$

11.4.6 Solved Problems:

1. Problem: Evaluate $\int \cos^6 x\, dx$

Solution: We take $n = 6$ and use the reduction formula 11.4.5 for $n = 6, 4, 2$ in that order.

Then we have $\mathrm{I}_6 = \int \cos^6 x dx = \frac{\cos^5 x \sin x}{6} + \frac{5}{6} \mathrm{I}_4.$

$$\mathrm{I}_4 = \int \cos^4 x dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \mathrm{I}_2.$$

$$\mathrm{I}_2 = \int \cos^2 x dx = \frac{\cos x \sin x}{2} + \frac{1}{2} \mathrm{I}_0.$$

and $\mathrm{I}_0 = \int \cos^0 x dx = \int 1 dx = x + c$

Hence $\mathrm{I}_6 = \int \cos^6 x dx = \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\left[\frac{\cos^3 x \sin x}{4} + \frac{3}{4}\mathrm{I}_2\right]$

$$= \frac{\cos^5 x \sin x}{6} + \frac{5\cos^3 x \sin x}{24} + \frac{15}{24}\mathrm{I}_2$$

$$= \frac{\cos^5 x \sin x}{6} + \frac{5\cos^3 x \sin x}{24} + \frac{15}{24}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\mathrm{I}_0\right]$$

$$= \frac{\cos^5 x \sin x}{6} + \frac{5\cos^3 x \sin x}{24} + \frac{15\cos x \sin x}{48} + \frac{15x}{48} + c$$

2. Problem: Evaluate $\int \cos^7 x\, dx$

Solution: We take $n = 7$ and use the reduction formula 11.4.5 for $n = 7, 5, 3$ in that order.

Then we have $\mathrm{I}_7 = \int \cos^7 x dx = \frac{\cos^6 x \sin x}{7} + \frac{6}{7} \mathrm{I}_5.$

$$I_5 = \int \cos^5 x dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} I_3.$$

$$I_3 = \int \cos^3 x dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3} I_1.$$

and $I_1 = \int \cos^1 x dx = \int \cos x dx = \sin x + c$

Hence $\quad I_7 = \int \cos^7 x dx = \frac{\cos^6 x \sin x}{7} + \frac{6}{7}\left[\frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2\right]$

$$= \frac{\cos^6 x \sin x}{7} + \frac{3\cos^3 x \sin x}{14} + \frac{9}{14}\left[\frac{\cos^4 x \sin x}{5} + \frac{4}{5} I_3\right]$$

$$= \frac{\cos^6 x \sin x}{7} + \frac{3\cos^3 x \sin x}{14} + \frac{9\cos^4 x \sin x}{70} + \frac{18}{35} I_3$$

$$= \frac{\cos^6 x \sin x}{7} + \frac{3\cos^3 x \sin x}{14} + \frac{9\cos^4 x \sin x}{70} + \frac{18}{35}\left[\frac{\cos^2 x \sin x}{3} + \frac{2}{3} I_1\right]$$

$$= \frac{\cos^6 x \sin x}{7} + \frac{3\cos^3 x \sin x}{14} + \frac{9\cos^4 x \sin x}{70} + \frac{6\cos^2 x \sin x}{35} + \frac{12}{35} I_1$$

$$= \frac{\cos^6 x \sin x}{7} + \frac{3\cos^3 x \sin x}{14} + \frac{9\cos^4 x \sin x}{70} + \frac{6\cos^2 x \sin x}{35} + \frac{12}{35}\sin x + c$$

11.4.7 Reduction formula for $\int \tan^n x dx$ for an integer $n \geq 2.$:

If $\quad I_n = \int \tan^n x \, dx$ then $\quad I_n = \frac{\tan^{n-1} x}{n} - I_{n-2}.$

This is called a 'reduction formula' for $\int \tan^n x dx$

If n is even, after successive reduction we get $I_0 = \int (\tan x)^0 \, dx = \int 1 dx = x + c$

If n is odd, after successive reduction we get $I_1 = \int (\tan x)^1 \, dx = \int \tan x \, dx = \log|\sec x| + c$

11.4.8 Solved Problems:

1. Problem: Evaluate $\int \tan^4 x \, dx$

Solution: We take $n = 4$ and use the reduction formula 11.4.7 for $n = 4, 2$ in that order.

Then we have $I_4 = \int \tan^4 x dx = \frac{\tan^3 x}{4} - I_2$

$$I_2 = \int \tan^2 x dx = \frac{\tan x}{2} - I_0$$

and $I_0 = \int \tan^0 x dx = \int 1 dx = x + c$

Hence $\quad I_4 = \int \tan^4 x dx = \frac{\tan^3 x}{4} - \left[\frac{\tan x}{2} - I_0 \right]$

$$= \frac{\tan^3 x}{4} - \frac{\tan x}{2} + I_0$$

$$= \frac{\tan^3 x}{4} - \frac{\tan x}{2} + x + c$$

2. Problem: Evaluate $\int \tan^5 x\, dx$

Solution: We take $n = 5$ and use the reduction formula 11.4.7 for $n = 5, 3$ in that order.

Then we have $I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{5} - I_3$

$$I_3 = \int \tan^3 x dx = \frac{\tan^2 x}{3} - I_1$$

and $I_1 = \int \tan^1 x dx = \int \tan x dx = \log|\sec x| + c$

Hence $\quad I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{5} - \left[\frac{\tan^2 x}{3} - I_1 \right]$

$$= \frac{\tan^4 x}{5} - \frac{\tan^2 x}{3} + I_1$$

$$= \frac{\tan^4 x}{5} - \frac{\tan^2 x}{3} + \log|\sec x| + c$$

11.4.9 Reduction formula for $\int \cot^n x dx$ for an integer $n \geq 2.$:

If $\quad I_n = \int \cot^n x\, dx$ then $\quad I_n = \frac{-\cot^{n-1} x}{n} - I_{n-2}.$

This is called a 'reduction formula' for $\int \cot^n x dx$

If n is even, after successive reduction we get $I_0 = \int (\cot x)^0\, dx = \int 1 dx = x + c$

If n is odd, after successive reduction we get $I_1 = \int (\cot x)^1\, dx = \int \cot x\, dx = \log|\sin x| + c$

11.4.10 Solved Problems:

1. Problem: Evaluate $\int \cot^6 x\,dx$

Solution: We take $n = 6$ and use the reduction formula 11.4.9 for $n = 6, 4, 2$ in that order.

Then we have $I_6 = \int \cot^6 x dx = \dfrac{-\cot^5 x}{6} - I_4.$

$$I_4 = \int \cot^4 x dx = \frac{-\cot^3 x}{4} - I_2.$$

$$I_2 = \int \cot^2 x dx = \frac{-\cot x}{2} - I_0.$$

and $I_0 = \int \cot^0 x dx = \int 1 dx = x + c$

Hence $\quad I_6 = \int \cot^6 x dx = \dfrac{-\cot^5 x}{6} - \left[\dfrac{-\cot^3 x}{4} - I_2\right]$

$$= \frac{-\cot^5 x}{6} + \frac{\cot^3 x}{4} + I_2$$

$$= \frac{-\cot^5 x}{6} + \frac{\cot^3 x}{4} + \left[-\frac{\cot x}{2} - I_0\right]$$

$$= \frac{-\cot^5 x}{6} + \frac{\cot^3 x}{4} - \frac{\cot x}{2} - x + c$$

2. Problem: Evaluate $\int \cot^7 x\,dx$

Solution: We take $n = 7$ and use the reduction formula 11.4.9 for $n = 7, 5, 3$ in that order.

Then we have $I_7 = \int \cot^7 x dx = \dfrac{-\cot^6 x}{7} - I_5.$

$$I_5 = \int \cot^5 x dx = \frac{-\cot^4 x}{5} - I_3.$$

$$I_3 = \int \cot^3 x dx = \frac{-\cot^2 x}{3} - I_1.$$

and $I_1 = \int \cot^1 x dx = \int \cot x dx = \log|\sin x| + c$

Hence $\quad I_7 = \int \cot^7 x dx = \dfrac{-\cot^6 x}{7} - \left[\dfrac{-\cot^4 x}{5} - I_3\right]$

$$= \frac{-\cot^6 x}{7} + \frac{\cot^4 x}{5} + I_3$$

$$= \frac{-\cot^6 x}{7} + \frac{\cot^4 x}{5} + \left[-\frac{\cot^2 x}{3} - I_1\right]$$

$$= \frac{-\cot^6 x}{7} + \frac{\cot^4 x}{5} - \frac{\cot^2 x}{3} - I_1$$

$$= \frac{-\cot^6 x}{7} + \frac{\cot^4 x}{5} - \frac{\cot^2 x}{3} - \log|\sin x| + c$$

11.4.11 Reduction formula for $\int \sec^n x dx$ for an integer $n \geq 2.$:

If $I_n = \int \sec^n x\, dx$ then $I_n = \frac{\sec^{n-2} x \tan x}{n} + \frac{n-2}{n-1} I_{n-2}.$

This is called a 'reduction formula' for $\int \sec^n x dx$

If n is even, after successive reduction we get $I_2 = \int \sec^2 x\, dx = \tan x + c$

If n is odd, after successive reduction we get $I_1 = \int (\sec x)^1\, dx = \log|\sec x + \tan x| + c$

11.4.12 Solved Problems:

1. Problem: Evaluate $\int \sec^4 x\, dx$

Solution: We take $n = 4$ and use the reduction formula 11.4.11 for $n = 4, 2$ in that order.

Then we have $I_4 = \int \sec^4 x dx = \frac{\sec^2 x \tan x}{4} + \frac{2}{3} I_2.$

and $I_2 = \int \sec^2 x dx = \tan x + c.$

Hence $I_4 = \int \sec^4 x dx = \frac{\sec^2 x \tan x}{4} + \frac{2}{3} \tan x + c$

2. Problem: Evaluate $\int \sec^5 x\, dx$

Solution: We take $n = 5$ and use the reduction formula 11.4.11 for $n = 5, 3$ in that order.

Then we have $I_5 = \int \sec^5 x dx = \frac{\sec^3 x \tan x}{5} + \frac{3}{4} I_3.$

$$I_3 = \int \sec^3 x dx = \frac{\sec x \tan x}{3} + \frac{1}{2} I_1.$$

and $I_1 = \int \sec x dx = \log|\sec x + \tan x| + c.$

Hence $\quad I_5 = \int \sec^5 x dx = \frac{\sec^3 x \tan x}{5} + \frac{3}{4}\left[\frac{\sec x \tan x}{3} + \frac{1}{2} I_1\right]$

$$= \frac{\sec^3 x \tan x}{5} + \frac{\sec x \tan x}{4} + \frac{3}{8} I_1$$

$$= \frac{\sec^3 x \tan x}{5} + \frac{\sec x \tan x}{4} + \frac{3}{8}\log|\sec x + \tan x| + c.$$

11.4.13 Reduction formula for $\int \operatorname{cosec}^n x dx$ for an integer $n \geq 2.$:

If $\quad I_n = \int \operatorname{cosec}^n x\, dx$ then $\quad I_n = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n} + \frac{n-2}{n-1} I_{n-2}.$

This is called a 'reduction formula' for $\int \operatorname{cosec}^n x dx$

If n is even, after successive reduction we get $I_2 = \int \operatorname{cosec}^2 x dx = -\cot x + c$

If n is odd, after successive reduction we get $I_1 = \int (\operatorname{cosec} x)^1 dx = \log|\operatorname{cosec} x - \cot x| + c$

11.4.14 Solved Problems:

1. Problem: Evaluate $\int \operatorname{cosec}^6 x\, dx$

Solution: We take $n = 6$ and use the reduction formula 11.4.13 for $n = 6, 4, 2$ in that order. Then we have $I_6 = \int \operatorname{cosec}^6 x dx = \frac{-\operatorname{cosec}^4 x \cot x}{6} + \frac{4}{5} I_4.$

$$I_4 = \int \operatorname{cosec}^4 x dx = \frac{-\operatorname{cosec}^2 x \cot x}{4} + \frac{2}{3} I_2.$$

and $I_2 = \int \operatorname{cosec}^2 x dx = -\cot x + c.$

Hence $\quad I_6 = \int \operatorname{cosec}^6 x dx = \frac{-\operatorname{cosec}^4 x \cot x}{6} + \frac{4}{5}\left[\frac{-\operatorname{cosec}^2 x \cot x}{4} + \frac{2}{3} I_2\right]$

$$= \frac{-\operatorname{cosec}^4 x \cot x}{6} - \frac{\operatorname{cosec}^2 x \cot x}{5} + \frac{8}{15} I_2$$

$$= \frac{-\operatorname{cosec}^4 x \cot x}{6} - \frac{\operatorname{cosec}^2 x \cot x}{5} - \frac{8}{15}\cot x + c.$$

2. Problem: Evaluate $\int \operatorname{cosec}^7 x\,dx$

Solution: We take $n=7$ and use the reduction formula 11.4.13 for $n=7,5,3$ in that order. Then we have $I_7 = \int \operatorname{cosec}^7 x dx = \frac{-\operatorname{cosec}^5 x \cot x}{7} + \frac{5}{6} I_5.$

$$I_5 = \int \operatorname{cosec}^5 x dx = \frac{-\operatorname{cosec}^3 x \cot x}{5} + \frac{3}{4} I_3.$$

$$I_3 = \int \operatorname{cosec}^3 x dx = \frac{-\operatorname{cosec} x \cot x}{3} + \frac{1}{2} I_1.$$

and $I_1 = \int \operatorname{cosec} x dx = \log\left|\operatorname{cosec} x - \cot x\right| + c.$

Hence $\quad I_7 = \int \operatorname{cosec}^7 x dx = \frac{-\operatorname{cosec}^5 x \cot x}{7} + \frac{5}{6}\left[\frac{-\operatorname{cosec}^3 x \cot x}{5} + \frac{3}{4} I_3\right]$

$$= \frac{-\operatorname{cosec}^5 x \cot x}{7} - \frac{\operatorname{cosec}^3 x \cot x}{6} + \frac{5}{8} I_3$$

$$= \frac{-\operatorname{cosec}^5 x \cot x}{7} - \frac{\operatorname{cosec}^3 x \cot x}{6} + \frac{5}{8}\left[\frac{-\operatorname{cosec} x \cot x}{3} + \frac{1}{2} I_1\right]$$

$$= \frac{-\operatorname{cosec}^5 x \cot x}{7} - \frac{\operatorname{cosec}^3 x \cot x}{6} - \frac{5\operatorname{cosec} x \cot x}{24} + \frac{5}{16} I_1$$

$$= \frac{-\operatorname{cosec}^5 x \cot x}{7} - \frac{\operatorname{cosec}^3 x \cot x}{6} - \frac{5\operatorname{cosec} x \cot x}{24} + \frac{5}{16}\log\left|\operatorname{cosec} x - \cot x\right| + c.$$

11.4.15 Reduction formula for $\int \sin^m x \cos^n x\,dx$ for a positive integer m and an integer $n \geq 2$. :

If $\quad I_{m,n} = \int \sin^m x \cos^n x\,dx$ then $\quad I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$

This is called a ‘reduction formula’ for $\int \sin^m x \cos^n x\,dx$

If n is even, after successive reduction we get $I_{m,0} = \int \sin^m x (\cos x)^0\,dx = \int \sin^m x\,dx = I_m$

If n is odd, after successive reduction we get $I_1 = \int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1} + c$

11.4.16 Solved Problems:

1. Problem: Evaluate $\int \sin^4 x\cos^3 x\,dx$

Solution: We take $m=4, n=3$ and use the reduction formula 11.4.15

. Then we have $I_{4,3}=\int \sin^4 x\cos^3 x\,dx=\dfrac{\sin^5 x\cos^2 x}{7}+\dfrac{2}{7}I_{4,1}$

$$I_{4,1}=\int \sin^4 x\cos x\,dx=\frac{\sin^5 x}{5}+c\left[\begin{array}{l}\because \text{ put } \sin x=t\Rightarrow \cos x\,dx=dt\\ \int \sin^4 x\cos x\,dx=\int t^4\,dt=\dfrac{t^5}{5}=\dfrac{\sin^5 x}{5}\end{array}\right]$$

Hence $\quad I_{4,3}=\int \sin^4 x\cos^3 x\,dx=\dfrac{\sin^5 x\cos^2 x}{7}+\dfrac{2\sin^5 x}{35}+c$

2. Problem: Evaluate $\int \sin^5 x\cos^4 x\,dx$

Solution: We take $m=5, n=4$ and use the reduction formula 11.4.15

. Then we have $I_{5,4}=\int \sin^5 x\cos^4 x\,dx=\dfrac{\sin^6 x\cos^3 x}{9}+\dfrac{1}{3}I_{5,2}$

$$I_{5,2}=\int \sin^5 x\cos^2 x\,dx=\frac{\sin^6 x\cos x}{7}+\frac{1}{7}I_{5,0}$$

$$I_{5,0}=\int \sin^5 x(\cos x)^0\,dx=\int \sin^5 x\;dx=\frac{-\sin^4 x\cos x}{5}-\frac{4\sin^2 x\cos x}{15}-\frac{8}{15}\cos x+c$$

Hence $\quad I_{5,4}=\int \sin^5 x\cos^4 x\,dx=\dfrac{\sin^6 x\cos^3 x}{9}+\dfrac{\sin^6 x\cos x}{21}+\dfrac{1}{21}I_{5,0}$

$$=\frac{\sin^6 x\cos^3 x}{9}+\frac{\sin^6 x\cos x}{21}+\frac{1}{21}\left[\frac{-\sin^4 x\cos x}{5}-\frac{4\sin^2 x\cos x}{15}-\frac{8}{15}\cos x\right]+c$$

$$=\frac{\sin^6 x\cos^3 x}{9}+\frac{\sin^6 x\cos x}{21}-\frac{1}{105}\sin^4 x\cos x-\frac{4}{315}\sin^2 x\cos x-\frac{8}{315}\cos x+c$$

Exercise 11(e)

I Evaluate the following integrals:

$(i)\int \sin^2 xdx$ $(ii)\int \cos^2 xdx$ $(iii)\int \sin^2 x\cos^2 xdx$ $(iv)\int \sin^3 xdx$ $(v)\int \sin^6 xdx$

$(vi)\int \sin^7 xdx$ $(vii)\int \cos^3 xdx$ $(viii)\int \cos^4 xdx$ $(ix)\int \cos^5 xdx$ $(x)\int \tan^6 xdx$

$(xi)\int \tan^7 x\,dx$ $(xii)\int \cot^4 x\,dx$ $(xiii)\int \cot^5 x\,dx$

Key concepts

I. Let E be a subset of R such that E contains a right or left neighbourhood of each of its points and let $f: E \to R$ be a function. If there is a function F on E such that $F'(x) = f(x)$ for all $x \in E,$ then we call F an *anti-derivative* of f or a *primitive* of f

II. Let $f: I \to R.$ Suppose that f has an anti-derivative F on $I.$ Then we say that f has an integral on I for any real number $c,$ we call $F + c$ is an *indefinite integral* of f over $I,$ denote it by $\int f(x)dx$ and read it as integral $f(x)dx.$ Thus we have $\int f(x)dx = F(x) + c.$ Here c is called a '*constant of integration'*. In the indefinite integral $\int f(x)dx, f$ is called the *'integrand'* and x is called the *'variable of integration'*.

III. The indefinite integrals of some functions.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c, n \neq -1$

2. $\int \dfrac{1}{x} dx = \log|x| + c$

3. $\int \dfrac{1}{2\sqrt{x}} dx = \sqrt{x} + c$

4. $\int e^x dx = e^x + c$

5. $\int a^x dx = \dfrac{a^x}{\log a} + c$

6. $\int \sin x dx = -\cos x + c$

7. $\int \cos x dx = \sin x + c$

8. $\int \sec^2(ax+b)dx = \dfrac{1}{a}\tan(ax+b) + c$

9. $\int \operatorname{cosec}^2(ax+b)dx = -\dfrac{1}{a}\cot(ax+b) + c$

10. $\int \sec(ax+b)\tan(ax+b)dx = \dfrac{1}{a}\sec(ax+b) + c$

11. $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\dfrac{1}{a}\operatorname{cosec}(ax+b) + c$

12. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c \ (or) \ -\cos^{-1} x + c$

13. $\int \frac{1}{1+x^2}dx = \tan^{-1} x + c \ (or) \ -\cot^{-1} x + c$

14. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1} x + c \ (or) \ -\text{cosec}^{-1} x + c$

15. $\int \sinh x dx = \cosh x + c$

16. $\int \cosh x dx = \sinh x + c$

17. $\int \text{sech}^2 x dx = \tanh x + c$

18. $\int \text{cosec}\, h^2 x dx = -\coth x + c$

19. $\int \text{sech}\, x \tanh x dx = -\text{sech}\, x + c$

20. $\int \text{cosec}\,\text{h}x \coth x dx = -\text{cosec}\,\text{h}x + c$

21. $\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x + c$

22. $\int \frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1} x + c$

23. $\int \frac{1}{1-x^2}dx = \tanh^{-1} x + c \ (or) \coth^{-1} x + c$

IV. If the functions f and g have integrals on I, then $f+g$ has an integral on I and $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx + c$, where c is a constant.

V. If f has an integral on I and a is a real number then, af also has an integral on I and $\int (af)(x)dx = a\int f(x)dx + c$, where c is a constant.

(i) If f and g have integrals on I, then $f-g$ has an integral on I and

$$\int (f-g)(x)dx = \int f(x)dx - \int g(x)dx + c.$$

(ii) If $f_1, f_2, f_3, \ldots f_n$ have integrals on I, then $f_1 + f_2 + f_3 + \ldots + f_n$ has an integral on I and $\int (f_1 + f_2 + f_3 + \ldots + f_n)(x)dx = \int f_1(x)dx + \int f_2(x)dx + \int f_3(x)dx + \ldots + \int f_n(x)dx + c$.

(iii) If $f_1, f_2, f_3, \ldots f_n$ have integrals on I and $k_1, k_2, k_3, \ldots k_n$ are constants, then $k_1 f_1 + k_2 f_2 + k_3 f_3 + \ldots + k_n f_n$ has an integral on I and

$$\int (k_1 f_1 + k_2 f_2 + k_3 f_3 + \ldots + k_n f_n)(x) dx$$
$$= k_1 \int f_1(x) dx + k_2 \int f_2(x) dx + k_3 \int f_3(x) dx + \ldots + k_n \int f_n(x) dx + c.$$

VI. Let $f : I \to R$ have an interval on I and F be a primitive of f on I. Let J be an interval in R and $g : J \to I$ be a differentiable function. Then $(f \circ g) g'$ has an integral on J and $\int f(g(x)) g'(x) dx = F(g(x)) + c$

$$i.e., \quad \int f(g(x)) g'(x) dx = \left[\int f(t)\, dt \right]_{t=g(x)} = F(g(x)) + c$$

VII. Let $f : I \to R$ have an interval on I and F be a primitive of f on I. Let $a, b \in R$ with $a \neq 0$. Then $\int f(ax+b) dx = \frac{1}{a} F(ax+b) + c$ for all $x \in J$, where

$J = \{ x \in R : ax + b \in I \}$ and c is an integral constant.

VIII. **Some important formulae:**

1. $\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c, n \neq -1$

2. $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

3. $\int \frac{1}{2\sqrt{ax+b}} dx = \frac{1}{a} \sqrt{ax+b} + c$

4. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

5. $\int a^{mx+n} dx = \frac{1}{m \log a} a^{mx+n} + c$

6. $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$

7. $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$

8. $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$

9. $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

10. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c$

11. $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

12. $\int \frac{1}{\sqrt{1-(ax+b)^2}}dx = \frac{1}{a}\sin^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\cos^{-1}(ax+b)+c$

13. $\int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}\tan^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\cot^{-1}(ax+b)+c$

14. $\int \frac{1}{|ax+b|\sqrt{(ax+b)^2-1}}dx = \frac{1}{a}\sec^{-1}(ax+b)+c \ (or) \ -\frac{1}{a}\operatorname{cosec}^{-1}(ax+b)+c$

15. $\int \sinh(ax+b)dx = \frac{1}{a}\cosh(ax+b)+c$

16. $\int \cosh(ax+b)dx = \frac{1}{a}\sinh(ax+b)+c$

17. $\int \operatorname{sech}^2(ax+b)dx = \frac{1}{a}\tanh(ax+b)+c$

18. $\int \operatorname{cosec} h^2(ax+b)dx = -\frac{1}{a}\coth(ax+b)+c$

19. $\int \operatorname{sech}(ax+b)\tanh(ax+b)dx = -\frac{1}{a}\operatorname{sech}(ax+b)+c$

20. $\int \operatorname{cosec}\mathrm{h}(ax+b)\coth(ax+b)dx = -\frac{1}{a}\operatorname{cosec}\mathrm{h}(ax+b)+c$

21. $\int \frac{1}{\sqrt{1+(ax+b)^2}}dx = \sinh^{-1}(ax+b)+c$

22. $\int \frac{1}{\sqrt{(ax+b)^2-1}}dx = \cosh^{-1}(ax+b)+c$

23. $\int \frac{1}{1-(ax+b)^2}dx = \frac{1}{a}\tanh^{-1}(ax+b)+c \ (or) \frac{1}{a}\cot^{-1}(ax+b)+c$

IX. Let $f: I \to R$ be a differentiable function. Then

(i) If f is never zero on I then $\frac{f'}{f}$ has an integral on I and $\int \frac{f'}{f} dx = \log|f(x)| + c$ on I

(ii) $\int (f(x))^{\alpha} f'(x) dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \alpha \neq -1$

(iii) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

(iv) $\int f'(ax+b) dx = \frac{1}{a} f(ax+b) + c, a \neq 0$

X. **Evaluation of integrals of algebraic functions of special forms:**

In the following integrals, a is a positive real number. Let us know that

1. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

2. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

3. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

4. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1} \frac{x}{a} + c$

5. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1} \frac{x}{a} + c$

6. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$

7. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + c$

8. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + c$

9. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

XI. Let u and v be real valued differentiable functions on I such that $u'v$ has an integrable on I then uv' has integrable on I and

$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx + c$ where c is a constant.

XII. For a given differentiable function f on I $\int e^x\left[f(x)+f'(x)\right]dx=e^x f(x)+c$ where c is a constant.

XIII. **Reduction formula for** $\int x^n e^{ax}dx, n$ **being a positive integer**

If $\quad I_n=\int x^n e^{ax}dx$ then $\quad I_n=\dfrac{x^n e^{ax}}{a}-\dfrac{n}{a}I_{n-1}.$

XIV. **Reduction formula for** $\int \sin^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \sin^n x\,dx$ then $\quad I_n=\dfrac{-\sin^{n-1}x\cos x}{n}+\dfrac{n-1}{n}I_{n-2}.$

XV. **Reduction formula for** $\int \cos^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \cos^n x\,dx$ then $\quad I_n=\dfrac{\cos^{n-1}x\sin x}{n}+\dfrac{n-1}{n}I_{n-2}.$

XVI. **Reduction formula for** $\int \tan^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \tan^n x\,dx$ then $\quad I_n=\dfrac{\tan^{n-1}x}{n}-I_{n-2}.$

XVII. **Reduction formula for** $\int \cot^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \cot^n x\,dx$ then $\quad I_n=\dfrac{-\cot^{n-1}x}{n}-I_{n-2}.$

XVIII. **Reduction formula for** $\int \sec^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \sec^n x\,dx$ then $\quad I_n=\dfrac{\sec^{n-2}x\tan x}{n}+\dfrac{n-2}{n-1}I_{n-2}.$

XIX. **Reduction formula for** $\int \operatorname{cosec}^n x dx$ **for an integer** $n\geq 2.$ **:**

If $\quad I_n=\int \operatorname{cosec}^n x\,dx$ then $\quad I_n=\dfrac{-\operatorname{cosec}^{n-2}x\cot x}{n}+\dfrac{n-2}{n-1}I_{n-2}.$

XX. **Reduction formula for** $\int \sin^m x\cos^n x\,dx$ **for a positive integer** m **and an integer** $n\geq 2.$ **:**

If $\quad I_{m,n}=\int \sin^m x\cos^n x\,dx$ then $\quad I_{m,n}=\dfrac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\dfrac{n-1}{m+n}I_{m,n-2}.$

Answers

Exercise 11(a)

I $(i) -\frac{1}{x}-\frac{4}{x^3}-\frac{8}{x^5}+c$ $(ii) \frac{x^4}{4}+\frac{3^x}{\log 3}+2x+c$ $(iii) \frac{x^5}{5}+\frac{5^x}{\log 5}+\frac{5}{2}x^2+c$

$(iv) \frac{x^4}{4}+\frac{7^x}{\log 7}+\frac{11}{2}x^2+c$ (v) $\sin x+\cos x+c$ $(vi) \tan x-e^x-\cos x+c$

$(vii) -e^{-x}+\frac{a^x}{\log a}+\log x+3x$ $(viii) 4\tan x-2e^x-3\cos x+c$ $(ix) \frac{x^4}{4}+\frac{4^x}{\log 4}+3x^2+c$

$(x) \frac{x^{10}}{10}+\frac{9^x}{\log 9}+\frac{9}{2}x^2+c$ $(xi) \frac{x^3}{3}+x^2+3x+c$ $(xii) \frac{x^6}{6}+3\sin x-4\log x+c$

$(xiii) \frac{x^8}{8}-3\log x-\cos x+c$ $(xiv) \tan x-\cot x+c$ $(xv) \sec x+\log|\cos ecx-\cot x|+c$

$(xvi) \log|\sec x+\tan x|-\cos\text{ec}\, x+c$ $(xvii) \frac{x^3}{3}+x^2+3x+c$ $(xviii) -\cot x+\cos\text{ec}x+c$

$(xix) \tan x+\sec x+c$ $(xx) \frac{x^4}{4}-\frac{1}{2x^2}+\frac{3}{2}x^2+3\log x+c$ $(xxi)\, x+c$ $(xxii) \sin x+\cos x+c$

$(xxiii) -\sqrt{2}\cos x+c$ $(xxiv) \log\sin x+c$ $(xxv) -\cot x-x+c$

Exercise 11(b)

I $(i)\, 2\sin\sqrt{x}+c$ $(ii) -\cos(\log x)+c$ $(iii) \log(\log x)+c$ $(iv) \frac{(\log x)^2}{2}+c$ (v) $\frac{-1}{\log x}+c$

$(vi) \frac{(\sin^{-1}x)^2}{2}+c$ $(vii)\, e^{\sin^{-1}x}+c$ $(viii) -\log\cos(\sin^{-1}x)+c$ $(ix) -\frac{(\cos^{-1}x)^6}{6}+c$

$(x) \log(\log(\log x))+c$ $(xi) \frac{(\tan^{-1}x)^4}{4}+c$ (xii) $e^{\tan^{-1}x}+c$ $(xiii) \frac{(\tan^{-1}x)^2}{2}+c$ $(xiv)\, e^{x^2}+c$

$(xv) \frac{1}{9}\tan^{-1}(x^9)+c$ $(xvi) -\cos(\tan^{-1}x)+c$

II $(i) \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]+c$ $(ii) \frac{1}{4}\left[3\sin x+\frac{\sin 3x}{3}\right]+c$ $(iii) \frac{1}{8}\left[3x+\frac{\sin 4x}{4}+2\sin 2x\right]+c$

$(iv) \frac{1}{2}\left[\log x-\frac{\sin(2\log x)}{2}\right]+c$

III $(i) -\frac{1}{2}\left[\frac{\cos 8x}{8}+\frac{\cos 4x}{4}\right]+c$ $(ii) -\frac{1}{2}\left[\frac{\cos 12x}{12}+\frac{\cos 2x}{2}\right]+c$

$(iii) -\frac{1}{2}\left[\frac{\cos 10x}{10}+\frac{\cos 4x}{4}\right]+c$ $(iv) \frac{1}{2}\left[\frac{\cos 5x}{5}-\frac{\cos 9x}{9}\right]+c$ $(v) \frac{1}{2}\left[\cos x-\frac{\cos 5x}{5}\right]+c$

$(vi)\frac{1}{2}\left[\frac{\sin 5x}{5}+\sin x\right]+c$ $(vii)-\frac{1}{2}\left[\frac{\cos 7x}{7}+\frac{\cos 3x}{3}\right]+c$ $(viii)-\frac{1}{2}\left[\frac{\cos 4x}{4}+\frac{\cos 2x}{2}\right]+c$

$(ix)\frac{1}{2}\left[\frac{\sin 3x}{3}+\sin x\right]+c$ $(x)-\frac{1}{2}\left[\frac{\cos 9x}{9}+\frac{\cos 5x}{5}\right]+c$

IV Evaluate the following integrals:

$(i)\frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}+c$ $(ii)\frac{\cos^{13} x}{13}-\frac{\cos^{11} x}{11}+c$ $(iii)\frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}+c$

$(iv)\frac{\cos^9 x}{9}-\frac{\cos^7 x}{7}+c$ $(v)\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+c$ $(vi)\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+c$

Exercise 11(c)

I Evaluate the following integrals:

(1) $(a)\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$ $(b)\frac{1}{6}\log\left(\frac{x-3}{x+3}\right)+c$ $(c)\frac{1}{6}\log\left(\frac{3+x}{3-x}\right)+c$

$(d)\sinh^{-1}\left(\frac{x}{3}\right)+c$ $(e)\cosh^{-1}\left(\frac{x}{3}\right)+c$ $(f)\sin^{-1}\left(\frac{x}{3}\right)+c$

$(g)\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\sinh^{-1}\left(\frac{x}{3}\right)+c$ $(h)\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\cosh^{-1}\left(\frac{x}{3}\right)+c$

$(i)\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+c$

(2) $(a)\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$ $(b)\frac{1}{8}\log\left(\frac{x-4}{x+4}\right)+c$ $(c)\frac{1}{8}\log\left(\frac{4+x}{4-x}\right)+c$

$(d)\sinh^{-1}\left(\frac{x}{4}\right)+c$ $(e)\cosh^{-1}\left(\frac{x}{4}\right)+c$ $(f)\sin^{-1}\left(\frac{x}{4}\right)+c$

$(g)\frac{x}{2}\sqrt{x^2+16}+8\sinh^{-1}\left(\frac{x}{4}\right)+c$ $(h)\frac{x}{2}\sqrt{x^2-16}-8\cosh^{-1}\left(\frac{x}{4}\right)+c$

$(i)\frac{x}{2}\sqrt{16-x^2}+8\sin^{-1}\left(\frac{x}{4}\right)+c$

(3) $(a)\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+c$ $(b)\frac{1}{10}\log\left(\frac{x-5}{x+5}\right)+c$ $(c)\frac{1}{10}\log\left(\frac{5+x}{5-x}\right)+c$

$(d)\sinh^{-1}\left(\frac{x}{5}\right)+c$ $(e)\cosh^{-1}\left(\frac{x}{5}\right)+c$ $(f)\sin^{-1}\left(\frac{x}{5}\right)+c$

$(g)\frac{x}{2}\sqrt{x^2+25}+\frac{25}{2}\sinh^{-1}\left(\frac{x}{5}\right)+c$ $(h)\frac{x}{2}\sqrt{x^2-25}-\frac{25}{2}\cosh^{-1}\left(\frac{x}{5}\right)+c$

$(i)\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)+c$

(4) $(a)\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+c$ $(b)\frac{1}{12}\log\left(\frac{x-6}{x+6}\right)+c$ $(c)\frac{1}{12}\log\left(\frac{6+x}{6-x}\right)+c$

$(d)\sinh^{-1}\left(\frac{x}{6}\right)+c$ $(e)\cosh^{-1}\left(\frac{x}{6}\right)+c$ $(f)\sin^{-1}\left(\frac{x}{6}\right)+c$

$(g)\frac{x}{2}\sqrt{x^2+36}+18\sinh^{-1}\left(\frac{x}{6}\right)+c$ $(h)\frac{x}{2}\sqrt{x^2-36}-18\cosh^{-1}\left(\frac{x}{6}\right)+c$

$(i)\frac{x}{2}\sqrt{36-x^2}+18\sin^{-1}\left(\frac{x}{6}\right)+c$

(5) $(a)\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$ $(b)\frac{1}{4\sqrt{2}}\log\left(\frac{x-2\sqrt{2}}{x+2\sqrt{2}}\right)+c$ $(c)\frac{1}{4\sqrt{2}}\log\left(\frac{2\sqrt{2}+x}{2\sqrt{2}-x}\right)+c$

$(d)\sinh^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$ $(e)\cosh^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$ $(f)\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$

$(g)\frac{x}{2}\sqrt{x^2+8}+4\sinh^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$ $(h)\frac{x}{2}\sqrt{x^2-8}-4\cosh^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$

$(i)\frac{x}{2}\sqrt{8-x^2}+4\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)+c$

(6) $(a)\frac{1}{\sqrt{7}}\tan^{-1}\sqrt{7}(x+1)+c$ $(b)\frac{1}{2\sqrt{7}}\log\left(\frac{\sqrt{7}(x+1)-1}{\sqrt{7}(x+1)-1}\right)+c$ $(c)\frac{1}{4\sqrt{2}}\log\left(\frac{1+\sqrt{7}(x+1)}{1-\sqrt{7}(x+1)}\right)+c$

$(d)\frac{1}{\sqrt{7}}\sinh^{-1}\sqrt{7}(x+1)+c$ $(e)\frac{1}{\sqrt{7}}\cosh^{-1}\sqrt{7}(x+1)+c$ $(f)\frac{1}{\sqrt{7}}\sin^{-1}\sqrt{7}(x+1)+c$

$(g)\frac{(x+1)}{2}\sqrt{7(x+1)^2+1}+\frac{1}{2\sqrt{7}}\sinh^{-1}\sqrt{7}(x+1)+c$

$(h)\frac{(x+1)}{2}\sqrt{7(x+1)^2-1}-\frac{1}{2\sqrt{7}}\cosh^{-1}\sqrt{7}(x+1)+c$

$(i)\frac{(x+1)}{2}\sqrt{1-7(x+1)^2}+\frac{1}{2\sqrt{7}}\sin^{-1}\sqrt{7}(x+1)+c$

Exercise 11(d)

I

(1) $(a)\frac{e^{3x}}{3}\left(x^2-\frac{2x}{3}+\frac{2}{9}\right)+c$ $(b)\frac{e^{-4x}}{-4}\left(x+\frac{1}{4}\right)+c$ $(c)\frac{e^{5x}}{5}\left(x^2-\frac{2x}{5}+\frac{2}{25}\right)+c$

$(d)\frac{e^{7x}}{7}\left(x^2-\frac{2x}{7}+\frac{2}{49}\right)+c$ $(e)\frac{e^{-5x}}{-5}\left(x^2+\frac{2x}{5}+\frac{2}{25}\right)+c$ $(f)\frac{x^3}{3}\left(\log x-\frac{1}{3}\right)+c$

$(g)\frac{x^4}{4}\left(\log x-\frac{1}{4}\right)+c$ $(h)-x\cos x+\sin x+c$ $(i)x\sin x+\cos x+c$

$(j)\dfrac{-x^2\cos 3x}{3}+\dfrac{2x\sin 3x}{9}+\dfrac{2\cos 3x}{27}+c$ $(k)\dfrac{x^2\sin 2x}{2}+\dfrac{x\cos 2x}{2}+\dfrac{\sin 2x}{8}+c$

$(l)\dfrac{x^3\sin 3x}{3}+\dfrac{x^2\cos 3x}{3}-\dfrac{2x\sin 3x}{9}-\dfrac{2\cos 3x}{27}+c$

(2) $(a)\ x\cos^{-1}x-\sqrt{1-x^2}+c$ $(b)\,x\cot^{-1}x+\log(\sqrt{1+x^2})+c$

$(c)\left(\dfrac{x^2}{2}-\dfrac{1}{4}\right)\sin^{-1}x+\dfrac{x}{4}\sqrt{1-x^2}+c$ $(d)\left(\dfrac{x^2+1}{2}\right)\tan^{-1}x-\dfrac{x}{2}+c$

$(e)\,2x\tan^{-1}x-2\log(\sqrt{1+x^2})+c$ $(f)\,3x\cos^{-1}x-3\sqrt{1-x^2}+c$

$(g)\,3x\tan^{-1}x-3\log(\sqrt{1+x^2})+c$

(3) $(a)-e^x\cos x+c$ $(b)\,e^x\cos ecx+c$ $(c)\,e^x\cos ecx+c$ $(d)\,e^x\tan x+c$

$(e)-e^x\cot x+c$ $(f)\,e^x\tan x+c$ $(g)-e^x\cot x+c$ $(h)\,x^2e^x+c$

$(i)\,e^x\log x+c$ $(j)\,e^x\sin^{-1}x+c$ $(k)\dfrac{e^x}{x+1}+c$

Exercise 11(e)

I

$(i)\dfrac{-\sin x\cos x}{2}+\dfrac{1}{2}x+c$ $(ii)\ \dfrac{\cos x\sin x}{2}+\dfrac{1}{2}x+c$ $(iii)\dfrac{\sin^3 x\cos x}{4}-\dfrac{\sin x\cos x}{8}+\dfrac{1}{8}x+c$

$(iv)\dfrac{-\sin^2 x\cos x}{3}-\dfrac{2}{3}\cos x+c$ $(v)\dfrac{-\sin^5 x\cos x}{6}+\dfrac{-5\sin^3 x\cos x}{24}-\dfrac{5\sin x\cos x}{16}+\dfrac{5}{16}x+c$

$(vi)\dfrac{-\sin^6 x\cos x}{7}-\dfrac{6\sin^4 x\cos x}{35}-\dfrac{8\sin^2 x\cos x}{35}-\dfrac{16}{35}\cos x+c$

$(vii)\dfrac{\cos^2 x\sin x}{3}+\dfrac{2}{3}\sin x+c$ $(viii)\dfrac{\cos^3 x\sin x}{4}+\dfrac{3\cos x\sin x}{8}+\dfrac{3}{8}x+c$

$(ix)\ \dfrac{\cos^4 x\sin x}{5}+\dfrac{4\cos^2 x\sin x}{15}+\dfrac{8}{15}\sin x+c$ $(x)\dfrac{\tan^5 x}{6}-\dfrac{\tan^3 x}{4}+\dfrac{\tan x}{2}-x+c$

$(xi)\dfrac{\tan^6 x}{7}-\dfrac{\tan^4 x}{5}+\dfrac{\tan^2 x}{3}-\log|\sec x|+c$ $(xii)\dfrac{-\cot^3 x}{4}+\dfrac{\cot x}{2}+x+c$

$(xiii)\dfrac{-\cot^4 x}{5}+\dfrac{\cot^2 x}{3}+\log|\sin x|+c$

12. DEFINITE INTEGRATION

Introduction:

Calculus originated to solve mainly two geometric problems: finding the tangent line to a curve and finding the area of a region under a curve. The first was studied by a limit process known as differentiation (which we studied in Intermediate first year) and the second by another limit process- integration- which we study now.

We recall from elementary calculus that to find the area of the region under the graph of a positive and continuous function f defined on $[a,b]$, we subdivide the interval $[a,b]$ into a finite number of subintervals, say n, the k^{th} subinterval having length Δx_k, and we consider sums of the form $\sum_{k=1}^{n} f(t_k)\Delta x_k$, where t_k is some point in the k^{th} subinterval. Such a sum is an approximation to the area by means of the sum of the areas of the rectangles. Suppose we make subdivisions finer and finer. It so happens that the sequence of the corresponding sums tends to a limit as $n \to \infty$. Thus, roughly speaking, this is Riemann's definition of the definite integral $\int_a^b f(x)dx.$

We discussed in the earlier chapter that indefinite integration is an inverse process of differentiation. We recall that, if f is the derivative of F, then $\int_a^b f(x)dx = F(x) + c,$ where c is a real constant. In this case F called a primitive of f.

12.1The Fundamental Theorem of Integral Calculus:

In the evaluation of the definite integral, the following 'Fundamental Theorem of Integral Calculus' is useful. This important theorem is stated without proof. You will learn its proof in higher classes.

12.1.1 Theorem: If f is integrable on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a).$

12.1.2 Note: We write $[F(x)]_b^a$ for $F(b) - F(a)$. $[F(x)]_b^a$ is not dependent on x. Also, we write $[F(x)]_b^a = -[F(x)]_a^b$.

12.1.3 Solved Problems:

1. Problem: Evaluate $\int_1^2 x^5 dx$

Solution: Let $f(x) = x^5$ on $[1,2]$

Take $F(x)=\frac{x^6}{6}$ is primitive of f on $[1,2]$

Hence from the Fundamental Theorem of Integral Calculus Theorem 12.1.1, we have
$$\int_1^2 x^5 dx=\int_1^2 f(x)dx=F(2)-F(1)=\frac{2^6}{6}-\frac{1^6}{6}=\frac{63}{6}=\frac{21}{2}.$$

2. Problem: Evaluate $\int_0^{\pi}\sin x\,dx$

Solution: Let $f(x)=\sin x$ on $[0,\pi]$

Take $F(x)=-\cos x$ is primitive of f on $[0,\pi]$

Hence from the Fundamental Theorem of Integral Calculus Theorem 12.1.1, we have
$$\int_0^{\pi}\sin x\,dx=\int_0^{\pi} f(x)dx=F(\pi)-F(0)=(-\cos\pi)-(-\cos 0)=(-(-1))-(-1)=2.$$

3. Problem: Evaluate $\int_0^{\pi}\sin x\,dx$

Solution: Let $f(x)=\sin x$ on $[0,\pi]$

Take $F(x)=-\cos x$ is primitive of f on $[0,\pi]$

Hence from the Fundamental Theorem of Integral Calculus Theorem 12.1.1, we have
$$\int_0^{\pi}\sin x\,dx=\int_0^{\pi} f(x)dx=F(\pi)-F(0)=(-\cos\pi)-(-\cos 0)=(-(-1))-(-1)=2.$$

Exercise 12(a)

I Evaluate the following integrals:

(1) $\int_0^{\pi}\cos x\ dx$ (2) $\int_0^{2\pi}(\cos x-\sin x)dx$ (3) $\int_0^1\frac{1}{\sqrt{1-x^2}}dx$

(4) $\int_0^2(x^2+2x+3)dx$ (5) $\int_0^1\frac{1}{1+x^2}dx$ (6) $\int_{1/\sqrt{3}}^1\frac{1}{1+x^2}dx$

12.2 Properties of Definite Integrals:

We now discuss certain properties of the definite integrals.

12.2.1 Theorem: Let $f:[a,b]\to R$ be an integrable on $[a,b]$. Then we define $\int_b^a f(x)dx$ as the negative of $\int_a^b f(x)dx$, and for any c in $[a,b]$, $\int_c^c f(x)dx$ is zero. Thus

$$\int_b^a f(x)dx = -\int_a^b f(x)dx,\quad \int_a^a f(x)dx = 0 \text{ and } \int_b^b f(x)dx = 0.$$

We state without proof, Theorems which will be used in the subsequent development of the theory and in solving problems.

12.2.2 Theorem: Suppose that f and g are integrable on $[a,b]$. Then

(i) $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(ii) For any $\alpha \in R, \alpha f$ is integrable on $[a,b]$ and $\int_a^b (\alpha f)(x)dx = \alpha\int_a^b f(x)dx.$

12.2.3 Theorem: Let $f:[a,b]\to R$ be bounded. Let $c\in(a,b)$. Then f is integrable on $[a,b]$ if and onl if it is integrable on $[a,c]$ as well as on $[c,b]$ and, in this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

12.2.4 Theorem: Let $f:[a,b]\to R$ is continuous, then $f([a,b])$ is closed and bounded interval in R.

12.2.5 Theorem: Let f be integrable on $[a,b]$, then the function h, defined on $[a,b]$ as $h(x) = f(a+b-x)$ for all $x\in[a,b]$, is integrable on $[a,b]$ and $\int_a^b h(x)dx = \int_a^b f(x)dx.$

12.2.6 Theorem: Let f be integrable on $[0,a]$, then the function h, defined on $[0,a]$ as $h(x) = f(a-x)$ for all $x\in[a,b]$, is integrable on $[a,b]$ and

$$\int_0^a f(a-x)dx = \int_0^a h(x)dx = \int_0^a f(x)dx.$$

12.2.7 Theorem: Let $f:[-a,a]\to R$ be integrable on $[0,a]$, then the function f is either odd or even. Then f is integrable on $[-a,a]$ and

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_{0}^{a} f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$$

12.2.8 Theorem: Let $f:[0,2a]\to R$ be integrable on $[0,a]$.

(i) If $f(2a-x)=f(x)$ for all $x\in[0,2a]$, then function f is integrable on $[0,2a]$ and

$$\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(i) If $f(2a-x)=-f(x)$ for all $x\in[0,2a]$, then function f is integrable on $[0,2a]$ and

$$\int_{0}^{2a} f(x)dx = 0.$$

12.2.9 Theorem: If f and g are integrable on $[a,b]$, then their product fg is integrable on $[a,b]$.

12.2.10 Theorem: Let u and v be real valued differentiable functions on $[a,b]$ such that u' and v' are integrable on $[a,b]$ and $\int_a^b u(x)v'(x)dx = u(b)v(b)-u(a)v(a)-\int_a^b u'(x)v(x)dx.$

12.2.11 Solved Problems:

1. Problem: Evaluate $\int_0^2 x^4 dx$

Solution: Let $\mathrm{I}=\int_0^2 x^4 dx = \left(\dfrac{x^5}{5}\right)_0^2 = \dfrac{2^5}{5}-\dfrac{0^5}{5}=\dfrac{32}{5}-0=\dfrac{32}{5}$

2. Problem: Evaluate $\int_1^e \dfrac{1}{x}dx$

Solution: Let $\mathrm{I}=\int_1^e \dfrac{1}{x}dx = \left(\log x\right)_1^e = \log e-\log 1=1-0=1$

3. Problem: Evaluate $\int_4^9 \dfrac{1}{2\sqrt{x}}dx$

Solution: Let $\mathrm{I}=\int_4^9 \dfrac{1}{2\sqrt{x}}dx = \left(\sqrt{x}\right)_4^9 = \sqrt{9}-\sqrt{4}=3-2=1$

4. Problem: Evaluate $\int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^2}}dx$

Solution: Let $\mathrm{I}=\int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^2}}dx = \left(\sin^{-1} x\right)_0^{\frac{1}{2}} = \sin^{-1}\dfrac{1}{2}-\sin^{-1}0=\dfrac{\pi}{6}-0=\dfrac{\pi}{6}$

5. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin 2x}dx$

Solution: Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1+\sin 2x}dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}dx$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(\cos x + \sin x)^2}dx = \int_0^{\frac{\pi}{2}} (\cos x + \sin x)dx = \int_0^{\frac{\pi}{2}} \cos xdx + \int_0^{\frac{\pi}{2}} \sin xdx$$

$$= (\sin x - \cos x)_0^{\frac{\pi}{2}} = \left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (\sin 0 - \cos 0)$$

$$= (1-0)-(0-1) = 1+1 = 2$$

6. Problem: Evaluate $\int_0^1 \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}dx$

Solution: Let $I = \int_0^1 \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}dx = \int_0^1 \frac{\cos x - \sin x}{\sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}}dx$

$$= \int_0^1 \frac{\cos x - \sin x}{\sqrt{(\cos x - \sin x)^2}}dx = \int_0^1 \frac{\cos x - \sin x}{\cos x - \sin x}dx$$

$$= \int_0^1 1\, dx = (x)_0^1 = 1-0 = 1$$

7. Problem: Evaluate $\int_0^1 \frac{(\sin^{-1} x)^4}{\sqrt{1-x^2}}dx$

Solution: Let $I = \int_0^1 \frac{(\sin^{-1} x)^4}{\sqrt{1-x^2}}dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

U.L: $t = \sin^{-1} 1 = \frac{\pi}{2}$, L.L: $t = \sin^{-1} 0 = 0$

$$\therefore I = \int_0^{\frac{\pi}{2}} t^4 dt = \left(\frac{t^5}{5}\right)_0^{\frac{\pi}{2}} = \frac{1}{5}\left[\left(\frac{\pi}{2}\right)^5 - 0\right] = \frac{\pi}{160}^{5}$$

8. Problem: Evaluate $\int_0^1 \frac{2x+1}{x^2+x+1}dx$

Solution: Let $I = \int_0^1 \frac{2x+1}{x^2+x+1}dx$

Put $x^2+x+1 = t \Rightarrow (2x+1)dx = dt$

U.L: $t = 1^2+1+1 = 3$, L.L: $t = 0^2+0+1 = 1$

$$\therefore \text{I} = \int_1^3 \frac{1}{t} dt = \left(\log t\right)_1^3 = \log 3 - \log 1 = \log 3 - 0 = \log 3$$

9. Problem: Evaluate $\int_0^1 \frac{3x^2}{x^6+1} dx$

Solution: Let $\text{I} = \int_0^1 \frac{3x^2}{x^6+1} dx = \int_0^1 \frac{3x^2}{(x^3)^2+1} dx$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

U.L: $t = 1^3 = 1$, L.L: $t = 0^3 = 0$

$$\therefore \text{I} = \int_0^1 \frac{1}{1+t^2} dt = \left(\tan^{-1} t\right)_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

10. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin 8x \cos 3x dx$

Solution: Let $\text{I} = \int_0^{\frac{\pi}{2}} \sin 8x \cos 3x dx$

We have $2\sin \text{A} \cos \text{B} = \sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})$

$$\Rightarrow \sin \text{A} \cos \text{B} = \frac{1}{2}\left(\sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})\right)$$

Now $\text{I} = \int_0^{\frac{\pi}{2}} \sin 8x \cos 3x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin(8x+3x) + \sin(8x-3x)\right) dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin 11x + \sin 5x\right) dx = \frac{1}{2}\left[-\frac{\cos 11x}{11} - \frac{\cos 5x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}\left[\frac{\cos 11x}{11} + \frac{\cos 5x}{5}\right]_0^{\frac{\pi}{2}} = -\frac{1}{2}\left[\left(\frac{\cos\frac{11\pi}{2}}{11} + \frac{\cos\frac{5\pi}{2}}{5}\right) - \left(\frac{\cos 0}{11} + \frac{\cos 0}{5}\right)\right]$$

$$= -\frac{1}{2}\left[\left(\frac{0}{11} + \frac{0}{5}\right) - \left(\frac{1}{11} + \frac{1}{5}\right)\right] = -\frac{1}{2}\left[-\left(\frac{1}{11} + \frac{1}{5}\right)\right] = \frac{1}{2}\left(\frac{16}{55}\right) = \frac{8}{55}$$

11. Problem: Evaluate $\int_0^2 \frac{1}{x^2+4} dx$

Solution: Let $\text{I} = \int_0^2 \frac{1}{x^2+4} dx = \int_0^2 \frac{1}{x^2+2^2} dx$

$$=\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)_0^2=\frac{1}{2}\tan^{-1}\frac{2}{2}-\frac{1}{2}\tan^{-1}\frac{0}{2}$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0=\frac{1}{2}.\frac{\pi}{4}-\frac{1}{2}.0=\frac{\pi}{8}-0=\frac{\pi}{8}$$

12. Problem: Evaluate $\int_0^4\frac{1}{\sqrt{16-x^2}}dx$

Solution: Let $\mathrm{I}=\int_0^4\frac{1}{\sqrt{16-x^2}}dx\ =\int_0^4\frac{1}{\sqrt{4^2-x^2}}dx$

$$=\left(\sin^{-1}\frac{x}{4}\right)_0^4=\sin^{-1}\frac{4}{4}-\sin^{-1}\frac{0}{4}$$

$$=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}-0=\frac{\pi}{2}$$

13. Problem: Evaluate $\int_0^6\sqrt{36-x^2}dx$

Solution: Let $\mathrm{I}=\int_0^6\sqrt{36-x^2}dx\ =\int_0^6\sqrt{6^2-x^2}dx=\left(\frac{x}{2}\sqrt{6^2-x^2}+\frac{6^2}{2}\sin^{-1}\frac{x}{6}\right)_0^6$

$$=\left(\frac{6}{2}\sqrt{6^2-6^2}+\frac{6^2}{2}\sin^{-1}\frac{6}{6}\right)-\left(\frac{0}{2}\sqrt{6^2-0^2}+\frac{6^2}{2}\sin^{-1}\frac{0}{6}\right)$$

$$=\left(\frac{6}{2}\sqrt{6^2-6^2}+\frac{6^2}{2}\sin^{-1}\frac{6}{6}\right)-\left(\frac{0}{2}\sqrt{6^2-0^2}+\frac{6^2}{2}\sin^{-1}\frac{0}{6}\right)$$

$$=\left(0+18\sin^{-1}1\right)-\left(0+18\sin^{-1}0\right)$$

$$=18\sin^{-1}1-18\sin^{-1}0=18.\frac{\pi}{2}-18.0=9\pi-0=9\pi$$

14. Problem: Evaluate $\int_0^\infty\frac{1}{(x+1)(x+2)}dx$

Solution: Let $\mathrm{I}=\int_0^\infty\frac{1}{(x+1)(x+2)}dx\ =\int_0^\infty\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx=\left(\log(x+1)-\log(x+2)\right)_0^\infty$

$$=\left[\log\left(\frac{x+1}{x+2}\right)\right]_0^\infty\ =\log1-\log\frac{1}{2}\quad\left[\because \text{As } x\to\infty\Rightarrow\log\left(\frac{x+1}{x+2}\right)\to\log1\right]$$

$$=0-(-\log2)\quad\left[\because\log1=0,\log\frac{1}{2}=-\log2\right]$$

$$=\log2$$

15. Problem: Evaluate $\int_0^{\infty} \frac{1}{(x^2+25)(x^2+36)} dx$

Solution: Let $\mathrm{I} = \int_0^{\infty} \frac{1}{(x^2+25)(x^2+36)} dx = \frac{1}{11}\int_0^{\infty}\left(\frac{1}{x^2+5^2} - \frac{1}{x^2+6^2}\right) dx$

$$= \frac{1}{11}\left(\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) - \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)\right)_0^{\infty}$$

$$= \frac{1}{11}\left(\frac{1}{5}.\frac{\pi}{2} - \frac{1}{6}.\frac{\pi}{2}\right) - \frac{1}{11}\left(\frac{1}{5}.0 - \frac{1}{6}.0\right) = \frac{1}{22}\left(\frac{1}{5} - \frac{1}{6}\right) - \frac{1}{11}(0) = \frac{1}{660}$$

16. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x} dx$

Solution: Let $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x} dx$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$

U.L: $t = \tan\frac{\pi}{4} = 1$, L.L: $t = \tan 0 = 0$

Now $\mathrm{I} = \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)}.\frac{2dt}{1+t^2} = \int_0^1 \frac{1}{\frac{5+5t^2+4-4t^2}{1+t^2}}.\frac{2dt}{1+t^2} = 2\int_0^1 \frac{1}{9+t^2} dt$

$$= 2\int_0^1 \frac{1}{3^2+t^2} dt = 2\left(\frac{1}{3}\tan^{-1}\frac{t}{3}\right)_0^1 = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right) - \frac{2}{3}\tan^{-1}(0) = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$$

17. Problem: Evaluate $\int_0^1 x^2 e^x dx$

Solution: Let $\mathrm{I} = \int_0^1 x^2 e^x dx = \left(e^x(x^2-2x+2)\right)_0^1 = \left(e^1(1^2-2.1+2) - e^0(0^2-2.0+2)\right)$

$$= \left(e(1-2+2) - 1(0-0+2)\right) = e-2$$

18. Problem: Evaluate $\int_1^e \log x dx$

Solution: Let $\mathrm{I} = \int_1^e \log x dx = \left(x(\log x - 1)\right)_1^e$

$$= \left(e(\log e - 1) - 1(\log 1 - 1)\right) = \left(e(1-1) - 1(0-1)\right) = 0+1 = 1$$

19. Problem: Evaluate $\int_0^{\pi/2} x\sin x dx$

Solution: Let $\mathrm{I} = \int_0^{\pi/2} x\sin x dx$

$$= (-x\cos x)_0^{\pi/2} + \int_0^{\pi/2}\cos x dx = \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + 0\cos 0\right) + (\sin x)_0^{\pi/2}$$

$$= \left(-\frac{\pi}{2}.0 + 0.1\right) + \left(\sin\frac{\pi}{2} - \sin 0\right) = 1 - 0 = 1$$

20. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx$

Solution: Let $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx$

We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^7\left(\frac{\pi}{2} - x\right)}{\sin^7\left(\frac{\pi}{2} - x\right) + \cos^7\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^7 x}{\cos^7 x + \sin^7 x} dx$$

Now $\mathrm{I+I} = \int_0^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^7 x}{\cos^7 x + \sin^7 x} dx$

$$\Rightarrow 2\mathrm{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\sin^7 x}{\sin^7 x + \cos^7 x} + \frac{\cos^7 x}{\cos^7 x + \sin^7 x}\right) dx = \int_0^{\frac{\pi}{2}} \left(\frac{\sin^7 x + \cos^7 x}{\sin^7 x + \cos^7 x}\right) dx = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$= (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \mathrm{I} = \frac{\pi}{4}$$

21. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^{5/2} x}{\sin^{5/2} x + \cos^{5/2} x} dx$

Solution: Let $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^{5/2} x}{\sin^{5/2} x + \cos^{5/2} x} dx$

We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \text{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^{5/2}\left(\frac{\pi}{2}-x\right)}{\sin^{5/2}\left(\frac{\pi}{2}-x\right)+\cos^{5/2}\left(\frac{\pi}{2}-x\right)}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{5/2} x}{\cos^{5/2} x+\sin^{5/2} x}dx$$

$$\text{Now I+I} = \int_0^{\frac{\pi}{2}} \frac{\sin^{5/2} x}{\sin^{5/2} x+\cos^{5/2} x}dx + \int_0^{\frac{\pi}{2}} \frac{\cos^{5/2} x}{\cos^{5/2} x+\sin^{5/2} x}dx$$

$$\Rightarrow 2\text{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\sin^{5/2} x}{\sin^{5/2} x+\cos^{5/2} x} + \frac{\cos^{5/2} x}{\cos^{5/2} x+\sin^{5/2} x}\right)dx = \int_0^{\frac{\pi}{2}} \left(\frac{\sin^{5/2} x+\cos^{5/2} x}{\sin^{5/2} x+\cos^{5/2} x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} 1\, dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \text{I} = \frac{\pi}{4}$$

22. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

Solution: Let $\text{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \text{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\text{Now I+I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2\text{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$= (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \text{I} = \frac{\pi}{4}$$

23. Problem: Evaluate $\int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}dx$

Solution: Let $I = \int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}dx$

We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore I = \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{3-(3-x)}}dx = \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}}dx$

Now $I+I = \int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}dx + \int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}}dx$

$\Rightarrow 2I = \int_0^3 \left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}} + \frac{\sqrt{3-x}}{\sqrt{3-x}+\sqrt{x}} \right) dx = \int_0^3 \left(\frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}} \right) dx = \int_0^3 1\, dx$

$= (x)_0^3 = 3-0 = \frac{3}{2}$

$\therefore I = \frac{3}{4}$

Exercise 12(b)

I Evaluate the following integrals:

(1) $\int_0^{\frac{\pi}{2}} (\cos x - \sin x)dx$ (2) $\int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx$ (3) $\int_0^{\frac{\pi}{4}} (\sec^2 x + \sin x)dx$

(4) $\int_0^2 (x^2 + 2x + 3)dx$ (5) $\int_0^1 \frac{\cos x - \sin x}{\sqrt{1-\sin 2x}}dx$ (6) $\int_0^{\frac{\pi}{2}} \sqrt{1-\sin 2x}dx$ (7) $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}dx$

II Evaluate the following integrals:

(1) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$ (2) $\int_0^1 \frac{\exp(\sin^{-1} x)}{\sqrt{1-x^2}}dx$ (3) $\int_0^1 \frac{\sin(\tan^{-1} x)}{1+x^2}dx$

III Evaluate the following integrals:

(1) $\int_0^{\frac{\pi}{2}} \sin 6x \cos 2x dx$ (2) $\int_0^{\frac{\pi}{2}} \sin 7x \cos 5x dx$ (3) $\int_0^{\frac{\pi}{2}} \cos 2x \cos x dx$ (4) $\int_0^{\frac{\pi}{2}} \sin 7x \sin 2x dx$

IV Evaluate the following integrals:

(1) $\int_0^3 \frac{1}{x^2+9}dx$ (2) $\int_0^2 \frac{1}{9-x^2}dx$ (3) $\int_0^7 \frac{1}{\sqrt{49-x^2}}dx$ (4) $\int_0^2 \sqrt{4-x^2}$

V Evaluate the following integrals:

(1) $\int_0^{\frac{\pi}{2}} \frac{\sin^{10} x}{\sin^{10} x+\cos^{10} x}dx$ (2) $\int_0^{\frac{\pi}{2}} \frac{\cos^{12} x}{\cos^{12} x+\sin^{12} x}dx$ (3) $\int_0^{\frac{\pi}{2}} \frac{\tan^{15} x}{\tan^{15} x+\cot^{15} x}dx$

(4) $\int_0^{\frac{\pi}{2}} \frac{\sin^{9} x}{\sin^{9} x+\cos^{9} x}dx$ (5) $\int_0^{\frac{\pi}{2}} \frac{\cos ec^{8} x}{\cos ec^{8} x+\sec^{8} x}dx$ (6) $\int_0^{5} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$

12.3 Reduction formulae:

In this section, we state some reduction formulae without proofs for evaluation of the definite integral of $\sin^n x, \cos^n x, \sin^m x\cos^n x$ between 0 and $\frac{\pi}{2}$ for positive integers $m, n.$

The following theorem gives a useful formula to evaluate the definite integral of $\sin^n x$ between 0 and $\frac{\pi}{2}$ when n is an integer $\geq 2.$

12.3.1 Theorem: Let n be an integer greater than or equal to $2.$ Then

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$$

12.3.2 Theorem: Let n be an integer greater than or equal to $2.$ Then

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$$

12.3.3 Theorem: Let m and n be positive integers. Then

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x dx = \begin{cases} \frac{1}{m+1} & \text{if } n=1 \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{2}{m+3}.\frac{1}{m+1}, & \text{if } n\neq 1 \text{ is odd} \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{1}{m+2}.\frac{m-1}{m}.\frac{m-3}{m-2}....\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even and } m \text{ is odd} \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{1}{m+2}.\frac{m-1}{m}.\frac{m-3}{m-2}....\frac{2}{3}, & \text{if } n \text{ is even and } n\neq 1 \text{ is odd.} \\ \frac{1}{m+1} & \text{if } m=1. \end{cases}$$

12.3.4 Solved Problems:

1. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x dx$

Solution: Let $I_7 = \int_0^{\frac{\pi}{2}} \sin^7 x dx$

We have $\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$

Since $n = 7$ which is odd

$\therefore I_7 = \int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{7-1}{7}.\frac{7-3}{7-2}.\frac{7-5}{7-4}.1 = \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1 = \frac{16}{35}$

2. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^{10} x dx$

Solution: Let $I_{10} = \int_0^{\frac{\pi}{2}} \sin^{10} x dx$

We have $\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$

Since $n = 10$ which is even

$\therefore I_{10} = \int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{10-1}{10}.\frac{10-3}{10-2}.\frac{10-5}{10-4}.\frac{10-7}{10-6}.\frac{10-9}{10-8}.\frac{\pi}{2}$

$= \frac{9}{10}.\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{63\pi}{512}$

3. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \cos^9 x dx$

Solution: Let $I_9 = \int_0^{\frac{\pi}{2}} \cos^9 x dx$

We have $\int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$

Since $n = 9$ which is odd

$$\therefore I_9 = \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{9-1}{9}.\frac{9-3}{9-2}.\frac{9-5}{9-4}.\frac{9-7}{9-6}.1 = \frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1 = \frac{128}{315}$$

4. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \cos^8 x dx$

Solution: Let $I_8 = \int_0^{\frac{\pi}{2}} \cos^8 x dx$

We have $$\int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$$

Since $n = 8$ which is even

$$\therefore I_8 = \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{8-1}{8}.\frac{8-3}{8-2}.\frac{8-5}{8-4}.\frac{8-7}{8-6}.\frac{\pi}{2}$$

$$= \frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{35\pi}{256}$$

5. Problem: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$

Solution: Let $I_7 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$

Take $f(x) = \sin^7 x \Rightarrow f(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x = -f(x)$

$\therefore f(x)$ is an odd function.

We have $$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$$

6. Problem: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{10} x dx$

Solution: Let $I_{10} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{10} x dx$

Take $f(x) = \sin^{10} x \Rightarrow f(-x) = (\sin(-x))^{10} = (-\sin x)^{10} = \sin^{10} x = f(x)$

$\therefore f(x)$ is an even function.

We have $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{10} x dx = 2\int_0^{\frac{\pi}{2}} \sin^{10} x dx = 2\left(\frac{63\pi}{512}\right) [\because \text{by problem (2)}]$$

$$= \frac{63\pi}{256}$$

7. Problem: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^9 x dx$

Solution: Let $I_9 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^9 x dx$

Take $f(x) = \cos^9 x \Rightarrow f(-x) = (\cos(-x))^9 = (\cos x)^9 = \cos^9 x = f(x)$

$\therefore f(x)$ is an even function.

We have $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^9 x dx = 2 \int_0^{\frac{\pi}{2}} \cos^9 x dx = 2\left(\frac{128}{315}\right) [\because \text{by problem (3)}]$$

$$= \frac{256}{315}$$

8. Problem: Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^8 x dx$

Solution: Let $I_8 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^8 x dx$

Take $f(x) = \cos^8 x \Rightarrow f(-x) = (\cos(-x))^8 = (\cos x)^8 = \cos^8 x = f(x)$

$\therefore f(x)$ is an even function.

We have $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function} \end{cases}$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^8 x dx = 2 \int_0^{\frac{\pi}{2}} \cos^8 x dx = 2\left(\frac{35\pi}{256}\right) [\because \text{by problem (4)}]$$

$$= \frac{35\pi}{128}$$

9. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x dx$

Solution: Let $I_{7,5} = \int_0^{\frac{\pi}{2}} \sin^7 x \cos^5 x dx$

We have $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

Now $I_{7,5} = \frac{5-1}{7+5} I_{7,5-2} = \frac{4}{12} I_{7,3} = \frac{4}{12}\left(\frac{3-1}{7+3} I_{7,3-2}\right) = \frac{1}{3}.\frac{2}{10} I_{7,1}$

$$= \frac{1}{15} I_{7,1} = \frac{1}{15}.\frac{1}{8}\left[\because I_{7,1} = \int_0^{\frac{\pi}{2}} \sin^7 x \cos x dx = \left(\frac{\sin^8 x}{8}\right)_0^{\pi/2} = \frac{1}{8}\left(\sin^8 \frac{\pi}{2} - \sin^8 0\right) = \frac{1}{8}\right]$$

$$= \frac{1}{120}$$

10. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx$

Solution: Let $I_{6,4} = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx$

We have $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

Now $I_{6,4} = \frac{4-1}{6+4} I_{6,4-2} = \frac{3}{10} I_{6,2} = \frac{3}{10}\left(\frac{2-1}{6+2} I_{6,2-2}\right) = \frac{3}{10}.\frac{1}{8} I_{6,0}$

$$= \frac{3}{80} I_6 = \frac{3}{80}.\frac{5\pi}{32}\left[\because I_6 = \int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{32}\right]$$

$$= \frac{3\pi}{512}$$

11. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx$

Solution: Let $I_{3,4} = \int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx$

We have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\Rightarrow I_{3,4} = \int_0^{\frac{\pi}{2}} \sin^3\left(\frac{\pi}{2} - x\right)\cos^4\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x dx = I_{4,3}$$

We have $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

Now $I_{3,4} = I_{4,3} = \frac{3-1}{4+3} I_{4,3-2} = \frac{2}{7} I_{4,1}$

$$= \frac{2}{7}.\frac{1}{5}\left[\because I_{4,1} = \int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx = \left(\frac{\sin^5 x}{5}\right)_0^{\pi/2} = \frac{1}{5}\left(\sin^5 \frac{\pi}{2} - \sin^5 0\right) = \frac{1}{5}\right]$$

$$= \frac{2}{35}$$

12. Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x dx$

Solution: Let $I_{4,4} = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x dx$

We have $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

Now $I_{4,4} = \frac{4-1}{4+4} I_{4,4-2} = \frac{3}{8} I_{4,2} = \frac{3}{8}\left(\frac{2-1}{4+2} I_{4,2-2}\right) = \frac{3}{8}.\frac{1}{6} I_{4,0} = \frac{1}{16} I_4$

$$= \frac{1}{16}.\frac{3\pi}{16}\left[\because I_{4,0} = I_4 = \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi}{16}\right]$$

$$= \frac{3\pi}{256}$$

13. Problem: Evaluate $\int_0^2 x^{3/2}\sqrt{2-x}dx$

Solution: Let $I = \int_0^2 x^{3/2}\sqrt{2-x}dx$

Put $x = 2\sin^2\theta \Rightarrow dx = 4\sin\theta\cos\theta d\theta,$

$\sqrt{2-x} = \sqrt{2-2\sin^2\theta} = \sqrt{2(1-\sin^2\theta)} = \sqrt{2\cos^2\theta} = \sqrt{2}\cos\theta$

U.L: $2\sin^2\theta = 2 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

L.L: $2\sin^2\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$$\therefore I = \int_0^{\pi/2} (2\sin^2\theta)^{3/2}.\sqrt{2}\cos\theta.4\sin\theta\cos\theta d\theta = 16\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta = 16 I_{4,2}$$

$$= 16.\frac{3\pi}{32}\left[\because I_{4,2} = \frac{4-1}{4+2} I_{4,2-2} = \frac{3}{6} I_{4,0} = \frac{3}{6} I_4 = \frac{3}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi}{32}\right] = \frac{3\pi}{2}$$

Exercise 12(c)

I Evaluate the following integrals:

(1) $\int_0^{\frac{\pi}{2}} \sin^4 x dx$ (2) $\int_0^{\frac{\pi}{2}} \sin^5 x dx$ (3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x dx$ (4) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6 x dx$ (5) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x dx$

(6) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x dx$ (7) $\int_0^{\frac{\pi}{2}} \cos^4 x dx$ (8) $\int_0^{\frac{\pi}{2}} \cos^5 x dx$ (9) $\int_0^{\frac{\pi}{2}} \sin^{11} x dx$ (10) $\int_0^{\frac{\pi}{2}} \cos^{10} x dx$

II Evaluate the following integrals:

(1) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x dx$ (2) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{10} x dx$ (3) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x dx$ (4) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^6 x dx$

(5) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x dx$ (6) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx$ (7) $\int_0^1 x^{5/2} \sqrt{1-x} dx$ (8) $\int_0^1 x^6 \sqrt{1-x^2} dx$

Key concepts

1. If f is the derivative of F, then $\int_a^b f(x)dx = F(x) + c$, where c is a real constant. In this case F called a primitive of f.

2. If f is integrable on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. Let $f:[a,b] \to R$ be an integrable on $[a,b]$. Then we define $\int_b^a f(x)dx$ as the negative of $\int_a^b f(x)dx$, and for any c in $[a,b]$, $\int_c^c f(x)dx$ is zero. Thus $\int_b^a f(x)dx = -\int_a^b f(x)dx$, $\int_a^a f(x)dx = 0$ and $\int_b^b f(x)dx = 0$.

4. Suppose that f and g are integrable on $[a,b]$. Then

(i) $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(ii) For any $\alpha \in R, \alpha f$ is integrable on $[a,b]$ and $\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx.$

5. Let $f:[a,b]\to R$ be bounded. Let $c\in(a,b)$. Then f is integrable on $[a,b]$ if and onl if it is integrable on $[a,c]$ as well as on $[c,b]$ and, in this case,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

6. Let $f:[a,b]\to R$ is continuous, then $f([a,b])$ is closed and bounded interval in R.

7. Let f be integrable on $[a,b]$, then the function $h,$ defined on $[a,b]$ as $h(x)=f(a+b-x)$ for all $x\in[a,b],$ is integrable on $[a,b]$ and $\int_a^b h(x)dx=\int_a^b f(x)dx.$

8. Let f be integrable on $[0,a]$, then the function $h,$ defined on $[0,a]$ as $h(x)=f(a-x)$ for all $x\in[a,b],$ is integrable on $[a,b]$ and

$$\int_0^a f(a-x)dx=\int_0^a h(x)dx=\int_0^a f(x)dx.$$

9. Let $f:[-a,a]\to R$ be integrable on $[0,a]$, then the function f is either odd or even. Then f is integrable on $[-a,a]$ and

$$\int_{-a}^{a} f(x)dx=\begin{cases}0 & \text{if } f(x) \text{ is an odd function}\\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function}\end{cases}$$

10. Let $f:[0,2a]\to R$ be integrable on $[0,a]$.

(i) If $f(2a-x)=f(x)$ for all $x\in[0,2a],$ then function f is integrable on $[0,2a]$ and

$$\int_0^{2a} f(x)dx=2\int_0^a f(x)dx.$$

(ii) If $f(2a-x)=-f(x)$ for all $x\in[0,2a],$ then function f is integrable on $[0,2a]$ and

$$\int_0^{2a} f(x)dx=0.$$

11. If f and g are integrable on $[a,b]$, then their product fg is integrable on $[a,b]$.

12. Let u and v be real valued differentiable functions on $[a,b]$ such that u' and v' are integrable on $[a,b]$ and $\int_a^b u(x)v'(x)dx=u(b)v(b)-u(a)v(a)-\int_a^b u'(x)v(x)dx.$

13. Let n be an integer greater than or equal to $2.$ Then

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$$

14. Let n be an integer greater than or equal to $2.$ Then

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}...\frac{2}{3}.1, & \text{if } n \text{ is odd.} \end{cases}$$

15. Let m and n be positive integers. Then

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \begin{cases} \frac{1}{m+1} & \text{if } n=1 \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{2}{m+3}.\frac{1}{m+1}, & \text{if } n \neq 1 \text{ is odd} \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{1}{m+2}.\frac{m-1}{m}.\frac{m-3}{m-2}....\frac{1}{2}.\frac{\pi}{2}, & \text{if } n \text{ is even and } m \text{ is odd} \\ \frac{n-1}{m+n}.\frac{n-3}{m+n-2}....\frac{1}{m+2}.\frac{m-1}{m}.\frac{m-3}{m-2}....\frac{2}{3}, & \text{if } n \text{ is even and } n \neq 1 \text{ is odd.} \\ \frac{1}{m+1} & \text{if } m=1. \end{cases}$$

Answers

Exercise 12(a)

I (1) 0 (2) 0 (3) $\frac{\pi}{2}$ (4) $\frac{38}{3}$ (5) $\frac{\pi}{4}$ (6) $\frac{\pi}{12}$

Exercise 12(b)

I (1) 0 (2) $\sqrt{2}-1$ (3) $2-\frac{1}{\sqrt{2}}$ (4) $\frac{38}{3}$ (5) 1 (6) 2 (7) $\sqrt{2}$

II (1) $\frac{\pi^2}{8}$ (2) $e^{\pi/2}-1$ (3) $1-\frac{1}{\sqrt{2}}$

III (1) $\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $\frac{2}{45}$

IV (1) $\frac{\pi}{12}$ (2) $\frac{1}{6}\log 5$ (3) $\frac{\pi}{2}$ (4) π

V (1) $\frac{\pi}{4}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{4}$ (5) $\frac{\pi}{4}$ (6) $\frac{5}{2}$

Exercise 12(c)

I (1) $\frac{3\pi}{16}$ (2) $\frac{8}{15}$ (3) $\frac{16}{15}$ (4) $\frac{5\pi}{16}$ (5) $\frac{16}{15}$ (6) $\frac{3\pi}{8}$ (7) $\frac{3\pi}{16}$ (8) $\frac{8}{15}$ (9) $\frac{256}{693}$ (10) $\frac{63\pi}{512}$

II (1) $\frac{2}{63}$ (2) $\frac{2}{143}$ (3) $\frac{1}{12}$ (4) $\frac{2}{63}$ (5) $\frac{1}{24}$ (6) $\frac{2}{15}$ (7) $\frac{5\pi}{128}$ (8) $\frac{5\pi}{256}$

13. DIFFERENTIAL EQUATIONS

Introduction

Differential equations have applications in many branches of applications in many branches of Physics, Physical chemistry etc. In this chapter we study some basic concepts of differential equations and learn how to solve simple differential equations.

13.1 Formation of differential equations- Degree and order of an ordinary differential equation:

The present section is aimed at defining an ordinary differential equation, forming such an equation from a given family of surfaces or curves. We also define two concepts, namely order and degree of an ordinary differential equation.

13.1.1 Definition: An equation involving derivatives with one dependent variable and two or more independent variables is called a **differential equation**.

Examples: (1) $x\frac{dz}{dx}+y\frac{dz}{dy}=0$

(2) $x\left(\frac{dz}{dx}\right)^2+\frac{d^2z}{dx^2}+\frac{dz}{dx}+\frac{dz}{dy}=4x+2y+2$

13.1.2 Definition: An equation involving derivatives with one dependent variable and exactly one independent variables is called an **ordinary differential equation**.

Examples: (1) $x\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4x\frac{dy}{dx}=3x+2y$

(2) $x\left(\frac{dz}{dx}\right)^2+\frac{d^2z}{dx^2}+\frac{dz}{dx}=4x+2y+2$

13.1.3 Definition: If a differential equation contains n^{th} and lower order partial derivatives then the **order of the differential equation** is n

Examples: (1) $x\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4x\frac{dy}{dx}=3x+2y$ order=2

(2) $x\left(\frac{dz}{dx}\right)^2+\frac{d^3z}{dx^3}+\frac{dz}{dx}=4x+2y+2$ order=3

13.1.4 Definition: The **degree of the differential equation** is the greatest power of the highest order derivative involved in it.

Examples: (1) $x\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4x\frac{dy}{dx}=3x+2y$ degree=1

(2) $x\left(\frac{d^3z}{dx^3}\right)^2+\left(\frac{d^3z}{dx^3}\right)^4+\frac{d^2z}{dx^2}+\frac{dz}{dx}=4x+2y+2$ degree=4

(3) $\left(\frac{d^3y}{dx^3}\right)^2+3\left(\frac{d^2z}{dx^2}\right)^4-3=0$ degree=2

13.1.5 Examples:

1. Order and degree of $\frac{dy}{dx}=\frac{x^{1/2}}{y^{1/2}\left(1+x^{1/2}\right)}$ are 1and 1 respectively.

2. Order and degree of $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$ are 2and 3 respectively.

3. Order and degree of $1+\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ are 2and 4 respectively.

4. Order and degree of $\left[\left(\frac{d^2y}{dx^2}\right)^{1/3}+\left(\frac{dy}{dx}\right)^{1/2}\right]^{1/4}=0$ are 2and 2 respectively.

5. Order of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\log\left(\frac{dy}{dx}\right)$ is 2and degree is not defined since the equation cannot be expressed as a polynomial equation in the derivatives.

13.1.6 Note: The general form of an ordinary differential equation of n^{th} order is

$$\mathrm{F}\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots\frac{d^ny}{dx^n}\right)=0 \text{ or } \mathrm{F}\left(x,y,y^{(1)},y^{(2)},\ldots y^{(n)}\right)=0.$$

13.1.7 Elimination of arbitrary constants:

Suppose that an equation $y=\phi\left(x,\alpha_1,\alpha_2,\ldots\alpha_n\right)$ ...(I) where $\alpha_1,\alpha_2,\ldots\alpha_n$ are parameters (or arbitrary constants), representing a family of curves is given. Then by successively differentiating (I), a differential equation of the form $\mathrm{F}\left(x,y,y^{(1)},y^{(2)},\ldots y^{(n)}\right)=0$ can be formed by eliminating the parameters $\alpha_1,\alpha_2,\ldots\alpha_n$. This process is called formation of a differential equation by elimination of parameters (or arbitrary constants**).**

13.1.8 Solved Problems:

1. Problem: Find the order and degree of the differential equation $\frac{d^2y}{dx^2}=-p^2y$

Solution: The given differential equation is $\frac{d^2y}{dx^2}=-p^2y$

Which is a polynomial equation in $\frac{d^2y}{dx^2}$. Hence degree is 1. Since $\frac{d^2y}{dx^2}$ is the highest derivative occurring in the equation, its order is 2.

2. Problem: Find the order and degree of the differential equation

$$\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 - e^x = 4.$$

Solution: The given differential equation is $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 - e^x = 4.$

Which is a polynomial equation in $\frac{dy}{dx}$ and $\frac{d^3y}{dx^3}$. The exponent of $\frac{d^3y}{dx^3}$ is 2.

Hence degree is 2. Since $\frac{d^3y}{dx^3}$ is the highest derivative occurring in the equation, its order is 3.

3. Problem: Find the order and degree of the differential equation

$$x^{1/2}\left(\frac{d^2y}{dx^2}\right)^{1/3} + x\frac{dy}{dx} + y = 0.$$

Solution: The given differential equation is $x^{1/2}\left(\frac{d^2y}{dx^2}\right)^{1/3} + x\frac{dy}{dx} + y = 0.$

It can be rewritten as $\left(x\frac{dy}{dx} + y\right)^3 = -x^{1/2}\left(\frac{d^2y}{dx^2}\right)$

Which is a polynomial equation in $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. The exponent of $\frac{d^2y}{dx^2}$ is 1.

Hence degree is 1. Since $\frac{d^2y}{dx^2}$ is the highest derivative occurring in the equation, its order is 2.

4. Problem: Form the differential equation **by** eliminate the arbitrary constants c from the equation $y = cx - 2c^2$

Solution: The given equation is $y = cx - 2c^2$ ....(I)

Differentiate the above equation w.r.t x on both sides we get

$\frac{dy}{dx} = c$ ....(II)

From equations (I) and (II) we get

$$y = x\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right)^2$$

Hence the required differential equation is $y = x\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right)^2$

5. Problem: Eliminate the arbitrary constants a, b from the equation $y = ae^x + be^{-x}$

Solution: The given equation is $y = ae^{x} + be^{-x}$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = ae^{x} - be^{-x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} = ae^{x} + be^{-x} \Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - y = 0$

6. Problem: Eliminate the arbitrary constants a, b from the equation $y = ae^{2x} + be^{-2x}$

Solution: The given equation is $y = ae^{2x} + be^{-2x}$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = 2ae^{2x} - 2be^{-2x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 4be^{-2x} \Rightarrow \frac{d^2y}{dx^2} = 4\left(ae^{2x} + be^{-2x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - 4y = 0$.

7. Problem: Eliminate the arbitrary constants a, b from the equation $y = ae^{3x} + be^{-3x}$

Solution: The given equation is $y = ae^{3x} + be^{-3x}$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = 3ae^{3x} - 3be^{-3x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} = 9ae^{3x} + 9be^{-3x} \Rightarrow \frac{d^2y}{dx^2} = 9\left(ae^{3x} + be^{-3x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 9y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 9y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - 9y = 0$.

8. Problem: Eliminate the arbitrary constants a, b from the equation $y = a\cos x + b\sin x$

Solution: The given equation is $y = a\cos x + b\sin x$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = -a\sin x + b\cos x$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x \Rightarrow \frac{d^2y}{dx^2} = -\left(a\cos x + b\sin x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} + y = 0$.

9. Problem: Eliminate the arbitrary constants a, b from the equation $y = a\cos 2x + b\sin 2x$

Solution: The given equation is $y = a\cos 2x + b\sin 2x$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = -2a\sin 2x + 2b\cos 2x$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} = -4a\cos 2x - 4b\sin 2x \Rightarrow \frac{d^2y}{dx^2} = -4\left(a\cos 2x + b\sin 2x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -4y$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} + 4y = 0$.

10. Problem: Eliminate the arbitrary constants a, b from the equation $y = a\cos 3x + b\sin 3x$

Solution: The given equation is $y = a\cos 3x + b\sin 3x$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = -3a\sin 3x + 3b\cos 3x$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2}=-9a\cos 3x-9b\sin 3x \Rightarrow \frac{d^2y}{dx^2}=-9\left(a\cos 3x+b\sin 3x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}=-9y$$

$$\Rightarrow \frac{d^2y}{dx^2}+9y=0$$

Hence the required differential equation is $\frac{d^2y}{dx^2}+9y=0$

11. Problem: Eliminate the arbitrary constants a,b from the equation $y=ae^{2x}+be^{3x}$

Solution: The given equation is $y=ae^{2x}+be^{3x}$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx}=2ae^{2x}+3be^{3x}$$

$$\Rightarrow \frac{dy}{dx}=2\left(ae^{2x}+be^{3x}\right)+be^{3x}$$

$$\Rightarrow \frac{dy}{dx}=2y+be^{3x}$$

$$\Rightarrow \frac{dy}{dx}-2y=be^{3x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}=3be^{3x} \Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}=3\left(\frac{dy}{dx}-2y\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}=3\frac{dy}{dx}-6y$$

$$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}-3\frac{dy}{dx}+6y=0$$

$$\Rightarrow \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Hence the required differential equation is $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$.

12. Problem: Eliminate the arbitrary constants a,b from the equation $y=ae^{4x}+be^{-3x}$

Solution: The given equation is $y=ae^{4x}+be^{-3x}$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx}=4ae^{4x}-3be^{-3x}$$

$$\Rightarrow \frac{dy}{dx}=4\left(ae^{4x}+be^{-3x}\right)-7be^{-3x}$$

$$\Rightarrow \frac{dy}{dx} = 4y - 7be^{-3x}$$

$$\Rightarrow \frac{dy}{dx} - 4y = -7be^{-3x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -7\left(-3be^{-3x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -3\left(-7be^{-3x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -3\left(\frac{dy}{dx} - 4y\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = -3\frac{dy}{dx} + 12y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3\frac{dy}{dx} - 12y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$.

13. Problem: Eliminate the arbitrary constants a, b from the equation $xy = ae^x + be^{-x} + x^2$

Solution: The given equation is $xy = ae^x + be^{-x} + x^2$

$$\Rightarrow xy - x^2 = ae^x + be^{-x}$$

Differentiate the above equation w.r.t x on both sides we get

$$x\frac{dy}{dx} - 2x = ae^x - be^{-x}$$

Again differentiate the above equation w.r.t x on both sides we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2 = ae^x + be^{-x}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2 = xy - x^2$$

Hence the required differential equation is $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2 = xy - x^2$

14. Problem: Eliminate the arbitrary constants a, b from the equation $y = ae^x + bxe^x$

Solution: The given equation is $y = ae^x + bxe^x$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = ae^x + b\left(xe^x + e^x.1\right)$$

$$\Rightarrow \frac{dy}{dx} = ae^x + bxe^x + be^x$$
$$\Rightarrow \frac{dy}{dx} = y + be^x$$
$$\Rightarrow \frac{dy}{dx} - y = be^x$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = be^x$$
$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y$$
$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$.

15. Problem: Eliminate the arbitrary constants a, b from the equation $y = e^x\left(a\cos 2x + b\sin 2x\right)$

Solution: The given equation is $y = e^x\left(a\cos 2x + b\sin 2x\right)$

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = \left(a\cos 2x + b\sin 2x\right).e^x + e^x\left(-2a\sin 2x + 2b\cos 2x\right)$$
$$\Rightarrow \frac{dy}{dx} = e^x\left(a\cos 2x + b\sin 2x\right) + e^x\left(-2a\sin 2x + 2b\cos 2x\right)$$
$$\Rightarrow \frac{dy}{dx} = y + e^x\left(-2a\sin 2x + 2b\cos 2x\right)$$
$$\Rightarrow \frac{dy}{dx} - y = e^x\left(-2a\sin 2x + 2b\cos 2x\right)$$

Again differentiate the above equation w.r.t x on both sides we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = \left(-2a\sin 2x + 2b\cos 2x\right).e^x + e^x\left(-4a\cos 2x - 4b\sin 2x\right)$$
$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x\left(-2a\sin 2x + 2b\cos 2x\right) - 4\, e^x\left(a\cos 2x + b\sin 2x\right)$$
$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \left(\frac{dy}{dx} - y\right) - 4\, y$$
$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - 4\, y$$
$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} + 5y = 0$$
$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

Hence the required differential equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$.

16. Problem: Eliminate the arbitrary constant a from the equation $y = a\sin^{-1} x$

Solution: The given equation is $y = a\sin^{-1} x$ (I)

Differentiate the above equation w.r.t x on both sides we get

$$\frac{dy}{dx} = a.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = a \qquad \text{(II)}$$

Eliminate a from the above equations we get

$$y = \sqrt{1-x^2}\,\frac{dy}{dx}\sin^{-1} x$$

$$\Rightarrow\ y\frac{dx}{dy} = \sqrt{1-x^2}\,\sin^{-1} x$$

Hence the required differential equation is $y\frac{dx}{dy} = \sqrt{1-x^2}\,\sin^{-1} x$.

Exercise 13(a)

I Find the order and degree of the following differential equations:

1. $\frac{dy}{dx} = \frac{x^{1/2}}{y^{1/2}\left(1+x^{1/2}\right)}$ 2. $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$ 3. $1+\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

4. $\left[\left(\frac{d^2y}{dx^2}\right)^{1/3} + \left(\frac{dy}{dx}\right)^{1/2}\right]^{1/4} = 0$ 5. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \log\left(\frac{dy}{dx}\right)$ 6. $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\right]^{6/5} = 6y.$

II Eliminate the arbitrary constants from the following equations:

1. $y = ae^{4x} + be^{-4x}$ 2. $y = ae^{5x} + be^{-5x}$ 3. $y = ae^{nx} + be^{-nx}$

4. $y = a\cos 4x + b\sin 4x$ 5. $y = a\cos 5x + b\sin 5x$ 6. $y = a\cos nx + b\sin nx$

7. $y = ae^{x} + be^{-2x}$ 8. $y = ae^{-4x} + be^{3x}$ 9. $y = e^{2x}\left(a\cos 3x + b\sin 3x\right)$

10. $y = e^{mx}\left(a\cos nx + b\sin nx\right)$ 11. $y = c\left(x-c\right)^2$

13.1.9 Solution of a differential equation:

Solution: A solution of a differential equation is a relation between dependent variable, independent variables and along with some arbitrary constants satisfying the differential equation.

General solution: A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution.

Particular solution: A particular solution of a differential equation is a solution obtained by giving particular values to the arbitrary constants in the general solution.

13.2 Solving differential equations:

In this section we discuss methods to solve some first order first degree differential equations. Since first order first degree differential equations contains terms like $\frac{dy}{dx}$ and some terms involving x and y, a general first order first degree differential equation is of the form $\frac{dy}{dx} = f(x, y)$, where f is a function of x and y.

Throughout our discussion in the rest of the chapter, unless otherwise mentioned, a differential equation means a first order first degree ordinary differential equation.

13.2(a) Variables separable method:

If the differential equation is of the form $\frac{dy}{dx} = f(x, y)$ can be expressed as $\frac{dy}{dx} = F(x)G(y)$ then it can be solved by the method of **variables separable**

Examples: (1) $\frac{dy}{dx} = \frac{2x+1}{3y+2}$

(2) $\frac{dy}{dx} = \sin 2x.\cos 3y$

13.2(a).1 Solved Problems:

1. Problem: Solve $\frac{dy}{dx} = \frac{x}{y}$

Solution: The given equation is $\frac{dy}{dx} = \frac{x}{y}$

Which is a first order and first degree differentialequation

It can be solved by the method of variables separable

$ydy = xdx$

$\Rightarrow xdx - ydy = 0$

Now taking integration on both sides we get

$\int xdx - \int ydy = \int 0$

$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = c$

$\Rightarrow x^2 - y^2 = 2c$

Hence the required solution is $x^2 - y^2 = 2c$.

2. Problem: Solve $\frac{dy}{dx} = \frac{y}{x}$

Solution: The given equation is $\frac{dy}{dx} = \frac{y}{x}$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$$\frac{dy}{y}=\frac{dx}{x}$$

$$\Rightarrow \frac{dy}{y}-\frac{dx}{x}=0$$

Now taking integration on both sides we get

$$\int\frac{dy}{y}-\int\frac{dx}{x}=\int 0$$

$$\Rightarrow \log y-\log x=\log c$$

$$\Rightarrow \log\frac{y}{x}=\log c \Rightarrow \frac{y}{x}=c \Rightarrow y=cx$$

Hence the required solution is $y=cx$.

3. Problem: Solve $\frac{dy}{dx}=\frac{x+1}{y+2}$

Solution: The given equation is $\frac{dy}{dx}=\frac{x+1}{y+2}$

Which is a first order and first degree differential equation
It can be solved by the method of variables separable

$$(y+2)dy=(x+1)dx$$

$$\Rightarrow (x+1)dx-(y+2)dy=0$$

Now taking integration on both sides we get

$$\int(x+1)dx-\int(y+2)dy=\int 0$$

$$\Rightarrow \left(\frac{x^2}{2}+x\right)-\left(\frac{y^2}{2}+2y\right)=c$$

$$\Rightarrow \frac{x^2}{2}-\frac{y^2}{2}+x-2y=c$$

$$\Rightarrow x^2-y^2+2x-4y=C$$

Hence the required solution is $x^2-y^2+2x-4y=C$.

4. Problem: Solve $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

Solution: The given equation is $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

Which is a first order and first degree differential equation
It can be solved by the method of variables separable

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}-\frac{dx}{\sqrt{1-x^2}}=0$$

Now taking integration on both sides we get

$$\int\frac{dy}{\sqrt{1-y^2}}-\int\frac{dx}{\sqrt{1-x^2}}=\int 0$$

$\Rightarrow \sin^{-1} y - \sin^{-1} x = c$

Hence the required solution is $\sin^{-1} y - \sin^{-1} x = c$.

5. Problem: Solve $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

Solution: The given equation is $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$

$\Rightarrow \frac{dy}{\sqrt{1+y^2}} - \frac{dx}{\sqrt{1+x^2}} = 0$

Now taking integration on both sides we get

$\int \frac{dy}{\sqrt{1+y^2}} - \int \frac{dx}{\sqrt{1+x^2}} = \int 0$

$\Rightarrow \sinh^{-1} y - \sinh^{-1} x = c$

Hence the required solution is $\sinh^{-1} y - \sinh^{-1} x = c$

6. Problem: Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution: The given equation is $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\Rightarrow \frac{dy}{1+y^2} - \frac{dx}{1+x^2} = 0$

Now taking integration on both sides we get

$\int \frac{dy}{1+y^2} - \int \frac{dx}{1+x^2} = \int 0$

$\Rightarrow \tan^{-1} y - \tan^{-1} x = c$

Hence the required solution is $\tan^{-1} y - \tan^{-1} x = c$.

7. Problem: Solve $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$

Solution: The given equation is $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$

$$\Rightarrow \left(1+y^2\right)dy=\left(1+x^2\right)dx$$
$$\Rightarrow \left(1+x^2\right)dx-\left(1+y^2\right)dy=0$$
Now taking integration on both sides we get
$$\Rightarrow \int\left(1+x^2\right)dx-\int\left(1+y^2\right)dy=\int 0$$
$$\Rightarrow \left(x+\frac{x^3}{3}\right)-\left(y+\frac{y^3}{3}\right)=c$$
$$\Rightarrow x^3-y^3+3x-3y=C$$
Hence the required solution is $x^3-y^3+3x-3y=C$

8. Problem: Solve $\frac{dy}{dx}=e^{x+y}$

Solution: The given equation is $\frac{dy}{dx}=e^{x+y}$

Which is a first order and first degree differential equation
It can be solved by the method of variables separable
$$\frac{dy}{dx}=e^x e^y$$
$$\Rightarrow \frac{dy}{e^y}=e^x dx$$
$$\Rightarrow e^{-y}dy=e^x dx$$
$$\Rightarrow e^x dx-e^{-y}dy=0$$
Now taking integration on both sides we get
$$\Rightarrow \int e^x dx-\int e^{-y}dy=\int 0$$
$$\Rightarrow e^x-(-e^{-y})=c$$
$$\Rightarrow e^x+e^{-y}=c$$
Hence the required solution is $e^x+e^{-y}=c$.

9. Problem: Solve $y(1+x)dx+x(1+y)dy=0$

Solution: The given equation is $y(1+x)dx+x(1+y)dy=0$

Which is a first order and first degree differential equation
It can be solved by the method of variables separable
$$y(1+x)dx+x(1+y)dy=0$$
$$\Rightarrow y(1+x)dx=-x(1+y)dy$$
$$\Rightarrow \frac{(1+x)}{x}dx=-\frac{(1+y)}{y}dy$$
$$\Rightarrow \left(1+\frac{1}{x}\right)dx+\left(1+\frac{1}{y}\right)dy=0$$
Now taking integration on both sides we get
$$\Rightarrow \int\left(1+\frac{1}{x}\right)dx+\int\left(1+\frac{1}{y}\right)dy=\int 0$$
$$\Rightarrow x+\log x+y+\log y=\log c$$
$$\Rightarrow x+y+\log xy=\log c$$

$$\Rightarrow x+y=\log\frac{c}{xy}$$

$$\Rightarrow e^{x+y}=\frac{c}{xy}$$

$$\Rightarrow xye^{x+y}=c$$

Hence the required solution is $xye^{x+y}=c$.

10. Problem: Solve $(1-x^2)\frac{dy}{dx}+xy=5x$

Solution: The given equation is $(1-x^2)\frac{dy}{dx}+xy=5x$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$$(1-x^2)\frac{dy}{dx}=5x-xy$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=\left(5-y\right)x$$

$$\Rightarrow \frac{dy}{5-y}=\frac{xdx}{1-x^2}$$

$$\Rightarrow \frac{dy}{5-y}-\frac{xdx}{1-x^2}=0$$

Now taking integration on both sides we get

$$\Rightarrow \int\frac{dy}{5-y}-\int\frac{xdx}{1-x^2}=\int 0$$

$$\Rightarrow -\log(5-y)+\frac{1}{2}\log(1-x^2)=\log c$$

$$\Rightarrow \log\left(\frac{\sqrt{1-x^2}}{5-y}\right)=\log c$$

$$\Rightarrow \frac{\sqrt{1-x^2}}{5-y}=c$$

$$\Rightarrow \sqrt{1-x^2}=c\left(5-y\right)$$

Hence the required solution is $\sqrt{1-x^2}=c\left(5-y\right)$.

11. Problem: Solve $3e^x\tan y dx+\left(1-e^x\right)\sec^2 y dy=0$

Solution: The given equation is $3e^x\tan y dx+\left(1-e^x\right)\sec^2 y dy=0$

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$$3e^x\tan y dx=-\left(1-e^x\right)\sec^2 y dy$$

$$\Rightarrow \frac{3e^x}{1-e^x}dx=-\frac{\sec^2 y dy}{\tan y}$$

$$\Rightarrow \quad \frac{3e^x}{1-e^x}dx + \frac{\sec^2 y dy}{\tan y} = 0$$

Now taking integration on both sides we get

$$\Rightarrow \quad \int \frac{3e^x}{1-e^x}dx + \int \frac{\sec^2 y dy}{\tan y} = \int 0$$

$$\Rightarrow -3\log(1-e^x) + \log(\tan y) = \log c$$

$$\Rightarrow \log\left(\frac{\tan y}{\left(1-e^x\right)^3}\right) = \log c$$

$$\Rightarrow \frac{\tan y}{\left(1-e^x\right)^3} = c$$

$$\Rightarrow \tan y = c\left(1-e^x\right)^3$$

Hence the required solution is $\tan y = c\left(1-e^x\right)^3$.

12. Problem: Solve $\frac{dy}{dx} = 1 + x\tan\left(y-x\right)$

Solution: The given equation is $\frac{dy}{dx} = 1 + x\tan\left(y-x\right)$ (I)

Which is a first order and first degree differential equation

It can be solved by the method of reducing to variables separable

put $y - x = z$

$$\left.\begin{array}{l} \Rightarrow \frac{dy}{dx} - 1 = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} + 1 \end{array}\right\} \quad \text{(II)}$$

From equations (I) and (II) we get

$$\frac{dz}{dx} + 1 = 1 + x\tan z$$

$$\Rightarrow \quad \frac{dz}{dx} = x\tan z$$

$$\Rightarrow \quad \frac{dz}{\tan z} = xdx$$

$$\Rightarrow \quad \cot z dz - xdx = 0$$

Now taking integration on both sides we get

$$\Rightarrow \quad \int \cot z dz - \int xdx = \int 0$$

$$\Rightarrow \log \sin z - \frac{x^2}{2} = c$$

$$\Rightarrow \log\left(\sin\left(y-x\right)\right) - \frac{x^2}{2} = c$$

$$\Rightarrow \log\left(\sin\left(y-x\right)\right) = \frac{x^2}{2} + c$$

Hence the required solution is $\log\left(\sin\left(y-x\right)\right) = \frac{x^2}{2} + c$

13. Problem: Solve $\frac{dy}{dx}=(3x+y+4)^2$

Solution: The given equation is $\frac{dy}{dx}=(3x+y+4)^2$ (I)

Which is a first order and first degree differential equation

It can be solved by the method of reducing to variables separable

$$\left.\begin{array}{l}\text{put } 3x+y+4=z\\ \Rightarrow 3+\frac{dy}{dx}=\frac{dz}{dx}\Rightarrow\frac{dy}{dx}=\frac{dz}{dx}-3\end{array}\right\} \quad \text{(II)}$$

From equations (I) and (II) we get

$$\frac{dz}{dx}-3=z^2$$

$$\Rightarrow \frac{dz}{dx}=z^2+3$$

$$\Rightarrow \frac{dz}{z^2+3}=dx$$

$$\Rightarrow \frac{dz}{z^2+3}-dx=0$$

Now taking integration on both sides we get

$$\Rightarrow \int\frac{dz}{z^2+3}-\int dx=\int 0$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}-x=c$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right)-x=c$$

Hence the required solution is $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right)-x=c$

14. Problem: Solve $\frac{dy}{dx}=\tan^2(x+y)$

Solution: The given equation is $\frac{dy}{dx}=\tan^2(x+y)$ (I)

Which is a first order and first degree differential equation

It can be solved by the method of reducing to variables separable

$$\left.\begin{array}{l}\text{put } x+y=z\\ \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx}\Rightarrow\frac{dy}{dx}=\frac{dz}{dx}-1\end{array}\right\} \quad \text{(II)}$$

From equations (I) and (II) we get

$$\frac{dz}{dx}-1=\tan^2 z$$

$$\Rightarrow \frac{dz}{dx}=\tan^2 z+1\Rightarrow \frac{dz}{dx}=\sec^2 z$$

$$\Rightarrow \frac{dz}{\sec^2 z}=dx$$

$\Rightarrow \cos^2 z dz - dx = 0$

Now taking integration on both sides we get

$\Rightarrow \int \cos^2 z dz - \int dx = \int 0$

$\Rightarrow \int \frac{1+\cos 2z}{2} dz - \int dx = \int 0$

$\Rightarrow \frac{1}{2}\left(\int 1dz + \int \cos 2z dz\right) - x = c$

$\Rightarrow \frac{1}{2}\left(z + \frac{\sin 2z}{2}\right) - x = c$

$\Rightarrow \frac{1}{2}\left(x + y + \frac{\sin 2(x+y)}{2}\right) - x = c$

$\Rightarrow \frac{1}{2}\left(y - x + \frac{\sin 2(x+y)}{2}\right) = c$

Hence the required solution is $\frac{1}{2}\left(y - x + \frac{\sin 2(x+y)}{2}\right) = c$

15. Problem: Solve $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution: The given equation is $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ (I)

Which is a first order and first degree differential equation

It can be solved by the method of variables separable

$(\sin y + y\cos y)dy = x(2\log x + 1)dx$

$\Rightarrow x(2\log x + 1)dx - (\sin y + y\cos y)dy = 0$

Now taking integration on both sides we get

$\Rightarrow \int x(2\log x + 1)dx - \int(\sin y + y\cos y)dy = \int 0$

$\Rightarrow \int \frac{d}{dx}(x^2 \log x)dx - \int \frac{d}{dy}(y\sin y)dy = \int 0$

$\Rightarrow x^2 \log x - y\sin y = c$

Hence the required solution is $x^2 \log x - y\sin y = c$ ..

Exercise 13(b)

I Solve the following differential equations:

1. $\frac{dy}{dx} = \frac{x+1}{y+1}$ 2. $\frac{dy}{dx} = \sqrt{\frac{1-x^2}{1-y^2}}$ 3. $\frac{dy}{dx} = \sqrt{\frac{1+x^2}{1+y^2}}$ 4. $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$ 5. $\frac{dy}{dx} = a^{x+y}$

6. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ 7. $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$

8. $e^x \cot y dx + \left(1-e^x\right)\operatorname{cosec}^2 y dy = 0$ 9. $\left(e^y+1\right)\cos x dx + e^y \sin x dy = 0$

10. $y - x\dfrac{dy}{dx} = 3\left(1+x^2\dfrac{dy}{dx}\right)$ 11. $\dfrac{dy}{dx} = e^{x-y}$ 12. $\dfrac{dy}{dx} = \left(3x+4y+1\right)^2$ 13. $\dfrac{dy}{dx} = \left(4x+y+1\right)^2$

14. $\dfrac{dy}{dx} = \left(9x+y+12\right)^2$ 15. $\left(x-y\right)^2\dfrac{dy}{dx} = a^2$ 16. $y\dfrac{dy}{dx} = xe^{x^2+y^2}$ 17. $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$

18. $\tan y dy + x\cos^2 y dx = 0$ 19. $\dfrac{dy}{dx} = e^{x+y} + x^2e^{x^3+y}$ 20. $\left(x+y\right)^2\dfrac{dy}{dx} = a^2$

21. $\sqrt{1-x^2}\sin^{-1} x dy + y dx = 0$ 22. $\dfrac{dy}{dx} = \left(x+y\right)^2$ 23. $\dfrac{dy}{dx} = xy + x + y + 1$

24. $\sqrt{\left(1+x^2\right)\left(1+y^2\right)}dx + xy dy = 0$ 25. $x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$

13.2(b) Homogeneous differential equation:

13.2(b).1Definition: A function $f(x,y)$ of two variables x and y is said to be a homogeneous function of degree n if $f(kx,ky) = k^n f(x,y)$ for all values of k for which both sides of the above equation are meaningful.

13.2(b).2Example: $f(x,y) = 4x^2y + 2xy^2$ is a homogeneous function of degree 3.

$$f(kx,ky) = 4(kx)^2(ky) + 2(kx)(ky)^2 = 4k^3x^2y + 2k^3xy^2$$

$$= k^3\left(4x^2y + 2xy^2\right) = k^3 f(x,y).$$

13.2(b).3Definition: A differential equation is of the form $\dfrac{dy}{dx} = f(x,y)$ of first order and first degree differential equation is called a homogeneous in x and y if the function $f(x,y)$ is a homogeneous function of degree zero in x and y

13.2(b).4Examples: **(1)** $\dfrac{dy}{dx} = \dfrac{x+3y}{2x+y}$

(2) $\dfrac{dy}{dx} = \dfrac{4x^2+2xy}{y^2}$

13.2(b).5Solved Problems:

1. Problem: Solve $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

Solution: The given equation is $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

$$\Rightarrow x^2\frac{dy}{dx} - xy\frac{dy}{dx} = -y^2$$

$$\Rightarrow \left(x^2 - xy\right)\frac{dy}{dx} = -y^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{x^2-xy} \qquad \text{(I)}$$

Which is a first order and first degree differential equation

take $f(x,y) = -\dfrac{y^2}{x^2 - xy}$

$$\Rightarrow f(kx,ky) = -\frac{(ky)^2}{(kx)^2 - (kx)(ky)}$$

$$\Rightarrow f(kx,ky) = -\frac{k^2y^2}{k^2x^2 - k^2xy} \Rightarrow f(kx,ky) = -\frac{k^2y^2}{k^2\left(x^2 - xy\right)}$$

$$\Rightarrow f(kx,ky) = -k^0\,\frac{y^2}{x^2 - xy} \Rightarrow f(kx,ky) = k^0 f(x,y)$$

Which is a homogeneous function of degree zero.
Hence equation (I) is a homogeneous differential equation.

put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ (II)

From equations (I) and (II) we get

$$v + x\frac{dv}{dx} = -\frac{(vx)^2}{x^2 - x(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{v^2x^2}{x^2(1-v)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{-v^2}{(1-v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^2}{(1-v)} - v \Rightarrow x\frac{dv}{dx} = \frac{-v^2 - v + v^2}{(1-v)} \Rightarrow x\frac{dv}{dx} = \frac{-v}{(1-v)}$$

$$\Rightarrow \frac{(1-v)\,dv}{v} = \frac{-dx}{x}$$

$$\Rightarrow \frac{(1-v)\,dv}{v} + \frac{dx}{x} = 0$$

Now taking integration on both sides we get

$$\int\frac{(1-v)\,dv}{v} + \int\frac{dx}{x} = \int 0$$

$$\Rightarrow \int\frac{1}{v}dv - \int 1dv + \int\frac{dx}{x} = \int 0$$

$$\Rightarrow \log v - v + \log x = c$$

$$\Rightarrow \log vx - v = c$$

$$\Rightarrow \log y - \frac{y}{x} = c$$

Hence the required solution is $\log y - \dfrac{y}{x} = c$.

.

2. Problem: Solve $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

Solution: The given equation is $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

$$\Rightarrow xdy = \sqrt{x^2 + y^2}\,dx + ydx$$

$$\Rightarrow xdy = \left(\sqrt{x^2+y^2} + y\right)dx$$
$$\Rightarrow x\frac{dy}{dx} = \left(\sqrt{x^2+y^2} + y\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2+y^2} + y}{x} \qquad \text{(I)}$$
Which is a first order and first degree differential equation

take $f(x,y) = \dfrac{\sqrt{x^2+y^2} + y}{x}$
$$\Rightarrow f(kx,ky) = \frac{\sqrt{(kx)^2+(ky)^2} + (ky)}{(kx)}$$
$$\Rightarrow f(kx,ky) = \frac{\sqrt{k^2x^2+k^2y^2} + ky}{kx} \Rightarrow f(kx,ky) = \frac{k\left(\sqrt{x^2+y^2} + y\right)}{kx}$$
$$\Rightarrow f(kx,ky) = k^0\,\frac{\sqrt{x^2+y^2} + y}{x} \quad \Rightarrow f(kx,ky) = k^0 f(x,y)$$
Which is a homogeneous function of degree zero.
Hence equation (I) is a homogeneous differential equation.

put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ (II)

From equations (I) and (II) we get
$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+(vx)^2} + vx}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{x\sqrt{1+v^2} + vx}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{x\left(\sqrt{1+v^2} + v\right)}{x} \qquad \Rightarrow v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$
$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$
$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$
$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} - \frac{dx}{x} = 0$$
Now taking integration on both sides we get
$$\int \frac{dv}{\sqrt{1+v^2}} - \int \frac{dx}{x} = \int 0$$
$$\Rightarrow \log(v + \sqrt{v^2+1}) - \log x = \log c$$
$$\Rightarrow \log\left(\frac{v + \sqrt{v^2+1}}{x}\right) = \log c$$
$$\Rightarrow \left(\frac{v + \sqrt{v^2+1}}{x}\right) = c \Rightarrow v + \sqrt{v^2+1} = cx \Rightarrow \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1} = cx$$

$$\Rightarrow \frac{y}{x}+\sqrt{\frac{y^2+x^2}{x^2}}=cx \Rightarrow \frac{y}{x}+\frac{\sqrt{y^2+x^2}}{x}=cx \Rightarrow \frac{y+\sqrt{y^2+x^2}}{x}=cx$$

$$\Rightarrow y+\sqrt{y^2+x^2}=cx^2$$

Hence the required solution is $y+\sqrt{y^2+x^2}=cx^2$.

3. Problem: Solve $xdy-ydx=x\cos^2\frac{y}{x}dx$

Solution: The given equation is $xdy-ydx=x\cos^2\frac{y}{x}dx$

$$\Rightarrow xdy=x\cos^2\frac{y}{x}dx+ydx$$

$$\Rightarrow xdy=\left(x\cos^2\frac{y}{x}+y\right)dx$$

$$\Rightarrow x\frac{dy}{dx}=\left(x\cos^2\frac{y}{x}+y\right) \Rightarrow \frac{dy}{dx}=\frac{x\cos^2\frac{y}{x}+y}{x} \Rightarrow \frac{dy}{dx}=\frac{x\cos^2\frac{y}{x}}{x}+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\cos^2\frac{y}{x}+\frac{y}{x} \qquad \text{(I)}$$

Which is a first order and first degree differential equation

take $f(x,y)=\cos^2\frac{y}{x}+\frac{y}{x}$

$$\Rightarrow f(kx,ky)=\cos^2\frac{ky}{kx}+\frac{ky}{kx}$$

$$\Rightarrow f(kx,ky)=\cos^2\frac{y}{x}+\frac{y}{x} \Rightarrow f(kx,ky)=k^0\left(\cos^2\frac{y}{x}+\frac{y}{x}\right)$$

$$\Rightarrow f(kx,ky)=k^0 f(x,y)$$

Which is a homogeneous function of degree zero.

Hence equation (I) is a homogeneous differential equation.

put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ (II)

From equations (I) and (II) we get

$$v+x\frac{dv}{dx}=\cos^2\frac{vx}{x}+\frac{vx}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=\cos^2 v+v$$

$$\Rightarrow x\frac{dv}{dx}=\cos^2 v \Rightarrow \frac{dv}{\cos^2 v}=\frac{dx}{x}$$

$$\Rightarrow \sec^2 v\,dv=\frac{dx}{x}$$

$$\Rightarrow \sec^2 v\,dv-\frac{dx}{x}=0$$

Now taking integration on both sides we get

$$\int \sec^2 v\,dv-\int\frac{dx}{x}=\int 0$$

$$\Rightarrow \tan v-\log x=c$$

$$\Rightarrow \tan\frac{y}{x} - \log x = c$$

Hence the required solution is $\tan\frac{y}{x} - \log x = c$.

4. Problem: Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution: The given equation is $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ (I)

Which is a first order and first degree differential equation

take $f(x, y) = \frac{y}{x} + \tan\frac{y}{x}$

$$\Rightarrow f(kx, ky) = \frac{ky}{kx} + \tan\frac{ky}{kx}$$

$$\Rightarrow f(kx, ky) = \frac{y}{x} + \tan\frac{y}{x} \Rightarrow f(kx, ky) = k^0\left(\frac{y}{x} + \tan\frac{y}{x}\right)$$

$$\Rightarrow f(kx, ky) = k^0 f(x, y)$$

Which is a homogeneous function of degree zero.

Hence equation (I) is a homogeneous differential equation.

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ (II)

From equations (I) and (II) we get

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \tan v + v$$

$$\Rightarrow x\frac{dv}{dx} = \tan v \Rightarrow \frac{dv}{\tan v} = \frac{dx}{x}$$

$$\Rightarrow \cot v dv = \frac{dx}{x}$$

$$\Rightarrow \cot v dv - \frac{dx}{x} = 0$$

Now taking integration on both sides we get

$$\int \cot v dv - \int \frac{dx}{x} = \int 0$$

$$\Rightarrow \log \sin v - \log x = \log c$$

$$\Rightarrow \log\frac{\sin v}{x} = \log c$$

$$\Rightarrow \frac{\sin v}{x} = c$$

$$\Rightarrow \sin\frac{y}{x} = cx$$

Hence the required solution is $\sin\frac{y}{x} = cx$.

.

5. Problem: Solve $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

The given equation is $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

Solution: $\Rightarrow \left(1+e^{x/y}\right)dx=-e^{x/y}\left(1-\frac{x}{y}\right)dy$

$$\Rightarrow \frac{dx}{dy}=-\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)} \qquad \text{(I)}$$

Which is a first order and first degree differential equation

take $f(x,y)=-\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)}$

$$\Rightarrow f(kx,ky)=-\frac{e^{kx/ky}\left(1-\frac{kx}{ky}\right)}{\left(1+e^{kx/ky}\right)}$$

$$\Rightarrow f(kx,ky)=-\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)}$$

$\Rightarrow f(kx,ky)=k^0 f(x,y)$

Which is a homogeneous function of degree zero.

Hence equation (I) is a homogeneous differential equation.

put $x=vy\Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$ (II)

From equations (I) and (II) we get

$$v+y\frac{dv}{dy}=-\frac{e^{vy/y}\left(1-\frac{vy}{y}\right)}{\left(1+e^{vy/y}\right)}$$

$$\Rightarrow v+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{\left(1+e^{v}\right)}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{\left(1+e^{v}\right)}-v\Rightarrow y\frac{dv}{dy}=-\frac{e^{v}(1-v)+v\left(1+e^{v}\right)}{\left(1+e^{v}\right)}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{\left(e^{v}-ve^{v}+v+ve^{v}\right)}{\left(1+e^{v}\right)}\Rightarrow y\frac{dv}{dy}=-\frac{e^{v}+v}{1+e^{v}}$$

$$\Rightarrow \frac{1+e^{v}}{v+e^{v}}dv=-\frac{dy}{y}$$

$$\Rightarrow \frac{1+e^v}{v+e^v}dv + \frac{dy}{y} = 0$$

Now taking integration on both sides we get

$$\int \frac{1+e^v}{v+e^v}dv + \int \frac{dy}{y} = \int 0$$

$$\Rightarrow \log(v+e^v) + \log y = \log c$$

$$\Rightarrow \log(v+e^v)y = \log c$$

$$\Rightarrow (v+e^v)y = c$$

$$\Rightarrow \left(\frac{x}{y} + e^{x/y}\right)y = c$$

$$\Rightarrow \left(\frac{x+ye^{x/y}}{y}\right)y = c$$

$$\Rightarrow \left(x+ye^{x/y}\right) = c$$

Hence the required solution is $\left(x+ye^{x/y}\right) = c$.

Exercise 13(c)

I Solve the following differential equations:

1. $x\,dx + ydy = \frac{x\,dy - ydx}{x^2+y^2}$ 2. $x\frac{dy}{dx} = y + xe^{y/x}$

Key concepts

1. An equation involving derivatives with one dependent variable and two or more independent variables is called a *differential equation.*

2. An equation involving derivatives with one dependent variable and exactly one independent variables is called an *ordinary differential equation*.

3. If a differential equation contains n^{th} and lower order partial derivatives then the *order of the differential equation* is n

4. The *degree of the differential equation* is the greatest power of the highest order derivative involved in it.

5. The general form of an ordinary differential equation of n^{th} order is

$$\mathrm{F}\left(x, y, \frac{dy}{dx}, \frac{d^2y}{d\,x^2}, \ldots \frac{d^ny}{d\,x^n}\right) = 0 \text{ or } \mathrm{F}\left(x, y, y^{(1)}, y^{(2)}, \ldots y^{(n)}\right) = 0.$$

6. A *solution* of a differential equation is a relation between dependent variable, independent variables and along with some arbitrary constants satisfying the differential equation.

7. A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the *general solution*.

8. A *particular solution* of a differential equation is a solution obtained by giving particular values to the arbitrary constants in the general solution.

9. If the differential equation is of the form $\frac{dy}{dx} = f(x,y)$ can be expressed as $\frac{dy}{dx} = F(x)G(y)$ then it can be solved by the method of *variables separable*

10. A function $f(x,y)$ of two variables x and y is said to be a homogeneous function of degree n if $f(kx,ky) = k^n f(x,y)$ for all values of k for which both sides of the above equation are meaningful.

11. A differential equation is of the form $\frac{dy}{dx} = f(x,y)$ of first order and first degree differential equation is called a homogeneous in x and y if the function $f(x,y)$ is a homogeneous function of degree zero in x and y

Answers

Exercise 13(a)

I 1. 1,1 2. 2,3 3. 2,4 4. 2,2 5. 2,0 6. 2,1

II 1. $y'' - 16y = 0$ 2. $y'' - 25y = 0$ 3. $y'' - n^2 y = 0$

4. $y'' + 16y = 0$ 5. $y'' + 25y = 0$ 6. $y'' + n^2 y = 0$

7. $y'' + y' - 2y = 0$ 8. $y'' + y' - 12y = 0$ 9. $y'' - 4y' + 13y = 0$

10. $y'' - 2my' + (m^2 + n^2)y = 0$ 11. $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$

Exercise 13(b)

I 1. $x + \frac{x^2}{2} = y + \frac{y^2}{2} + c$ 2. $\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x = \frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1} y + c$

3. $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1} x = \frac{y}{2}\sqrt{1+y^2} + \frac{1}{2}\sinh^{-1} y + c$ 4. $x + \frac{x^3}{3} = y + \frac{y^3}{3} + c$

5. $\frac{a^x}{\log a} + \frac{a^{-y}}{\log a} = c$ 6. $\frac{y}{1-ay} = c\left(a + x\right)$ 7. $1 + \tan\left(\frac{x+y}{2}\right) = ce^x$

8. $\left(e^x - 1\right)\cot y = c$ 9. $\left(e^y + 1\right)\sin x = c$ 10. $\frac{x}{(3x+1)} = c(y-3)$ 11. $e^x - e^{-y} = c$ 12. $3x + 4y + 1 = \sqrt{3}\tan\left(\sqrt{3}x + c\right)$ 13. $4x + y + 1 = 2\tan\left(2x + c\right)$ 14. $9x + y + 12 = 3\tan\left(3x + c\right)$ 15. $2y = a\log\left|\frac{x-y-a}{x-y+a}\right|$ 16. $e^{x^2}\left(1 + e^{-x^2-y^2}\right) = c$

17. $\frac{x^3}{3}+e^x-e^y=c$ 18. $x^2+\tan^2 y=2c$ 19. $\frac{e^{x^3}}{3}+e^x-e^y=c$

20. $y=a\tan^{-1}\left(\frac{x+y}{2}\right)+c$ 21. $y\sin^{-1}x=c$ 22. $x+y=\tan(x+c)$ 23. $e^{x^2/2+x}=c(y+1)$

24. $\log x+\log\left(\sqrt{1+x^2}-1\right)+\sqrt{1+x^2}+\sqrt{1+y^2}=c$ 25. $\sqrt{1+x^2}+\sqrt{1+y^2}=c$

Exercise 13(c)

I 1. $x^2+y^2+2\tan^{-1}\left(\frac{x}{y}\right)=c$ 2. $\log x+e^{-y/x}=c$

14. PERMUTATIONS AND COMBINATIONS

Introduction:

The first known use of permutations combinations goes back to 6^{th} century B.C. when *Susruta* in his medical work *Susruta Samhita* finds 63 combinations out of 6 different tastes by taking one at a time, two at a time etc. Later in the 3^{rd} century B.C., a Sanskrit scholar *Pingala* in his book *Chandassastra* used permutations and combinations to determine the number of combinations of a given number of letters by taking one at a time, two at a time etc. The concept of permutations and combinations was treated as a self contained topic in mathematics under the name of *Vikalpa* by renowned mathematician *Mahavira* in 9^{th} century A.D. The credit of stating several important theorems and results on the subject matter of permutations and combinations goes to the by renowned scholar *Bhaskaracharya*. He treated this topic under the name *Anka Vyasha* in his famous book *Leelavathi Ganitham*.

The theory of permutations and combinations in the present sense first appeared in the book *Ars Conjectandi* written by the renowned mathematician *Jakob Bernoulli* in 17^{th} century A.D. which was published in 1713 A.D. after his death.

We must have come across situations like choosing five questions out of eight questions in a question paper or which items to be chosen from the menu card in a hotel etc. We discuss such situations in this chapter. This chapter permutations and combinations is an important chapter in algebra in view of a number of applications in a day- to- day life and in the theory of probability. While learning *permutations and combinations* we should be in a position to clearly see weather the concept of a permutation or the concept of a combination is applicable in a given situation. In general, a combination is only a selection while a permutation involves two steps, namely, selection and arrangement. For example, forming a three digit number using the digits $1,2,3,4,5$ is a *permutation.* This involves two steps. In the first step we select three digits, say $2,4,5$. In the second step, we arrange them to form a three digit number such as $245, 452, 542$ etc. Forming a set with three elements using the digits $1,2,3,4,5$ is a *combination.* This involves only one process, namely, selection of three elements, say $2,4,5.$ Then the element set formed is $\{2,4,5\}$ which is same as the sets $\{4,5,2\},\{5,4,2\}$ etc. Thus whenever there is importance to the arrangement or order in which the objects are placed, then it is a *permutation* and if there is no importance to the arrangement or order , but only selection is required, then it is a *combination*. These notations will help us to arrive the number of arrangements or combinations without actually counting them.

Before going into formal definitions, we introduce *factorial notation*, which is required to calculate the number of permutations and combinations. If n is a positive integer, we define $n!$ (read as n factorial) by mathematical induction as follows.

$1! = 1$ and $n! = n.(n-1)!$ if $n > 1$.

For example, $2! = 2.(1!) = 2$

$3! = 3.(2!) = 3.2 = 6$

$4! = 4.(3!) = 4.6 = 24$

$5! = 5.(4!) = 5.24 = 120$ etc.

By convention, we define $0! = 1$

Throughout this chapter the letters n, r denote nonnegative integers unless otherwise mentioned.

14.1 Fundamental Principle of Counting- Linear and Circular Permutations:

Before giving formal definitions of linear and circular permutations, we first learnt about the *Fundamental principle of Counting*, which plays a very crucial role in the development of the theory of permutations and combinations.

14.1.1 Fundamental Principle of Counting:

If a work w_1 can be performed in m different ways and second work w_2 can be performed (after the work w_1 has been performed in any one of the m ways) in n different ways, then the two works (one after the other) can be performed in mn different ways.

This principle can be understood help of the following two examples.

14.1.2 Example: If a man has 4 different coloured trousers T_1, T_2, T_3, T_4 and 3 different coloured shirts S_1, S_2, S_3, then he can select a pair (a trouser and a shirt) in $4 \times 3 = 12$ different ways as explained below.

In this example, we take w_1 as selecting a trouser which can be performed in 4 different ways and w_2 as selecting a shirt which can be performed in 3 different ways. They are

$T_1S_1 \quad T_1S_2 \quad T_1S_3$

$T_2S_1 \quad T_2S_2 \quad T_2S_3$

$T_3S_1 \quad T_3S_2 \quad T_3S_3$

$T_4S_1 \quad T_4S_2 \quad T_4S_3$

14.1.3 Example: If there are 4 different modes of transport available to travel from Hyderabad to Chennai, namely, bus, car, train and aeroplane (we denote these by M_1, M_2, M_3, M_4 respectively) and 3 different modes of transport from Chennai to Bangalore, namely, bus, train and aeroplane (we denote these by N_1, N_2, N_3 respectively), then how many different modes of transport are available for a person to travel from Hyderabad to Bangalore (via Chennai)?

Here the work w_1 is to travel from Hyderabad to Chennai, which can be performed in 4 different ways and the work w_2 is to travel Chennai to Bangalore, which can be performed in 3 different ways. Therefore, by the fundamental principle, the two works can be done in $4 \times 3 = 12$ different ways. That is a person can travel from Hyderabad to Bangalore in 12 different ways. They are

$M_1N_1 \quad M_1N_2 \quad M_1N_3$

$M_2N_1 \quad M_2N_2 \quad M_2N_3$

$M_3N_1 \quad M_3N_2 \quad M_3N_3$

$M_4N_1 \quad M_4N_2 \quad M_4N_3$

In this, M_1N_1 means that the person travels by bus from Hyderabad to Chennai and again by bus from Chennai to Bangalore. M_2N_3 means that the person travels by car from Hyderabad to Chennai and by aeroplane from Chennai to Bangalore etc.

Now we give the definition of a *linear permutation.*

14.1.4 Definition: From a given set of elements (similar or not) selecting some or all of them and arranging them in a line is called a *linear permutation* or simply a *permutation.*

14.1.5 Example: (i) From the letters of the word MINT, two letter permutations are MI, IM, MN, NM, MT, TM, IN, NI, IT, TI, NT, TN.

(ii) From the letters of the word RUNNING, permutations with three letters are RUN, UNN, GUR, GUN, NNN etc. and permutations with four letters are RUNN, UNIG, GNUN, NNIN etc.

(iii) Using the digits 1,2,3,4,5 permutations with two digits are 12,13,32 etc. permutations with three digits are 123, 324, 513 etc. and permutations with four digits are 1234, 4351, 5124 etc.

Next we define the circular permutation in the following.

14.1.6 Definition: From a given set of things (similar or not) choosing some or all of them and arranging them around a circle is called a *circular permutation.*

14.1.7 Example: Some of the circular permutations formed using the digits 1,2,3,4 are 1234,2134,3142 etc. The important difference between a circular permutation and a linear permutation is that a linear permutation has a first place (also a last place), where as a circular permutation has no starting place. It can be treated as starting from any one of the elements in it. But how the other elements are arranged relative to this (starting) element is to be taken into consideration.

Similarly, the linear permutations 1234,4321,3214,2143 give rise to only one circular permutation.

Thus in a circular permutation where the first element is placed is not important but how the remaining elements are arranged relative to that element is important.

In some cases like garlands of flowers, chains of beads etc., there is no distinction between the clockwise and anti- clockwise arrangements of the same circular permutation. They will be treated as a single circular permutation. In such cases, the two circular permutations described above will be treated as a single circular permutation.

14.1.8 Example: Write all possible (i) linear (ii) circular permutations using the digits 1,2,3,4 taken three at a time.

Solution: (i) Required linear permutations are

123	124	134	234
231	241	341	342
312	412	413	423
132	142	143	243
321	421	431	432
213	214	314	324

Thus the number of linear permutations using the digits 1,2,3,4 taken three at a time is 24.

(i) Required circular permutations are

123	124	134	234
132	142	143	243

Thus the number of circular permutations using the digits 1,2,3,4 taken three at a time is 8.

14.1.9 Permutations of n dissimilar things taken r at a time:

Here after by a permutation we mean a linear permutation in which no object is used more than once (that is, a permutation without repetition). If repetition is allowed any where, it will be clearly mentioned. In example 14.1.8, we have exhibited that the number of permutations of 4 dissimilar things taken 3 at a time 24. But if the number of given things and (or) the number of things to be arranged is large, then it is not eas to enumerate the permutations like in example 14.1.8.

Hence we develop a formula without proof to find the number of such permutations in the following.

14.1.10 Theorem: If n, r are positive integers and $r \le n$, then the number of permutations of n dissimilar things taken r at a time is $n(n-1)(n-2)...(n-r+1)$

14.1.11 Notation: The number of permutations of n dissimilar things taken r at a time is denoted by n_{P_r} or $P(n,r)$. However, we use the notation n_{P_r} only. Thus, for $1 \le r \le n$,

$n_{P_r} = n(n-1)(n-2)...(n-r+1)$ and we write $n_{P_0} = 1$ by convention.

14.1.12 Formula: If $n \ge 1$ and $0 \le r \le n$, then $n_{P_r} = \dfrac{n!}{(n-r)!}$

For $1 \le r \le n$, from the Theorem 14.1.10, we get

$$n_{P_r} = n(n-1)(n-2)...(n-r+1)$$

$$= \frac{n(n-1)(n-2)...(n-r+1)(n-r)(n-r-1)...3.2.1}{(n-r)(n-r-1)...3.2.1}$$

$$= \frac{n!}{(n-r)!}$$

By convention, $n_{P_0} = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$

14.1.13 Note: $n_{P_n} = n!$ and $n_{P_0} = 1$

14.1.14 Theorem: For $1 \le r \le n$, $n_{P_r} = n.\ (n-1)_{P_{(r-1)}}$

$$n.\ (n-1)_{P_{(r-1)}} = n.\frac{(n-1)!}{\left[(n-1)-(r-1)\right]!} = \frac{n.(n-1)!}{\left[(n-r)!\right]} = \frac{n!}{(n-r)!} = n_{P_r}.$$

14.1.15 Note: From the above formula we also get

$$n_{P_r} = n.\ (n-1)_{P_{(r-1)}}$$

$$= n.\ (n-1)(n-2)_{P_{(r-2)}}$$

$$= n.\ (n-1)(n-2)(n-3)_{P_{(r-3)}}$$

$$= ....$$

14.1.16 Theorem: Let n, r be positive integers and $1 \le r \le n$, then

$$n_{P_r} = (n-1)_{P_r} + r.\ (n-1)_{P_{(r-1)}}$$

14.1.17 Permutations when repetitions are allowed:

We have learnt about the number of permutations of n dissimilar things taken r at a time, when repetition of things not allowed. Now we learnt about the number of permutations of n dissimilar things taken r at a time when each thing is can be repeated any number of times (That is, when repetition of things is allowed).

14.1.18 Theorem: Let n, r be positive integers. If the repetition of things is allowed, then the number of permutations of n dissimilar things taken r at a time is n^r.

14.1.19 Theorem: The number of circular permutations of n dissimilar things taken all at a time is $(n-1)!$

14.1.20 Note: In case of like garlands of flowers, chains of beads etc., there is no distinction between the clockwise and anti- clockwise arrangements of the same circular permutation. They will be treated as a single circular permutation. In such cases, number of circular permutations of n dissimilar things taken all at a time is $\frac{1}{2}(n-1)!$

14.1.21 Theorem: The number of linear permutations of n things in which p things are alike and the rest are different is $\frac{n!}{p!}$.

14.1.22 Theorem: The number of linear permutations of n things in which there are p like things of one type, q like things of second type, r like things of third type and the rest are different is $\frac{n!}{p!q!r!}$.

14.1.23 Combinations:

At the beginning of this chapter, we have exhibited the difference between a permutation and a combination. A combination is only a selection. There is no importance to the order or arrangement of things in a combination. Thus a combination of n things taken r at a time can be regarded as a subset with r elements of a set containing n elements. The number of combinations of n dissimilar things taken r at a time is denoted by n_{C_r} or $C(n,r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$

14.1.24 Theorem: The number of combinations of n dissimilar things taken r at a time is denoted by $\frac{n_{P_r}}{r!}$. That is $n_{C_r} = \frac{n_{P_r}}{r!} = \frac{n!}{r!(n-r)!}$

14.1.25 Corollary: The number of different subsets of r elements of a set containing n elements is n_{C_r}.

14.1.26 Theorem: If n, r are integers with $0 \le r \le n,$ then $n_{C_r} = n_{C_{n-r}}$

14.1.27 Corollary: For any positive integer $n,$ $n_{C_0} = n_{C_n} = 1$

14.1.28 Theorem: If m, n are distinct positive integers then the number of ways of dividing things $(m+n)$ into two groups containing m things and n things is $\frac{(m+n)!}{m!n!}$

14.1.29 Corollary: If m, n, p are distinct positive integers then the number of ways of dividing things $(m+n+p)$ into three groups containing m things, n things and p things is $\frac{(m+n+p)!}{m!n!p!}$

14.1.30 Corollary: The number of ways of dividing $2n$ dissimilar things into two equal groups containing n things in each is $\frac{2n!}{2!n!n!}$

14.1.31 Corollary: The number of ways of dividing mn dissimilar things into m equal groups containing n things in each is $\frac{mn!}{m!(n!)^m}$

14.1.32 Corollary: The number of ways of distributing mn dissimilar things equally among m persons is $\frac{mn!}{(n!)^m}$

14.1.33 Theorem: For $0 \le r, s \le n,$ if $n_{C_r} = n_{C_s}$, then either $r = s$ or $r + s = n$

14.1.34 Theorem: If $1 \le r \le n,$ then $n_{C_{r-1}} + n_{C_r} = (n+1)_{C_r}$

14.1.35 Theorem: If $2 \le r \le n,$ then $n_{C_{r-2}} + 2.n_{C_{r-1}} + n_{C_r} = (n+2)_{C_r}$

14.1.36 Theorem: If p things are alike of one kind, q things are alike of second kind and r things are alike of third kind, then the number of ways of selecting any number of things (one or more) out of these $(p+q+r)$ things is $(p+1)(q+1)(r+1)-1.$

14.1.37 Corollary: The number of ways of selecting one or more things out of n dissimilar things is $2^n - 1.$

14.2 Simple Problems:

In this section we find the solutions of some simple problems related to permutations and combinations.

14.2.1 Solved Problems:

1. Problem: If $n_{P_4} = 1680,$ find $n.$

Solution: We know that n_{P_4} is the product of 4 consecutive integers of which n is the largest.

That is $n_{P_4} = n(n-1)(n-2)(n-3)$ and $1680 = 8 \times 7 \times 6 \times 5$

On comparing the largest integers, we get $n = 8.$

2. Problem: If $(n+1)_{P_5} : n_{P_5} = 3:2,$ find $n.$

Solution: We know that $(n+1)_{P_5} = (n+1)n(n-1)(n-2)(n-3)$

and $n_{P_5} = n(n-1)(n-2)(n-3)(n-4)$

Given $(n+1)_{P_5} : n_{P_5} = 3:2 \Rightarrow \frac{(n+1)_{P_5}}{n_{P_5}} = \frac{3}{2} \Rightarrow \frac{(n+1)n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)(n-4)} = \frac{3}{2}$

$$\Rightarrow \frac{(n+1)}{(n-4)} = \frac{3}{2} \Rightarrow 2n+2 = 3n-12 \Rightarrow n = 14.$$

3. Problem: Find the number of ways of permuting the letters of the word, PICTURE so that

(i) all vowels come together

(ii) no two vowels come together

(iii) The relative position of vowels and consonants are not disturbed.

Solution: The word PICTURE has 3 vowels (I,E,U) and 4 consonants (P,C,T,R).

(i) Treat the 3 vowels (I,E,U) as one unit. Then we can arrange 4 consonants and 1 unit of vowels in $5!$ ways. Hence the number of permutations in which the 3 vowels come together is $5! \times 3! = 720$.

(ii) First arrange the 4 consonants (P,C,T,R) in $4!$ ways. Then in between the vowels, in the beginning and in the ending, there are 5 gaps as shown below by the letter X.

X □ X □ X □ X □ X

In these 5 gaps we can arrange the 3 vowels in 5_{P_3} ways. Thus the number of words in which no two vowels come together is $4! \times 5_{P_3} = 24 \times 60 = 1440$.

(iii) The 3 vowels can be arranged in their relative positions in $3!$ ways and the 4 consonants can be arranged in their relative positions in $4!$ ways.

V	C	C	V	C	V	C

The required number of arrangements is $3! \times 4! = 144$.

4. Problem: If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'PRISON'.

Solution: The letters of the word PRISON are P, R, I, S, O, N

The letters of the word PRISON in dictionary order are I, N, O, P, R, S. In the dictionary order, first we write all words that begin with I. If we fill the first place with I, then the remaining 5 places can be filled with remaining 5 letters in $5!$ ways. That is there are $5!$ words that begin with I. Proceeding like this, after writing all words that begin with I, N, O, we have to write the words that begin with P. Among them first come the words with first two letters P, I. As above there are $4!$ such words. On proceeding like this, we get

$I - - - - - \rightarrow 5!$ words

$N - - - - - \rightarrow 5!$ words

$O - - - - - \rightarrow 5!$ words

$P\,I - - - - \rightarrow 4!$ words

$P\,N - - - - \rightarrow 4!$ words

$PO----\to 4!$ words
$PRIN--\to 2!$ words
$PRIO--\to 2!$ words
$PRISN-\to 1!$ words
$PRISON\to 1$ word.

Hence the rank of the word PRISON is $3\times 5!+3\times 4!+2\times 2!+1!+1=438.$

5. Problem: Find the sum of all 4-digit numbers that can be formed using the digit $1,3,5,7,9.$

Solution: We know that the number of 4-digit numbers that can be formed using the given 5 digits is $5_{P_4}=120.$ Now we find their sum.

We first find the sum of the digits in the units place of all these 120 numbers. If we fill the units place with 1 as shown below,

			1

then the remaining 3 places can be filled with the remaining 4 digits in 4_{P_3} ways. This means, the number of 4 digits having 1 in units place is 4_{P_3}. Similarly, each of the digits $3,5,7,9$ appear 24 times in units place. By adding all these digits we get the sum of digits in units place of all these 120 numbers as

$$4_{P_3}\times 1+4_{P_3}\times 3+4_{P_3}\times 5+4_{P_3}\times 7+4_{P_3}\times 9=4_{P_3}\times 25.$$

Similarly, we get the sum of digits in tens place as $4_{P_3}\times 25.$

Since it is in 10's place, its value is $4_{P_3}\times 25\times 10.$

Similarly, the value of the sum of digits in 100's place as $4_{P_3}\times 25\times 100$ and the value of the sum of digits in 1000's place as $4_{P_3}\times 25\times 1000.$

Since it is in 10's place, its value is $4_{P_3}\times 25\times 10.$

Hence the sum of all 4-digit numbers that can be formed using the digit $1,3,5,7,9$ is

$$4_{P_3}\times 25+4_{P_3}\times 25\times 10+4_{P_3}\times 25\times 100+4_{P_3}\times 25\times 1000=4_{P_3}\times 25\times 1111=24\times 25\times 1111=6,66,600.$$

6. Problem: Find the number of 4 letter words that can be formed using the letters of the word PISTON in which at least one letter is repeated.

Solution: The word PISTON has 6 letters. The number of 4 letter words that can be formed using these 6 letters when repetition allowed is 6^4 and repetition allowed is 6_{P_4} .

Hence the number of 4 letter words that can be formed using the letters of the word PISTON in which at least one letter is repeated is $6^4-6_{P_4}=1296-360=936.$

7. Problem: Find the number of ways of seating 5 Indians, 4 Americans and 3 Russians at a round table so that

(i) all Indians sit together

(ii) no two Russians sit together

(iii) Persons of same nationality sit together.

Solution: (i) Treat the 5 Indians as one unit. Then we have 4 Americans, 3 Russians and 1 unit of Indians =8 entities.

They can be arranged at a round table in $(8-1)! = 7!$ ways.

Now the 5 Indians among themselves can be arranged in 5! ways. Hence, the required number of arrangements is $7! \times 5!$.repeated is $6^4 - 6_{P_4} = 1296 - 360 = 936.$

(ii) First we arrange 5 Indians and 4 Americans around a table in $(9-1)! = 8!$ ways. Now there are 9 gaps in between these 9 persons (one gap between any two consecutive persons). The 3 Russians can be arranged in these 9 gaps in 9_{P_3} ways. Hence, the required number of arrangements is $8! \times 9_{P_3}$.

(iii) Treat the 5 Indians as one unit, 4 Americans as one unit and 3 Russians as one unit. These 3 units can be arranged at a round table in $(3-1)! = 2!$ ways. Now the 5 Indians among themselves can be arranged in 5! ways. Similarly, the 4 Americans among themselves can be arranged in 4! ways and 3 Russians among themselves can be arranged in 3! ways. Hence, the required number of arrangements is $2! \times 5! \times 4! \times 3!$.

8. Problem: Find the number of different chains that can be prepared using 7 different coloured beads.

Solution: We know that the number of circular permutations of hanging type that can be formed using n things is $\frac{1}{2}(n-1)!$. as one unit. Hence, the number of different ways of preparing the chains is $\frac{1}{2}(7-1)! = \frac{6!}{2} = 360.$

9. Problem: Find the number of ways of arranging the letters of the words

(i) INDEPENDENCE (ii) MATHEMATICS (iii) SINGING

(iv) PERMUTATION (v) COMBINATION (vi) INTERMEDIATE

Solution: (i) The word INDEPENDENCE has 12 letters in which there are 2 D's, 3 N's and 4 E's. Hence, they can be arranged in $\frac{12!}{2! \times 3! \times 4!}$ ways.

(ii) The word MATHEMATICS has 11 letters in which there are 2 M's, 2 A's and 2 T's. Hence, they can be arranged in $\frac{11!}{2! \times 2! \times 2!}$ ways.

(iii) The word SINGING has 7 letters in which there are 2 I's, 2 N's and 2 G's. Hence, they can be arranged in $\frac{7!}{2! \times 2! \times 2!}$ ways.

(iv) The word PERMUTATION has 11 letters in which there are 2 T's. Hence, they can be arranged in $\frac{11!}{2!}$ ways.

(v) The word COMBINATION has 11 letters in which there are 2 C's, 2 O's and 2 N's. Hence, they can be arranged in $\frac{11!}{2! \times 2! \times 2!}$ ways.

(vi) The word INTERMEDIATE has 12 letters in which there are 3 E's, 2 I's and 2 T's. Hence, they can be arranged in $\frac{12!}{3! \times 2! \times 2!}$ ways.

10. Problem: Find the number of ways of selecting 4 English, 3 Telugu and 2 Hindi books out of 7 English, 6 Telugu and 5 Hindi books.

Solution: The number of ways of selecting

4 English books out of 7 English books $= 7_{C_4}$

3 Telugu books out of 6 Telugu books $= 6_{C_3}$

2 Hindi books out of 5 Hindi books $= 5_{C_2}$

Hence, the number of required ways $= 7_{C_4} \times 6_{C_3} \times 6_{C_3} = 35 \times 20 \times 10 = 7000.$

11. Problem: Find the number of ways of selecting11member cricket team from 7 batsmen, 6 bowlers and 2 wicket keepers so that team contains 2 wicket keepers and at least 4 bowlers.

Solution: The required cricket team have the following compositions.

Bowlers	Wicket Keepers	Bats man	Number of ways of selecting team
4	2	5	$6_{C_4} \times 2_{C_2} \times 7_{C_5} = 15 \times 1 \times 21 = 315$
5	2	3	$6_{C_5} \times 2_{C_2} \times 7_{C_4} = 6 \times 1 \times 35 = 210$
6	2	5	$6_{C_6} \times 2_{C_2} \times 7_{C_3} = 1 \times 1 \times 35 = 35$

Hence, the number of ways of selecting the cricket team is $= 315 + 210 + 35 = 560.$

12. Problem: Prove that $25_{C_4} + \sum_{r=0}^{4} (29-r)_{C_3} = 30_{C_4}$.

Solution: We have by theorem ...... $n_{C_{r-1}} + n_{C_r} = (n+1)_{C_r}$

$$25_{C_4} + \sum_{r=0}^{4} (29-r)_{C_3}$$

$$= 25_{C_4} + \left[25_{C_3} + 26_{C_3} + 27_{C_3} + 28_{C_3} + 29_{C_3}\right]$$

$$= \left(25_{C_3} + 25_{C_4}\right) + 26_{C_3} + 27_{C_3} + 28_{C_3} + 29_{C_3}$$

$$= 26_{C_4} + 26_{C_3} + 27_{C_3} + 28_{C_3} + 29_{C_3} \left[\because 25_{C_3} + 25_{C_4} = 26_{C_4}\right]$$

$$= \left(26_{C_3} + 26_{C_4}\right) + 27_{C_3} + 28_{C_3} + 29_{C_3}$$

$$= 27_{C_4} + 27_{C_3} + 28_{C_3} + 29_{C_3} \left[\because 26_{C_3} + 26_{C_4} = 27_{C_4}\right]$$

$$= 28_{C_4} + 28_{C_3} + 29_{C_3} \left[\because 27_{C_3} + 27_{C_4} = 28_{C_4}\right]$$

$$= 29_{C_4} + 29_{C_3} \left[\because 28_{C_3} + 28_{C_4} = 29_{C_4}\right]$$

$$= 30_{C_4} \left[\because 29_{C_3} + 29_{C_4} = 30_{C_4}\right]$$

13. Problem: If $n_{C_{21}} = n_{C_{27}}$, then find 50_{C_n}.

Solution: We have by theorem ...... If $n_{C_r} = n_{C_s}$, then $n = r + s.$

If $n_{C_{21}} = n_{C_{27}}$, then $n = 21 + 27 = 48.$

Now $50_{C_n} = 50_{C_{48}} = \dfrac{50 \times 49}{2 \times 1} = 25 \times 49 = 1225.$

14. Problem: Find the number of ways of forming a committee of 5 members out of 6 Indians, and 5 Americans so that always the Indians will be in majority in the committee.

Solution: The required committee have the following compositions.

Indians(6)	Americans(5)	Number of ways of selecting committee
5	0	$6_{C_5} \times 5_{C_0} = 6 \times 1 = 6$
4	1	$6_{C_4} \times 5_{C_1} = 15 \times 5 = 75$
3	2	$6_{C_3} \times 5_{C_2} = 20 \times 10 = 200$

Hence, the number of ways of selecting the committee of 5 members out of 6 Indians, and 5 Americans so that always the Indians will be in majority in the committee is $= 6 + 75 + 200 = 281.$

Exercise 14(a)

1. If $n_{P_3} = 1320,$ then find $n.$

2. If $n_{P_7} = 42 n_{P_5},$ then find $n.$

3. If $(n+1)_{P_5} : n_{P_6} = 2:7,$ then find $n.$

4. If $18_{P_{r-1}} : 17_{P_{r-1}} = 9:7,$ then find $r.$

5. Find the number of ways of permuting the letters of the word, TRIANGLE so that
   (i) all vowels come together
   (ii) no two vowels come together
   (iii) The relative position of vowels and consonants are not disturbed.

6. Find the number of ways of permuting the letters of the word, MONDAY so that
   (i) all vowels come together
   (ii) no two vowels come together
   (iii) The relative position of vowels and consonants are not disturbed.
   (iv) No two vowels occupies even places.

7. If the letters of the word BRING are permuted in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'BRING'.
8. If the letters of the word REMAST are permuted in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'REMAST'.
9. If the letters of the word MASTER are permuted in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'MASTER'.
10. Find the number of ways of seating 8 Men and 4 Women around a table so that
   (i) all Women sit together
   (ii) no two Women sit together

11.Find the sum of all 4-digit numbers that can be formed using the digit $1, 2, 4, 5, 6.$

12.Find the number of ways of arranging the letters of the words

(i) ASSOCIATIONS (ii) CHEESE (iii) MISSING

(iv) MISSISSIPPI (v) SPECIFIC (vi) INTELLECTUAL

13. Find the number of ways of selecting 4 Boys and 3 girls from a group of 8 Boys and 5 girls.

14. Find the number of ways of selecting 3 Boys and 3 girls from a group of 6 Boys and 7 girls.

15. Simplify $34_{C_5} + \sum_{r=0}^{4} (38-r)_{C_4}$

16. If $n_{C_4} = 210,$ then find $n.$

17. If $10.n_{C_2} = 3(n+1)_{C_3},$ then find $n.$

18. If $.n_{C_5} = n_{C_6},$ then find $13_{C_n}.$

19. Find the number of ways of selecting11member cricket team from 7 batsmen and 6 bowlers so that team contains at least 5 bowlers.

Key concepts

1. From a given set of elements (similar or not) selecting some or all of them and arranging them in a line is called a *linear permutation* or simply a *permutation.*

2. From a given set of things (similar or not) choosing some or all of them and arranging them around a circle is called a *circular permutation.*

3. If n, r are positive integers and $r \le n,$ then the number of permutations of n dissimilar things taken r at a time is $n(n-1)(n-2)...(n-r+1)$

4. The number of permutations of n dissimilar things taken r at a time is denoted by n_{P_r} or $P(n,r).$ However, we use the notation n_{P_r} only. Thus, for $1 \le r \le n,$

$n_{P_r} = n(n-1)(n-2)...(n-r+1)$ and we write $n_{P_0} = 1$ by convention.

5. If $n \ge 1$ and $0 \le r \le n,$ then $n_{P_r} = \dfrac{n!}{(n-r)!}$

6. $n_{P_r} = n(n-1)(n-2)...(n-r+1)$

7. $n_{P_n} = n!$ and $n_{P_0} = 1$

8. Let n, r be positive integers and $1 \le r \le n,$ then

$$n_{P_r} = (n-1)_{P_r} + r.\,(n-1)_{P_{(r-1)}}$$

9. Let n, r be positive integers. If the repetition of things is allowed, then the number of permutations of n dissimilar things taken r at a time is $n^r.$

10. The number of circular permutations of n dissimilar things taken all at a time is $(n-1)!$

11. In case of like garlands of flowers, chains of beads etc., there is no distinction between the clockwise and anti- clockwise arrangements of the same circular permutation. They will be treated as a single circular permutation. In such cases, number of circular permutations of n dissimilar things taken all at a time is $\frac{1}{2}(n-1)!$

12. The number of linear permutations of n things in which p things are alike and the rest are different is $\frac{n!}{p!}$.

1. The number of linear permutations of n things in which there are p like things of one type, q like things of second type, r like things of third type and the rest are different is $\frac{n!}{p!q!r!}$.

13. The number of combinations of n dissimilar things taken r at a time is denoted by n_{C_r} or $C(n,r)$ or $\binom{n}{r}$

14. The number of combinations of n dissimilar things taken r at a time is denoted by $\frac{n_{P_r}}{r!}$. That is $n_{C_r} = \frac{n_{P_r}}{r!} = \frac{n!}{r!(n-r)!}$

15. The number of different subsets of r elements of a set containing n elements is n_{C_r}.

16. If n, r are integers with $0 \le r \le n$, then $n_{C_r} = n_{C_{n-r}}$

17. For any positive integer n, $n_{C_0} = n_{C_n} = 1$

18. If m, n are distinct positive integers then the number of ways of dividing things $(m+n)$ into two groups containing m things and n things is $\frac{(m+n)!}{m!n!}$

19. If m, n, p are distinct positive integers then the number of ways of dividing things $(m+n+p)$ into three groups containing m things, n things and p things is $\frac{(m+n+p)!}{m!n!p!}$

20. The number of ways of dividing $2n$ dissimilar things into two equal groups containing n things in each is $\frac{2n!}{2!n!n!}$

21. The number of ways of dividing mn dissimilar things into m equal groups containing n things in each is $\frac{mn!}{m!(n!)^m}$

22. The number of ways of distributing mn dissimilar things equally among m persons is $\frac{mn!}{(n!)^m}$

23. For $0 \le r, s \le n$, if $n_{C_r} = n_{C_s}$, then either $r = s$ or $r + s = n$

24. If $1 \le r \le n$, then $n_{C_{r-1}} + n_{C_r} = (n+1)_{C_r}$

25. If $2 \le r \le n$, then $n_{C_{r-2}} + 2.n_{C_{r-1}} + n_{C_r} = (n+2)_{C_r}$

26. If p things are alike of one kind, q things are alike of second kind and r things are alike of third kind, then the number of ways of selecting any number of things (one or more) out of these $(p+q+r)$ things is $(p+1)(q+1)(r+1)-1$.

27. The number of ways of selecting one or more things out of n dissimilar things is 2^n-1.

Answers

Exercise 14(a)

1. 12 2. 12 3. 11 4. 5 5. (i) $6!\times 3!$ (ii) $5!\times 6_{P_3}$ (iii) $5!\times 3!$

6. (i) $5!\times 3!$ (ii) $4!\times 5_{P_2}$ (iii) $4!\times 2!$ (iv) $3_{P_2}\times 4!$ 7. 59 8. 391 9. 257

10. (i) $8!\times 4!$ (ii) $7!\times 8_{P_4}$ 11. 4,79,952 12. (i) $\frac{12!}{2!\times 3!\times 2!\times 2!}$ (ii) $\frac{6!}{3!}$ (iii) $\frac{7!}{2!\times 2!}$

(iv) $\frac{11!}{4!\times 4!\times 2!}$ (v) $\frac{8!}{2!\times 2!}$ (vi) $\frac{12!}{2!\times 2!\times 3!}$ 13. $8_{C_4}\times 5_{C_3}$ 14. $6_{C_3}\times 7_{C_3}$ 15. 39_{C_5}

16. 10 17. 9 18. 78 19. $6_{C_5}\times 7_{C_6}+6_{C_6}\times 7_{C_5}=63$

15. PROBABILITY

Introduction:

Probability is an important branch of mathematics that deals with the phenomena of chance or randomness. In daily life, we talk informally about the possibility of some event to happen. For example, while leaving the house in the morning on a cloudy day, one may have to decide to take an umbrella even if it is not raining because it may possibly rain later in the day. The theory of probability is developed initially to explain such type of decisions mathematically. In other words, Probability is a measure of uncertainty.

Historically the theory of probability has its origin to gambling and games of chance. Chevalier de Mere $(1607-1685)$, a French gambler approached a French mathematician Blaise Pascal $(1623-1662)$ to find a solution to a problem related to gambling. After giving a solution to Chevalier's problem, Pascal made correspondence with another French mathematician Fermat $(1601-1665)$ to lay the foundations of the theory of probability.

The first attempt towards giving some mathematical rigour to this subject was done by the French mathematician, astronomer and physicist Laplace $(1749-1827)$. In his monumental work *"Theorie analytique des probabilities"*, Laplace gave the classical definition for the probability of an event.

The theory of probability, as we learn today is due to the contribution of Anderi Nikolaevich Kolmogorov $(1903-1987)$ a Russian topologist and probabilist. He laid the set theoretic foundation to probability in his classic work "*foundations of theory of probability*", in 1933.

Although probability theory was initiated in the field of gambling, it now plays an essential role in several branches of Science and Engineering. It is extensively used in the study of genetics to help understand the inheritance of traits. In Computer science, probability theory plays an important role in the study of the complexity of algorithms. This extensive application makes it an important branch of study.

In this chapter, we shall restrict our study to random experiments that result in finitely many, equally likely outcomes. We shall define probability in a classical way and also through the axiomatic approach. Then we introduce some key concepts like equally likely, mutually exclusive, independent events and conditional probability. We shall state without proof the addition theorem, the multiplication theorem and illustrate their applications through some examples.

15.1 Random Experiments and events:

Theory of probability is a study of random or unpredictable nature of experiments. It is helpful in investigating the important features of these experiments. A preliminary knowledge of set theory and permutations and combinations is a pre-requisite for the study of this topic. We shall first introduce some key concepts and terminology that is necessary to develop the theory.

15.1.1 Definition: An experiment that can be repeated any number of times under identical conditions in which:

(i) All possible outcomes of the experiment known in advance,

(ii) The actual outcome in a particular case is not known in advance, is called a random experiment.

We shall now give some examples of random experiments.

15.1.2 Examples: (i) In an experiment of tossing an unbiased coin, we have only two possible outcomes: Head (H) and Tail (T). In particular trial, one does not know in advance, the outcome. This experiment can be performed any number of times under identical conditions are rational fractions. Therefore it is a random experiment.

(ii) Rolling a fair die is also a random experiment. If we denote the six faces of the cubic die with the numbers $1, 2, 3, 4, 5$ and $6,$ of tossing an unbiased coin, then the possible outcome of the experiment is one of the numbers $1, 2, 3, 4, 5$ or 6 appearing on the uppermost face of the die. The six faces of a die may also contain dots in numbers $1, 2, 3, 4, 5, 6.$ In any case, we shall identify the faces of a die, hereafter with the numbers $1, 2, 3, 4, 5, 6.$

(iii) Tossing a fair coin till a tail appears is also a random experiment.

The experiments such as measuring the acceleration due to gravity using a simple pendulum and measuring the volume of a gas by increasing the pressure, keeping the temperature fixed are not random experiments.

In the discussions that follow in this chapter, by a coin and a die, we mean always an unbiased (or fair) coin and unbiased (or fair) die unless specified otherwise.

15.1.3 Definitions: (i) Any possible outcome of a random experiment $\wp$ is called an elementary or simple event.

(ii) The set of all elementary events (possible outcomes) of a random experiment $\wp$ is called the sample space $\mathrm{S},$ associated with $\wp.$ That is, S is the sample space of a random experiment $\wp$ if every element of S is an outcome and every performance of $\wp$ results in an outcome that corresponds to exactly one element of $\mathrm{S}.$

(iii) An elementary event is a point of the sample space.

(iv) A subset E of $\mathrm{S},$ is called an event. That is, a set of elementary events is called an event.

(v) An event E is said to happen (or occur) if an outcome of the experiment belongs to $\mathrm{E}.$ Otherwise, we say that E has not happened (or not occurred)

(vi) The complement of an event $\mathrm{E},$ denoted by $\mathrm{E}^c,$ is the event given by $\mathrm{E}^c = \mathrm{S} - \mathrm{E},$ which is called the complementary event of $\mathrm{E}.$

(vii) The empty set ϕ and set $\mathrm{S},$ being trivial subsets of $\mathrm{S},$ are called impossible event and certain (definite) event respectively.

15.1.4 Examples: (i) In the experiment of 'tossing a coin' the sample space S is given by $S=\{H,T\}$, where H stands for head and T stands for tail. Occurrence of H and occurrence of T are the only two elementary (simple) events, while the occurrence of either H or T is a certain (definite) event.

(ii) In the experiment of rolling a die, the sample space $S=\{1,2,3,4,5,6\}$. Occurrence of a number ≤ 6 on the uppermost face of the die is a certain (definite) event, where as occurrence of an even number on the uppermost face of the die is an event.

15.1.5 Remark: Note that the sample space S of a random experiment $\wp$ may or may not be finite. Throughout this chapter S is taken to be either finite or countably infinite (A set S is said to be countably infinite if there exists a bijection from S to N, the set of natural numbers)

Observe that the sample space S of the experiment given in examples $(i),(ii)$ of 15.1.2 are finite, while that of example (iii) is countably infinite.

15.1.5 Definitions: (i) Two or more events are said to be *mutually exclusive* if the occurrence of one of the events prevents the occurrence of any of the remaining events. Thus events $E_1, E_2, E_3, ..., E_k$ are said to be mutually exclusive if $E_i \cap E_j = \phi$ for $i \neq j, 1 \leq i, j \leq k$.

(ii) Two or more events are said to be *equally likely (or equiprobable)* if there is no reason to expect any one of them to happen reference to the others.

(iii) Two or more events are said to be *exhaustive* if the performance of the experiment always results in the occurrence of at least one of them.

Thus events $E_1, E_2, E_3, ..., E_k$ are said to be exhaustive if $E_1 \cup E_2 \cup E_3 \cup ... \cup E_k = S$.

The following examples illustrate these concepts.

15.1.6 Examples: (i) In the experiment of rolling a die, the event

E_1 : Occurrence of an even number (on the face of the die) and

E_2 : Occurrence of an odd number (on the face of the die) are number mutually exclusive events. They are also exhaustive.

(ii) In the experiment of rolling a pair of dice, let us consider the following events:

E_1 : A sum 7 (of the numbers that appear on the uppermost faces of the dice)

E_2 : A sum 6 (of the numbers that appear on the uppermost faces of the dice)

$E_3:$ A sum 5 (of the numbers that appear on the uppermost faces of the dice) and

$E_4:$ A sum ≥ 7 (of the numbers that appear on the uppermost faces of the dice)

Observe that the events E_1, E_2, E_3 are mutually exclusive events, while the events E_1 and E_4 are not mutually exclusive.

(iii) If a coin is tossed, occurrence of a head H and occurrence of a tail T are equally likely. (iv) In the experiment of rolling a die, the events:

$E_1:$ Occurrence of an even number

$E_2:$ Occurrence of an even number and

$E_3:$ Occurrence of 1 on the face that shows up are exhaustive, but not mutually exclusive.

15.2 Definition of probability:

In this section, we shall start with the classical definition of probability given by Laplace. Next we give the statistical (empirical) definition. We shall then present the Kolmogorov's axiomatic approach to probability which overcomes the short comings of the earlier versions.

15.2.1 Definition: If a random experiment results in n exhaustive, mutually exclusive and equally likely elementary events and m of them are favourable to the happening of an event E, then the probability of occurrence of E (or simply occurrence of E) denoted by $P(E)$ is defined by $P(E) = \frac{m}{n}$

From the definition it is clear that for any event E, $0 \leq P(E) \leq 1$

Since the number of elementary events or outcomes not favourable to this event is $(n-m)$, the probability of non-occurrence of the event E, denoted by $P(E^C)$, is given by

$$P(E^C) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

$$\therefore P(E) + P(E^C) = 1$$

15.2.2 Examples:

1. Example: Two dice are thrown. We now find the probability of getting the same number on the both faces.

Solution: Let E be the event of getting the same number on both the faces of the two dice. So, the number of cases favourable to $\mathrm{E}=6.$ The total number of elementary events or the points on the sample space $=6\times6=36.$ Hence $P(\mathrm{E})=\dfrac{6}{36}=\dfrac{1}{6}.$

2. Example: An integer is picked from 1 to 20, both inclusive. Let us find the probability that it is a prime.

Solution: The sample space S consists of 20 points. Let E be the event that the number picked is a prime. Then $\mathrm{E}=\{2,3,5,7,11,13,17,19\}$. Then the number of elements in E is $8.$

Hence the required probability is $P(\mathrm{E})=\dfrac{8}{20}=\dfrac{2}{5}.$

3. Example: A bag contains 4 red, 5 black and 6 blue balls. Let us find the probability that two balls drawn at random simultaneously from the bag are a red and a black ball.

Solution: Total number of balls in the bag $=4+5+6=15.$ From out of these balls, two balls can be drawn in $15_{C_2}=\dfrac{15\times14}{2}=105$ ways.

Out of 4 red balls, one ball can be drawn in $4_{C_1}=4$ ways and out of 5 black balls, one ball can be drawn in $5_{C_1}=5$ ways. If E is the event of getting a red and a black ball in a draw, the number of cases favourable to $\mathrm{E}=4\times5=20.$

Hence the required probability is $P(\mathrm{E})=\dfrac{20}{105}=\dfrac{4}{21}.$

4. Example: Ten dice are thrown. Find the probability that none of the dice shows the number $1.$

Solution: We can express the sample space of this experiment as a list of 10^{-} tuples formed with the symbols $1,2,3,4,5$ and $6.$ (An n^{-} tuple is an orderly arrangement of n numbers). There are 6^{10} such entries, all of which are equally likely.

Let A be the event that none of the dice shows the number $1.$ The number of outcomes that do not have the number 1 is the number of 10^{-} tuples whose elements are chosen from the symbols $2,3,4,5$ and $6.$ The number of such 10^{-} tuples is $5^{10}.$

Hence the required probability is $P(\mathrm{A})=\dfrac{5^{10}}{6^{10}}=\left(\dfrac{5}{6}\right)^{10}.$

15.2.3 Limitations of classical definition of probability:

The classical definition of the probability has the following limitations:

(i) The outcomes of a random experiment are not equally likely, then the probability of an event in such an experiment is not defined. For instance, the probability of a student passing an examination is not $\frac{1}{2}$, as the outcomes of pass and failure in an examination are not equally likely.

(ii) If the random experiment contains infinitely many outcomes, then this definition cannot be applied to find the probability of an event in such experiment. For example, either in the random experiment of tossing a coin until tail appears or choosing a natural number, there are infinite number of outcomes.

In order to overcome these deficiencies we now consider the relative frequency approach to the definition of probability.

15.2.4 Relative frequency (or statistical or empirical) definition of probability:

Suppose a random experiment is n repeated times, out of which an event E occurs $m(n)$ times. Then the ratio $r_n = \frac{m(n)}{n}$ is called the n^{-}th relative frequency of the event E. Now consider the sequence $r_1, r_2, r_3, ..., r_n, ...$ If r_n tends to a definite limit, say l, as n tends to infinity $i.e., \lim_{n\to\infty} r_n = l,$ then l is defined to be the probability of an event E and we write $P(\mathrm{E}) = \lim_{n\to\infty} r_n = l.$

15.2.5 Deficiencies of the relative frequency definition of probability:

From the above definition we observe the following deficiencies:

(i) Repeating a random experiment infinitely many times is practically impossible.

(ii) The sequence of relative frequencies is assumed to tend to a definite limit, which may not exist.

(iii) The values $r_1, r_2, r_3, ..., r_n$ are not real variables. It is therefore not possible to prove the existence and the uniqueness limit of r_n as $r_n \to \infty$, by applying the methods used in calculus.

15.2.6 Axiomatic approach to probability:

Let S be the sample space of a random experiment $\wp$. Then the set S of all subsets of is called the *power set of* S and is denoted by $P(\mathrm{S})$. Observe that the set of all possible events of this experiment is the power set of S.

We now introduce Kolmogorov's axiomatic approach to the theory of probability, which overcomes the short comings (deficiencies)of both the definitions 15.2.1 and 15.2.4

15.2.7 Definition: Let S be the sample space of a random experiment $\wp$ which is finite. Then a function $\mathrm{P}: P(\mathrm{S}) \to \mathrm{R}$ satisfying the following axioms is called a *probability function*:

(i) $\mathrm{P(E)} \geq 0 \forall \mathrm{E} \in P(\mathrm{S})$ (axiom of non-negativity)

(ii) $\mathrm{P(S)} = 1$ (axiom of certainity)

(iii) If $\mathrm{E}_1, \mathrm{E}_2 \in P(\mathrm{S})$ and $\mathrm{E}_1 \cap \mathrm{E}_2 = \phi$ then $\mathrm{P}(\mathrm{E}_1 \cup \mathrm{E}_2) = \mathrm{P}(\mathrm{E}_1) + \mathrm{P}(\mathrm{E}_2)$

(axiom of additivity)

For each $\mathrm{E} \in P(\mathrm{S})$, the real number $\mathrm{P(E)}$ is called the probability of the event E.

15.2.8 Note: (i) $\mathrm{P}(\phi) = 0$ for any sample space $\mathrm{S}, \mathrm{S} \cup \phi = \mathrm{S}, \mathrm{S} \cap \phi = \phi$,

$$\mathrm{P(S)} = \mathrm{P}(\mathrm{S} \cup \phi) = \mathrm{P(S)} + \mathrm{P}(\phi) \Rightarrow \mathrm{P}(\phi) = 0.$$

(ii) If S is countably infinite, then axiom (iii) of the above definition is to be replaced by $(iii)^*$: If $\{\mathrm{E}_n\}_{n=1}^{\infty}$ is a sequence of pairwise mutually exclusive events, then

$$\mathrm{P}\left(\bigcup_{n=1}^{\infty} \mathrm{E}_n\right) = \sum_{n=1}^{\infty} \mathrm{P}(\mathrm{E}_n) \qquad \text{(axiom of countable additivity)}$$

(ii) Suppose S is the sample space of a random experiment $\wp$. Let P be a probability function. If $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, ..., \mathrm{E}_n$ are finitely many pairwise mutually exclusive events, then

$$\mathrm{P}(\mathrm{E}_1 \cup \mathrm{E}_2 \cup \mathrm{E}_3 \cup ... \cup \mathrm{E}_n) = \mathrm{P}(\mathrm{E}_1) + \mathrm{P}(\mathrm{E}_2) + \mathrm{P}(\mathrm{E}_3) + ... + \mathrm{P}(\mathrm{E}_n)$$

Without proof we give the following theorem.

15.2.9 Theorem: Let S be the sample space of a random experiment $\wp$ and P be a probability function on $P(\mathrm{S})$. Then

(i) $\mathrm{P}(\phi) = 0$

(ii) If E^C is the complementary event of E, then $P(\mathrm{E}^C) = 1 - P(\mathrm{E})$

(iii) $0 \leq \mathrm{E} \leq 1, \forall \mathrm{E} \subseteq \mathrm{S}$.

(iv) If $\mathrm{E}_1 \subseteq \mathrm{E}_2$ then $\mathrm{P}(\mathrm{E}_2 - \mathrm{E}_1) = \mathrm{P}(\mathrm{E}_2) - \mathrm{P}(\mathrm{E}_1)$

(v) If $\mathrm{E}_1 \subseteq \mathrm{E}_2$ then $\mathrm{P}(\mathrm{E}_1) \leq \mathrm{P}(\mathrm{E}_2)$

15.3 Addition and multiplication theorems on probability:

In this section we shall state the addition theorem of probability without proof. Next we define a conditional event, conditional probability and state the multiplication theorem of probability.

15.3.1 Addition theorem on probability: If $\mathrm{E}_1, \mathrm{E}_2$ are any two events of a random experiment and P is a probability function, then

$$\mathrm{P}(\mathrm{E}_1 \cup \mathrm{E}_2) = \mathrm{P}(\mathrm{E}_1) + \mathrm{P}(\mathrm{E}_2) - \mathrm{P}(\mathrm{E}_1 \cap \mathrm{E}_2).$$

15.3.2 Remarks:

(i) In the discussion that follows, unless otherwise specified, by a probability function we mean a function that satisfies all the properties mentioned in the definition (15.2.7).

(ii) Suppose that the sample space of a random experiment is a finite set S and $p : \mathrm{S} \to \mathrm{R}$ be such that $p(a) \geq 0$ for all $a \in \mathrm{S}$ and $\sum_{a \in \mathrm{S}} p(a) = 1.$ For any event

$$\mathrm{P}(\mathrm{E}) = \begin{cases} \sum_{a \in \mathrm{S}} p(a) & \text{if } \mathrm{E} \neq \phi \\ 0 & \text{if } \mathrm{E} = \phi \end{cases}$$

Thus any function p defined from S into the set of non-negative real numbers satisfying $\sum_{a \in \mathrm{S}} p(a) = 1$ defines a probability function.

(iii) Some set-theoretic descriptions of various events, useful in solving problems on probability are listed below.

(a) If event A or event B to occur, set theoretic description: $\mathrm{A} \cup \mathrm{B}$

(b) If both events A and B to occur, set theoretic description: $\mathrm{A} \cap \mathrm{B}$

(c) If neither A nor B to occur, set theoretic description: $(\mathrm{A} \cup \mathrm{B})^C = \mathrm{A}^C \cap \mathrm{B}^C$

(d) If A occurs and B does not occur, set theoretic description: $\mathrm{A} \cap \mathrm{B}^C$

15.3.3 Definition: Suppose A and B are two events of a random experiment. If the event B to occurs after the occurrence of the event A, then the event: happening of B after the happening of the event A, is called a *conditional event* and it is denoted by $\mathrm{B}|\mathrm{A}$. Similarly happening of A after the happening of the event B is denoted by $\mathrm{A}|\mathrm{B}$.

15.3.4 Definition: If A and B are two events of a sample space S and $P(A) \neq 0$, then the probability of event B after the occurrence of the event A is called a *conditional probability* of B given A and it is denoted by $P(B|A)$:

We define $P(B|A) = \frac{P(A \cap B)}{P(A)}; P(A) \neq 0$ Similarly $P(A|B) = \frac{P(A \cap B)}{P(B)}; P(B) \neq 0$

15.3.5 Multiplication theorem on probability: If A and B are two events of a random experiment with $P(A)>0$ and $P(B)>0$, then $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$

15.3.6 Solved Problems:

1. Problem: Find the probability of throwing a total score of 7 with 2 dice

Solution: The sample space S of this experiment is given by

$$S = \left\{ \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{array} \right\}$$

In a typical element the first coordinate represents the score on the first die and the second coordinate represents the score on the second die. There are 36 points in S and all these points are equally likely *i.e.*, $n(S) = 36$. Let E be the event of getting a total score of 7 is given by $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ so that $n(E) = 6$.

The probability of throwing a total score of 7 with 2 dice is $P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$.

2. Problem: Find the probability of obtaining 2 tails and 1 head when 3 coins are tossed.
Solution: The sample space S of this experiment is given by
$S = \{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$. There are 8 points in S and all these points are equally likely *i.e.*, $n(S) = 8$. Let E be the event of obtaining 2 tails and 1 head when 3 coins are tossed is given by $E = \{HTT, THT, TTH\}$ so that $n(E) = 3$.
The probability of obtaining 2 tails and 1 head when 3 coins are tossed is
$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$.

3. Problem: A page is opened at random from a book containing 200 pages what is the probability that the number on the page is a perfect square.
Solution: The sample space S of this experiment is given by $S = \{1,2,3,...,200\}$, so that the number of points in the sample space is $n(S) = 200$. Let E be the event of drawing a page whose number on the page is a perfect square. Then $E = \{1,4,9,...,196\}$ so that

$n(\text{E}) = 14.$ The probability of drawing a page whose number on the page is a perfect squar $\text{P(E)} = \frac{n(\text{E})}{n(\text{S})} = \frac{14}{200} = \frac{7}{100}.$

4. Problem: If A and B are events with P(A)=0.5,P(B)=0.4 and $\text{P(A} \cap \text{B)=0.3}$ find the probability that (i) A does not occur (ii) neither A nor B occurs.

Solution: Given P(A)=0.5,P(B)=0.4 and $\text{P(A} \cap \text{B)=0.3}$.

We know that A^C denotes the A does not occur and $(\text{A} \cup \text{B})^C$ denotes neither A nor B occurs.

(i) We have $\text{P(A}^C\text{)=1} - \text{P(A)=1} - 0.5\text{=0.5}.$

(ii) By addition theorem $\text{P(A} \cup \text{B)=P(A)} + \text{P(B)} - \text{P(A} \cap \text{B)=0.5} + 0.4 - 0.3\text{=0.6}$

Now $\text{P((A} \cup \text{B)}^C\text{)=1} - \text{P(A} \cup \text{B)=1} - 0.6\text{=0.4}.$

5. Problem: Two dice are rolled. What is the probability that none of the dice shows the number '2'?

Solution: The sample space S of this experiment is given by

$$\text{S} = \left\{ \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{array} \right\}$$

In a typical element the first coordinate represents the score on the first die and the second coordinate represents the score on the second die. There are 36 points in S and all these points are equally likely *i.e.,* $n(\text{S}) = 36.$ Let E be the event that none of the dice shows the number '2'? is given by

$$\text{E} = \left\{ \begin{array}{l} (1,1),(1,3),(1,4),(1,5),(1,6),(3,1),(3,3),(3,4),(3,5),(3,6),(4,1),(4,3), \\ (4,4),(4,5),(4,6),(5,1),(5,3),(5,4),(5,5),(5,6),(6,1),(6,3),(6,4),(6,5),(6,6) \end{array} \right\}$$

so that $n(\text{E}) = 25.$

The probability of none of the dice shows the number '2'? is $\text{P(E)} = \frac{n(\text{E})}{n(\text{S})} = \frac{25}{36}.$

6. Problem: Find the probability that a non-leap year contains (i) 53 Sundays (ii) 52 Sundays only.

Solution: The sample space S of this experiment is given by $\text{S} = \{S, M, T, W, Th, F, Sa\}$ so that the number of points in the sample space is $n(\text{S}) = 7.$

(i) Let A be the event that a non-leap year contains 53 Sundays. Then $\text{A} = \{S\}$ so that $n(\text{A}) = 1.$ The probability that a non-leap year contains 53 Sundays is $\text{P(A)} = \frac{n(\text{A})}{n(\text{S})} = \frac{1}{7}.$

(ii) Let B be the event that a non-leap year contains 52 Sundays. Then $B=\{M,T,W,Th,F,Sa\}$ so that $n(B)=6.$ The probability that a non-leap year contains 52 Sundays is $P(B)=\frac{n(B)}{n(S)}=\frac{6}{7}.$

7. Problem: Find the probability that a leap year contains (i) 53 Sundays (ii) 52 Sundays only.

Solution: The sample space S of this experiment is given by $S=\{SM,MT,TW,WTh,ThF,FSa,SaS\}$ so that the number of points in the sample space is $n(S)=7.$

(i) Let A be the event that a leap year contains 53 Sundays. Then $A=\{SM,SaS\}$ so that $n(A)=2.$ The probability that a leap year contains 53 Sundays is $P(A)=\frac{n(A)}{n(S)}=\frac{2}{7}.$

(ii) Let B be the event that a non-leap year contains 52 Sundays. Then $B=\{MT,TW,WTh,ThF,FSa\}$ so that $n(B)=5.$ The probability that a leap year contains 52 Sundays is $P(B)=\frac{n(B)}{n(S)}=\frac{5}{7}.$

8. Problem: Find the probability of drawing an Ace or a Spade from a well shuffled pack of 52 playing cards.

Solution: Let A be the event of drawing an Ace and B be the event of drawing a Spade. We have $n(S)=52, n(A)=4, n(B)=13$ and $n(A\cap B)=1.$

Also $P(A)=\frac{n(A)}{n(S)}=\frac{4}{52}=\frac{1}{13}$, $P(B)=\frac{n(B)}{n(S)}=\frac{13}{52}=\frac{1}{4}$, $P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{52}.$

By addition theorem $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow P(A\cup B)=\frac{1}{13}+\frac{1}{4}-\frac{1}{52}=\frac{4+13-1}{52}=\frac{16}{52}=\frac{4}{13}$$

9. Problem: In an experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probabilities of $A, B, A\cap B$ and $A\cup B.$

Solution: Let S be the sample space of the experiment of drawing a card from a pack, *i.e.*, $n(S)=52.$ Let A be the event of drawing a Spade and B be the event of drawing a pictured card (King, Queen or Jack). We have $n(A)=13, n(B)=12$ and $n(A\cap B)=3.$

Now $P(A)=\frac{n(A)}{n(S)}=\frac{13}{52}=\frac{1}{4}$, $P(B)=\frac{n(B)}{n(S)}=\frac{12}{52}=\frac{3}{13}$, $P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{3}{52}.$

By addition theorem $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow P(A\cup B)=\frac{1}{4}+\frac{1}{13}-\frac{3}{52}=\frac{13+4-3}{52}=\frac{14}{52}=\frac{7}{26}.$$

10. Problem: A, B, C are 3 news papers from a city. 20% of the population read A, 16% read B, 14% read C, 8% both A and B, 5% both A and C, 4% both B and C and 2% all the three. Find the percentage of the population who read at least one news paper.

Solution: The probability that the population read news paper A is

$P(A) = 20\% = \frac{20}{100} = \frac{1}{5}$, Similarly, $P(B) = 16\% = \frac{16}{100} = \frac{4}{25}$, $P(C) = 16\% = \frac{14}{100} = \frac{7}{50}$,

$P(A \cap B) = 8\% = \frac{8}{100} = \frac{2}{25}$,

$P(B \cap C) = 4\% = \frac{4}{100} = \frac{1}{25}$, $P(C \cap A) = 5\% = \frac{5}{100} = \frac{1}{20}$,

$P(A \cap B \cap C) = 2\% = \frac{2}{100} = \frac{1}{50}$. We have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$\Rightarrow P(A \cup B \cup C) = \frac{1}{5} + \frac{4}{25} + \frac{7}{50} - \frac{2}{25} - \frac{1}{25} - \frac{1}{20} + \frac{1}{50}$$

$$= \frac{20+16+14-8-4-5+2}{100} = \frac{35}{100} = 35\%.$$

11. Problem: Let A and B be independent events with $P(A) = 0.2, P(B) = 0.5$.

Find (i) $P(A|B)$ (ii) $P(B|A)$ (iii) $P(A \cup B)$

Solution: Given that A and B be independent events such that $P(A) = 0.2, P(B) = 0.5$.

We have $P(A \cap B) = P(A)P(B) \Rightarrow P(A \cap B) = 0.2 \times 0.5 = 0.1$

(i) We have $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.5} = \frac{1}{5}$.

(ii) We have $P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = \frac{2}{5}$.

(iii) By addition theorem $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cup B) = 0.2 + 0.5 - 0.1 = 0.6.$$

12. Problem: A pair of dice is thrown. Find the probability that either of the dice shows 2 when their sum is 6

Solution: Given that a pair of dice is thrown so the sample space contains 36 points *i.e.*, $n(S) = 36$. Let A be 2 appears on either of the dice.

$A = \{((1,2),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),(6,2)\}$ *i.e.*, $n(A) = 11$.

So that $P(A) = \frac{n(A)}{n(S)} = \frac{11}{36}$.

Let B be the sum on the two sides is 6 then $B=\{(1,5),(2,4),(3,3),(4,2),(5,1)\}.\ i.e., n(B)=5.$

So that $P(B)=\frac{n(B)}{n(S)}=\frac{5}{36}.$

Clearly, $A\cap B=\{(2,4),(4,2)\}\ i.e., n(A\cap B)=2.$ So that $P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{2}{36}.$

We have $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{2/36}{5/36}=\frac{2}{5}.$

13. Problem: A pair of dice is thrown. Find the probability that neither of the dice shows 2 when their sum is 7

Solution: Given that a pair of dice is thrown so the sample space contains 36 points $i.e., n(S)=36.$ Let A be 2 appears on neither of the dice.

$$A=\left\{\begin{matrix}(1,1),(1,3),(1,4),(1,5),(1,6),(3,1),(3,3),(3,4),(3,5),(3,6),(4,1),(4,3),\\(4,4),(4,5),(4,6),(5,1),(5,3),(5,4),(5,5),(5,6),(6,1),(6,3),(6,4),(6,5),(6,6)\end{matrix}\right\}$$

$i.e., n(A)=25.$ So that $P(A)=\frac{n(A)}{n(S)}=\frac{25}{36}.$

Let B be the sum on the two sides is 7 then

$B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.\ i.e., n(B)=6.$

So that $P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}.$

Clearly, $A\cap B=\{(1,6),(3,4),(4,3),(6,1)\},\ i.e., n(A\cap B)=4.$

So that $P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{4}{36}.$

We have $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{4/36}{6/36}=\frac{2}{3}.$

14. Problem: If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

Solution: Let S be a sample space that two numbers are selected randomly from 20 consecutive natural numbers $i.e., n(S)=20_{C_2}=190.$

(i) Let A be an event that the sum of the two numbers is even which is possible only when either both are even or odd $i.e., n(A)=10_{C_2}+10_{C_2}=45+45=90.$

The probability that the sum of the two numbers is an even number is $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{90}{190}=\dfrac{9}{19}$.

(ii) Let B be an event that the sum of the two numbers is odd which is possible only when one is even and another one is odd *i.e.,* $n(B)=10_{C_1}\times 10_{C_1}=10\times 10=100$.

The probability that the sum of the two numbers is an odd number is $P(B)=\dfrac{n(B)}{n(S)}=\dfrac{100}{190}=\dfrac{10}{19}$.

15. Problem: A bag contains 12 two rupee coins 7 one rupee coins and 4 half rupee coins. If three coins are selected at random then find the probability that

(i) The sum of three coins is Maximum

(ii) The sum of three coins is minimum

(iii) Each coin is of different value.

Solution: Given that a bag contains 12 two rupee coins 7 one rupee coins and 4 half rupee coins so that the number of coins in the bag is 23. Let S be a sample space that three coins are selected at random *i.e.,* $n(S)=23_{C_3}$.

(i) Let A be an event that the sum of the three coins is Maximum which is possible only when all are selected from 12 two rupee coins *i.e.,* $n(A)=12_{C_3}$.

The probability that the sum of the three coins is Maximum is $P(A)=\dfrac{n(A)}{n(S)}=\dfrac{12_{C_3}}{23_{C_3}}$.

(ii) Let B be an event that the sum of the sum of the three coins is minimum which is possible only when all are selected from 4 half rupee coins *i.e.,* $n(B)=4_{C_3}$.

The probability that the sum of the three coins is Maximum is $P(B)=\dfrac{n(B)}{n(S)}=\dfrac{4_{C_3}}{23_{C_3}}$.

(iii) Let C be an event that each coin is of different value *i.e.,* $n(C)=12_{C_1}\times 7_{C_1}\times 4_{C_1}$.

The probability that each coin is of different value is $P(C)=\dfrac{n(C)}{n(S)}=\dfrac{12_{C_1}\times 7_{C_1}\times 4_{C_1}}{23_{C_3}}$.

16. Problem: If A,B are two events such that $P(A\cup B)=0.65$, $P(A\cap B)=0.15$. Then find the value of $P(A^C)+P(B^C)$.

Solution: Given that A,B are two events such that $P(A\cup B)=0.65$, $P(A\cap B)=0.15$.

$P(B \cap C) = 4\% = \frac{4}{100} = \frac{1}{25}$, $P(C \cap A) = 5\% = \frac{5}{100} = \frac{1}{20}$,

$P(A \cap B \cap C) = 2\% = \frac{2}{100} = \frac{1}{50}$. We have

$$P(A^C) + P(B^C) = 1 - P(A) + 1 - P(B)$$

$$= 2 - [P(A) + P(B)] = 2 - [P(A \cup B) + P(A \cap B)]$$

$$= 2 - [0.65 + 0.15] = 2 - 0.8 = 1.2$$

17. Problem: If A, B are independent events such that $P(A) = 0.6$, $P(B) = 0.7$. Then compute (i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A|B)$ (iv) $P(A^C \cap B^C)$.

Solution: Given that A, B are two independent events such that $P(A) = 0.6$, $P(B) = 0.7$.

(i) We have $P(A \cap B) = P(A)P(B) \Rightarrow P(A \cap B) = 0.6 \times 0.7 = 0.42$

(ii) By addition theorem $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cup B) = 0.6 + 0.7 - 0.42 = 1.3 - 0.42 = 0.88$$

(iii) We have $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.42}{0.7} = 0.6$

(iv) We have $P(A^C \cap B^C) = P(A \cup B)^C = 1 - P(A \cup B) = 1 - 0.88 = 012$

18. Problem: A problem in calculus is given to two students A and B whose chances of solving it are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability of the problem being solved if both of them try independently.

Solution: Let E_1 and E_2 denote the events that the problem is solved by two students A and B respectively. We are given that $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{1}{4}$. Since E_1 and E_2 are two independent events so that $P(E_1 \cap E_2) = P(E_1)P(E_2) = \frac{1}{3}.\frac{1}{4} = \frac{1}{12}$.

The probability that the problem being solved is $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{4+3-1}{12} = \frac{6}{12} = \frac{1}{2}.$$

19. Problem: The probability that a boy A will get a scholarship is 0.9 and that another boy B will get is 0.8. What is the probability that at least one of them will get the scholarship.

Solution: Let $P(A)$ be the probability that a boy A will get a scholarship *i.e.,* $P(A) = 0.9$ and $P(B)$ be the probability that a boy B will get a scholarship *i.e.,* $P(B) = 0.8$.

Since A and B are two independent events so that $P(A \cap B) = P(A)P(B)$

$= 0.9 \times 0.8 = 0.72.$

The probability that at least one of them will get the scholarship is $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9 + 0.8 - 0.72 = 0.98.$

20. Problem: If A, B are two events then show that $P(A|B)P(B) + P(A|B^C)P(B^C) = P(A)$

Solution: Given that A, B are two events.

Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ and $P(A|B^C) = \dfrac{P(A \cap B^C)}{P(B^C)}$

we have $P(A|B)P(B) + P(A|B^C)P(B^C) = \dfrac{P(A \cap B)}{P(B)}.P(B) + \dfrac{P(A \cap B^C)}{P(B^C)}P(B^C)$

$$= P(A \cap B) + P(A \cap B^C) = P(A)$$

21. Problem: A fair die is rolled. Consider the events $A = \{1,3,5\}, B = \{2,3\}$ and $C = \{2,3,4,5\}$ find

(i) $P(A \cap B), P(A \cup B)$ (ii) $P(A|B), P(B|A)$

(iii) $P(A|C), P(C|A)$ (iv) $P(B|C), P(C|B)$

Solution: Given that fair die is rolled.

Also $A = \{1,3,5\}, B = \{2,3\}$ and $C = \{2,3,4,5\}$

$\therefore P(A) = \dfrac{3}{6} = \dfrac{1}{2}, P(B) = \dfrac{2}{6} = \dfrac{1}{3}, P(C) = \dfrac{4}{6} = \dfrac{2}{3},$

(i) We have $P(A \cap B) = P(A)P(B) \Rightarrow P(A \cap B) = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6}$

Also $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{6} = \dfrac{3+2-1}{2} = \dfrac{4}{6} = \dfrac{2}{3}$

(ii) We have $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/6}{1/3} = \dfrac{1}{2}$

Also $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$

(iii) We have $P(A \cap C) = P(A)P(C) = \dfrac{1}{2}.\dfrac{2}{3} = \dfrac{1}{3}$

Now $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

Also $P(C|A) = \dfrac{P(A \cap C)}{P(A)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$

(iv) We have $P(B\cap C) = P(B)P(C) = \frac{1}{3}.\frac{2}{3} = \frac{2}{9}$

Now $P(B|C) = \frac{P(B\cap C)}{P(C)} = \frac{2/9}{2/3} = \frac{1}{3}$

Also $P(C|B) = \frac{P(B\cap C)}{P(B)} = \frac{2/9}{1/3} = \frac{2}{3}$

22. Problem: Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of event that all 3 screws are non-defective assuming that the drawing is

(i) with replacement (ii) without replacement.

Solution: Given that there are 50 screws out of which 5 are defective, number of non-defective are 45.

(i) Three screws are drawn at random from a lot of 50 screws with replacement $i.e., n(S) = 50_{C_1} \times 50_{C_1} \times 50_{C_1} = 50\times 50\times 50$ are defective.

Let A be an event that the three of the screws are drawn at random are non-defective $i.e., n(A) = 45_{C_1} \times 45_{C_1} \times 45_{C_1} = 45\times 45\times 45$

The probability that three of the screws are drawn at random with replacement are non-defective is $P(A) = \frac{n(A)}{n(S)} = \frac{45\times 45\times 45}{50\times 50\times 50} = \frac{9\times 9\times 9}{10\times 10\times 10} = \left(\frac{9}{10}\right)^3$

(ii) Three screws are drawn at random from a lot of 50 screws without replacement $i.e., n(S) = 50_{C_3}$ are defective.

Let B be an event that the three of the screws are drawn at random are non-defective $i.e., n(B) = 45_{C_3}$

The probability that three of the screws are drawn at random without replacement are non-defective is

$$P(B) = \frac{n(B)}{n(S)} = \frac{45_{C_3}}{50_{C_3}} = \frac{45\times 44\times 43}{50\times 49\times 48} = \frac{3\times 11\times 43}{10\times 49\times 4} = \frac{1419}{1960}$$

Exercise 15(a)

1. Find the probability of throwing a total score of 8 with 2 dice

2. Find the probability of throwing a total score of 10 with 2 dice

3. Find the probability of throwing a total score of 11 with 2 dice

4. Find the probability of obtaining 1 tails and 2 head when 3 coins are tossed.

5. Find the probability of obtaining at least 1 head when 3 coins are tossed.

6. Find the probability of obtaining at least 1 tail when 3 coins are tossed.

7. Find the probability of obtaining exactly 2 heads when 3 coins are tossed.

8. Find the probability of obtaining exactly 2 tails when 3 coins are tossed.

9. A page is opened at random from a book containing 1000 pages what is the probability that the number on the page is a perfect square.

10. A page is opened at random from a book containing 100 pages what is the probability that the number on the page is a prime number.

11. A page is opened at random from a book containing 150 pages what is the probability that the number on the page is an odd number.

12. A page is opened at random from a book containing 50 pages what is the probability that the number on the page is an even number.

13. Find the probability that a non-leap year contains (i) 53 Mondays (ii) 52 Mondays only.

14. Find the probability that a leap year contains (i) 53 Mondays (ii) 52 Mondays only.

15. Find the probability of drawing an Ace or a King from a well shuffled pack of 52 playing cards.

16. Find the probability of drawing a King or a Queen from a well shuffled pack of 52 playing cards.

17. Find the probability of drawing a Diamond from a well shuffled pack of 52 playing cards.

18. Find the probability of drawing a Red card from a well shuffled pack of 52 playing cards.

19. Find the probability of drawing a Black from a well shuffled pack of 52 playing cards.

20. A pair of dice is thrown. Find the probability that either of the dice shows 3 when their sum is 7

21. A pair of dice is thrown. Find the probability that neither of the dice shows 3 when their sum is 7

22. If two numbers are selected randomly from 30 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

Key concepts

1. An experiment that can be repeated any number of times under identical conditions in which all possible outcomes of the experiment known in advance, the actual outcome in a particular case is not known in advance, is called a random experiment.

2. Any possible outcome of a random experiment $\wp$ is called an elementary or simple event.

3. The set of all elementary events (possible outcomes) of a random experiment $\wp$ is called the sample space S, associated with $\wp$.

4. Two or more events are said to be *mutually exclusive* if the occurrence of one of the events prevents the occurrence of any of the remaining events. Thus events $E_1, E_2, E_3, ..., E_k$ are said to be mutually exclusive if $E_i \cap E_j = \phi$ for $i \neq j, 1 \leq i, j \leq k.$

5. Two or more events are said to be *equally likely (or equi probable)* if there is no reason to expect any one of them to happen reference to the others.

6. Two or more events are said to be *exhaustive* if the performance of the experiment always results in the occurrence of at least one of them. Thus events $E_1, E_2, E_3, ..., E_k$ are said to be exhaustive if $E_1 \cup E_2 \cup E_3 \cup ... \cup E_k = S.$

7. If a random experiment results in n exhaustive, mutually exclusive and equally likely elementary events and m of them are favourable to the happening of an event E, then the probability of occurrence of E (or simply occurrence of E) denoted by $P(E)$ is defined by $P(E) = \dfrac{m}{n}$

8. For any event E, $0 \leq P(E) \leq 1$

9. $P(E) + P(E^C) = 1$

10. $P(E) \geq 0 \forall E \in P(S)$ (axiom of non-negativity)

11. $P(S) = 1$ (axiom of certainity)

12. If $E_1, E_2 \in P(S)$ and $E_1 \cap E_2 = \phi$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ (axiom of additivity)

13. Suppose S is the sample space of a random experiment $\wp$. Let P be a probability function. If $E_1, E_2, E_3, ..., E_n$ are finitely many pairwise mutually exclusive events, then

$$P\left(E_1 \cup E_2 \cup E_3 \cup ... \cup E_n\right) = P\left(E_1\right) + P\left(E_2\right) + P\left(E_3\right) + ... + P\left(E_n\right)$$

14. Let S be the sample space of a random experiment $\wp$ and P be a probability function on $P(S)$. Then

(i) $P(\phi) = 0$

(ii) If E^C is the complementary event of E, then $P(E^C) = 1 - P(E)$

(iii) $0 \leq E \leq 1, \forall E \subseteq S.$

(iv) If $E_1 \subseteq E_2$ then $P\left(E_2 - E_1\right) = P\left(E_2\right) - P\left(E_1\right)$

(v) If $E_1 \subseteq E_2$ then $P(E_1) \le P(E_2)$

15. If E_1, E_2 are any two events of a random experiment and P is a probability function, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

16. Suppose A and B are two events of a random experiment. If the event B to occurs after the occurrence of the event A, then the event: happening of B after the happening of the event A, is called a *conditional event* and it is denoted by $B|A$. Similarly happening of A after the happening of the event B is denoted by $A|B$.

17. If A and B are two events of a sample space S and $P(A) \neq 0$, then the probability of event B after the occurrence of the event A is called a *conditional probability* of B given A and it is denoted by $P(B|A)$:

We define $P(B|A) = \dfrac{P(A \cap B)}{P(A)}; P(A) \neq 0$ Similarly $P(A|B) = \dfrac{P(A \cap B)}{P(B)}; P(B) \neq 0$

18. If A and B are two events of a random experiment with $P(A)>0$ and $P(B)>0$, then $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$

Answers

Exercise 15(a)

1. $\frac{5}{36}$ 2. $\frac{1}{12}$ 3. $\frac{1}{18}$ 4. $\frac{3}{8}$ 5. $\frac{7}{8}$ 6. $\frac{7}{8}$ 7. $\frac{3}{8}$ 8. $\frac{3}{8}$ 9. $\frac{31}{1000}$ 10. $\frac{1}{4}$ 11. $\frac{1}{2}$
12. $\frac{1}{2}$ 13. (i) $\frac{1}{7}$ (ii) $\frac{6}{7}$ 14. (i) $\frac{2}{7}$ (ii) $\frac{5}{7}$ 15. $\frac{2}{13}$ 16. $\frac{2}{13}$ 17. $\frac{1}{4}$ 18. $\frac{1}{2}$ 19. $\frac{1}{2}$
20. $\frac{1}{3}$ 21. $\frac{5}{6}$ 22. (i) $\frac{42}{87}$ (ii) $\frac{15}{29}$.

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

[redacted]

MATHEMATICS – Second Year (w.e.f. 2019-2020)

MODEL QUESTION PAPER

Time: 3 Hours **Max.Marks: 75**

Section – A **10x3=30**

Note:

i) *Answer **all** questions*

*ii) Each question carries **3** marks.*

1. Express $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$ in the form of $a+bi$.

2. Obtain the quadratic equation whose roots are $\frac{m}{n},\frac{n}{m}(m\neq 0, n\neq 0)$

3. Find the Coefficient of x^{11} in $\left(2x^2+\frac{3}{x^3}\right)^{13}$

4. Evaluate $\int\frac{1+\cos 2x}{1-\cos 2x}dx$

5. Evaluate $\int x^2 e^x dx$

6. Evaluate $\int_0^1\frac{1}{1+x^2}dx$

7. Find the equation of the circle whose extremities of a diameter are (7, −3) and (3,5).

8. Find the eccentricity and latus rectum of the hyperbola $x^2 - 4y^2 = 4$

9. Find the mean deviation about the mean for the data 3, 6, 10, 4, 9, 10.

10. Two dice are rolled. Find the probability that none of the dice shows the number 2

Section – B **3x15=45**

Note:

i) *Answer any **3** questions*

*ii) Each question carries **15** marks.*

11.I(a) If the coefficient of x^{10} in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-10} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$, find the relation between a and b where a and b are real numbers.

11.I(b) Resolve $\frac{5x+1}{(x-1)(x+2)}$ into partial fractions.

(or)

11.II(a) If α,β are the roots of the equation $ax^2+bx+c=0$ then find the following

$(i)\,\alpha^2+\beta^2 \quad (ii)\,\alpha^3+\beta^3 \quad (iii)\,\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

11.II(b) If $(n+1)_{P_5} : n_{P_5} = 3:2,$ find $n.$

12.I(a) Evaluate $\int \frac{\sec^2 x}{(1+\tan x)^3}dx$

12.I(b) Evaluate $\int_0^1 \frac{1}{\sqrt{3-2x}}dx$

(or)

12.II(a) Evaluate $\int_0^{\pi/2} \frac{dx}{4+5\cos x}dx$

12.II(b) Solve $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

13.I(a) Show that the line of the line $x+y+1=0$ touches the circle

$x^2+y^2-3x+y+14=0.$ also find the point of contact.

13.I(b) Find the equation of the parabola whose focus is $(3,5)$ and vertex is $(1,3).$

(or)

13.II(a) Find the equation of ellipse whose focus is $(1,-1),$ eccentricity $\frac{2}{3}$ and directrix $x+y+2=0.$

13.II(b) Find the equation of the hyperbola, whose asymptotes are the straight lines $x+2y+3=0,\ 3x+4y+5=0$ and passing through the point $(1,-1)$.

14.I(a) Find the probability that a non-leap year contains (i) 53 Mondays (ii) 52 Mondays.
14.I(b) Find the mean deviation about the median of the following frequency distributions:

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

(or)

14.II(a) If A,B are two events such that $P(A\cup B)=0.65,\ P(A\cap B)=0.15$. Then find the value of $P(A^C)+P(B^C)$.

14.II(b) Find the variance and standard deviation of the following data: $5,12,3,18,6,8,2,10$.

15.I(a) Resolve $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}$ into partial fractions.

15.I(b) Solve $(x^2-y^2)\dfrac{dy}{dx}=xy$

(or)

15.II(a) Find the internal and external centre of similitudes for the following circles: $x^2+y^2=9$ and $x^2+y^2-16x+2y+9=0$

15.II(b) Find the variance and standard deviation of the following frequency distribution:

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA
VOCATIONAL BRIDGE COURSE

Second Year - Paper – II (w.e.f. 2019-2020)

MATHEMATICS SCEME OF EXIMATION (WEIGHTAGE)

Total Questions : 15

Time: 3 Hours **Max.Marks: 75**

Note: In section A – Answer all Questions
In section B – Answer any three Questions

Section – A **10x3=30**

Note:

i) *Answer all the questions*
ii) Each question carries ***2*** *marks.*

1. From Algebra
2. From Algebra
3. From Algebra
4. From Calculus
5. From Calculus
6. From Calculus
7. From Co-ordinate Geometry
8. From Co-ordinate Geometry
9. From Measures of Dispersion
10. From Probability

Section – B **3x15=45**

Note:

i) *Answer any* ***3*** *questions*

ii) Each question carries ***15*** *marks.*

11. From Algebra with internal choice
12. From Calculus with internal choice

13. From Co-ordinate Geometry with internal choice

14. From Probability and Measures of Dispersion with internal choice

15. I(a) – FromAlgebra
I(b) –FromCalculus
OR

II(a) – from Co-ordinate Geometry

II(b) – from Probability / Measures of Dispersion.

.

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

Second Year - Paper – II (w.e.f. 2019-2020)

MATHEMATICS WEIGHTAGE OF MARKS

S.No.	Unit	Weightage of Marks
1.	Algebra	31
2.	Calculus	32
3.	Coordinate Geometry	28
4.	Probability / Measures of Dispersion	29
	Total	**120 Marks**

***

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

Second Year - Paper – II (w.e.f. 2017-2018)

MATHEMATICS WEIGHTAGE OF MARKS

S.No.	Unit	Weightage of Marks
1.	Algebra	31
2.	Calculus	32
3.	Coordinate Geometry	28
4.	Probability / Measures of Dispersion	29
	Total	**120 Marks**

***

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA
VOCATIONAL BRIDGE COURSE

Second Year - Paper – II (w.e.f. 2017-2018)

MATHEMATICS SCEME OF EXIMATION (WEIGHTAGE)
Total Questions : 15

Time: 3 Hours **Max.Marks: 75**

Note: In section A – Answer all Questions
In section B – Answer any three Questions

Section – A **10x3=30**

Note:

i) *Answer all the questions*
ii) *Each question carries **2** marks.*

1. From Algebra
2. From Algebra
3. From Algebra
4. From Calculus
5. From Calculus
6. From Calculus
7. From Co-ordinate Geometry
8. From Co-ordinate Geometry
9. From Measures of Dispersion
10. From Probability

Section – B **3x15=45**

Note:

i) *Answer any **3** questions*
ii) *Each question carries **15** marks.*

11. From Algebra with internal choice
12. From Calculus with internal choice
13. From Co-ordinate Geometry with internal choice
14. From Probability and Measures of Dispersion with internal choice
15. I(a) – From Algebra
I(b) – From Calculus

OR

II(a) – from Co-ordinate Geometry
II(b) – from Probability / Measures of Dispersion.

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

MATHEMATICS – Second Year (w.e.f. 2017-2018)

MODEL QUESTION PAPER

Time: 3 Hours **Max.Marks: 75**

Section – A **10x3=30**

Note:

i) *Answer **all** questions*

ii) *Each question carries **3** marks.*

1. Express $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$ in the form of $a+ib$.
2. Obtain the quadratic equation whose roots are $\frac{m}{n}, \frac{-n}{m} (m \neq 0, n \neq 0)$.
3. Find the Coefficient of x^{11} in $\left(2x^2+\frac{3}{x}\right)^{13}$.
4. Evaluate $\int \frac{1+\cos 2x}{1-\cos 2x} dx$
5. Evaluate $\int x^2 e^x \, dx$
6. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$
7. Find the equation of the circle whose extremities of a diameter are (7, −3) and (3,5).
8. Find the eccentricity and latus rectum of the hyperbola $x^2 - 4y^2 = 4$.
9. Find the mean deviation about the mean for the data 3, 6, 10, 4, 9, 10.
10. Two dice are rolled. Find the probability that none of the dice shows the number 2.

Section – B **3x15=45**

Note:

i) *Answer any **3** questions*

*ii) Each question carries **15** marks.*

11. I (a) Show that the coefficients of x^{11} and x^{12} in the expansion $\left(2+\frac{8x}{3}\right)^{20}$ are equal.

I (b) Resolve $\frac{1}{(1-3x)(1-2x)^2}$ into Partial fractions.

OR

II (a) If α, β are the roots of $ax^2+bx+c=0$, then find (i) $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$ (iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

II (b) If $(n+1)P_5 : nP_5 = 3:2$, then find the value of 'n'.

12. I (a) Evaluate $\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx$.

I (b) Evaluate $\int_0^1 \frac{dx}{\sqrt{3-2x}}$.

OR

II (a) Evaluate $\int_0^{\pi/2} \frac{dx}{4+5\cos x}$.

II (b) Solve: $\frac{dy}{dx}=\frac{xy+y}{xy+x}$.

13. I (a) Show that the line $x+y+1=0$ touches the circle $x^2+y^2-3x+7y+14=0$ and find its point of contact.

I (b) Find the equation of the parabola whose focus is (3,5) and vertex is at the point (1,3)

OR

II (a) Find the equation of the ellipse whose focus is $(1,-1)$, eccentricity is $\frac{2}{3}$ and directrix in $x+y+2=0$.

II (b) Find the equation of the hyperbola passing through the point $(1,-1)$ and whose asymptotes are the lines $x+2y+3=0$ and $3x+4y+5=0$.

14. I (a) Find the probability that a non-leap year contains (i) 52 Mondays (ii) 53 Mondays I (b) Find the mean deviation about the median for the following data:

xi :	5	7	9	10	12	15
fi :	8	6	2	2	2	6

OR

II (a) A and B are two events such that $P(A \cup B) = 0.65$ and $P(A \cap B) = 0.15$. Then find the value of $P(A^e) + P(B^e)$.

II (b) Find the variance and standard deviation of the data: 5,12, 3,18, 6, 8, 2,10 .

15. I (a) Resolve into Partial Fractions: $\dfrac{2x^2 + 3x + 4}{(x - 1)(x^2 + 2)}$

I (b) Solve: $(x^2 - y^2)\dfrac{dy}{dx} = xy$

OR

II (a) Find the internal and external center of similitude of the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 16x + 2y + 9 = 0$,.

II (b) Find the standard Deviation for the data:

xi :	6	10	14	18	24	28	30
fi :	2	4	7	12	8	4	3.

***

BOARD OF INTERMEDIATE EDUCATION- ANDHRA PRADESH, VIJAYAWADA

VOCATIONAL BRIDGE COURSE

MATHEMATICS – Second Year (w.e.f. 2017-2018)

QUESTION BANK

1. COMPLEX NUMBERS

1. If $Z_1 = (3,5)$ and $Z_2 = (2, 6)$ find (i) $Z_1 Z_2$ (ii) $\frac{Z_1}{Z_2}$.

2. If $Z_1 = (6, 3)$ and $Z_2 = (2, -1)$ find (i) $Z_1 Z_2$ (ii) $\frac{Z_1}{Z_2}$.

3. If $Z = (\cos\theta, \sin\theta)$ find $Z - \frac{1}{2}$.

4. Find the multiplicative inverse of (i) $(\sin\theta, \cos\theta)$ (ii) $(7, 24)$ (iii) $(-2,1)$.

5. Express the following complex numbers in the form of $a + ib$:

   (i) $(2 - 3i)(3 + 4i)$ (ii) $\frac{a - ib}{a + ib}$ (iii) $\frac{4 + 3i}{(2 + 3i)(4 - 3i)}$ (iv) $\frac{2 + 5i}{3 - 2i} + \frac{2 - 5i}{3 + 2i}$

6. Find the conjugate of the following complex numbers:

   (i) $(15 + 3i) - (4 - 20i)$ (ii) $(2 + 5i)(-4 + 6i)$ (iii) $\frac{5i}{7 + i}$

7. Find a square root of the following complex numbers:

   (i) $7 + 24i$ (ii) $-8 - 6i$ (iii) $-47 + i.8\sqrt{3}$ (iv) $-5 + 12i$

8. If $(a + ib)^2 = x + iy$, find $x^2 + y^2$.

9. If $(x - iy)^{1/3} = a - ib$, then show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$.

10. Express $\left(\frac{a + ib}{a - ib}\right)^2 - \left(\frac{a - ib}{a + ib}\right)^2$ in the form of $x + iy$.

11. Express the following complex numbers in modules – amplitude form:

    (i) $1 - i$ (ii) $1 + i\sqrt{3}$ (iii) $-1 - i\sqrt{3}$ (iv) $\sqrt{3} + i$

12. Express $-\sqrt{7} + i\sqrt{21}$ in polar form.

13. Express $-1 - i$ in polar form.

2. QUADRATIC EXPRESSIONS

1. Find the value of K, if the equation $x^2 + 2(K+2)x + 9K = 0$ has equal roots.

2. Find the nature of the roots of the following equations

   (i) $4x^2 - 20x + 25 = 0$ (ii) $3x^2 + 7x + 2 = 0$ (iii) $2x^2 - 8x + 3 = 0$

   (iv) $9x^2 - 30x + 25 = 0$ (v) $x^2 - 12x + 32 = 0$ (iv) $2x^2 - 7x + 10 = 0$

3. Obtain the quadratic equations whose roots are given below:

   (i) $\frac{m}{n}, -\frac{n}{m}$ $(m \neq 0, n \neq 0)$ (ii) $-3 \pm 5i$

4. If the following equations have equal roots, find the value of m:

   (i) $(m+1)x^2 = 2(m+3)x + (m+8) = 0$ (ii) $(3m+1)x^2 = 2(m+1)x + m = 0$

   (iii) $(2m+1)x^2 + 2(m+3)x + (m+5) = 0$ (iv) $x^2 - m(2x-8) - 15 = 0$

5. Find the minimum & maximum values of the following quadratic expressions:

   (i) $3x^2 + 2x + 11$ (ii) $2x - 7 - 5x^2$ (iii) $4x - 5x^2 + 2$

   (iv) $x^2 - 5x + 6$ (v) $15 + 4x - 3x^2$ (vi) $x^2 - x + 7$

   (vii) $12x - x^2 - 32$ (viii) $2x + 5 - 3x^2$

   (ix) $ax^2 + bx + a$ $(a \neq 0, a \in R, b \in R)$

6. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the following:

   (i) $\alpha^3 + \beta^3$ (ii) $\alpha^2 + \beta^2$ (iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

   (iv) $\alpha^4\beta^7 + \alpha^7\beta^4$ (v) $\frac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}}$ $(c \neq 0)$

7. If α, β, υ are the roots of equations $4x^3 - 6x^2 + 7x + 3 = 0$, then find $\alpha\beta + \beta\upsilon + \upsilon\alpha$.

8. If the product of the roots of the equation $4x^3 + 16x^2 - 9x - a = 0$, then find '$a$'.

9. If $-1, 2, \alpha$ are the roots of the equation $2x^3 + x^2 - 7x - 6 = 0$, then find 'α'.

10. If $1, \alpha, \beta$ are the roots of $x^3 - 2x^2 - 5x + 6 = 0$, then find α and β.

11. If α, β, υ are the roots of $x^3 + px^2 + qx + r = 0$, then find $\sum \frac{1}{\alpha^2\beta^2}$ and $\sum \frac{1}{\alpha}$.

12. Solve the equation $x^3 - 3x^2 - 16x + 48 = 0$, given that the sum of two of its roots is zero.

13. Find the condition that $x^3 - px^2 - qx - r = 0$ may have the sum of two of its roots is zero.

14. Find the relation between the roots and the coefficients of the equation $3x^3 - 10x^2 + 7x + 10 = 0$.

15. If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, then find the values of a, b, c, d.

16. From a polynomial equation of the lowest degree, whose roots are

    (i) −2, −2, 2, 2 and (ii) 1, 3, 5, 7

3. BINOMIAL THEOREM

1. Write and simplify the first three terms of the expansions

   (i) $\left(\frac{2x}{3} + \frac{7y}{4}\right)^5$ (ii) $\left(\frac{2p}{5} - \frac{3q}{7}\right)^6$ (iii) $(3x - 14y)^7$

2. Write down the last three terms of the expansions

   (i) $(4x + 5y)^7$ (ii) $(3x - 4y)^{10}$ (iii) $(2a + 5b)^8$

3. Find the number of terms in the following expansions. Also, find the middle term (s) in each expansion

   (i) $\left(\frac{3a}{4} + \frac{b}{2}\right)^9$ (ii) $\left(\frac{3p}{4} - 5q\right)^{14}$ (iii) $(2x + 3y)^7$

4. Find the term independent of x in the expansions:

   (i) $\left(\sqrt{x/3} + \frac{3}{2x^3}\right)^{10}$ (ii) $\left(4x^3 + \frac{7}{x^2}\right)^{14}$ (iii) $\left(\frac{2x^2}{3} + \frac{15}{4x}\right)^9$ (iv) $\left(\frac{\sqrt{x}}{3} - \frac{4}{x^2}\right)^{10}$

5. Find the coefficient of

   (i) x^{-6} in $\left(3x - \frac{4}{x}\right)^{10}$ (ii) x^{11} in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$ (iii) x^{-7} in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$

   (iv) x^2 in $\left(7x^3 - \frac{2}{x^2}\right)^9$ (v) x^9 in $\left(2x^2 - \frac{1}{x}\right)^{20}$ (vi) x^{10} in $\left(ax^2 + \frac{1}{bx}\right)^{11}$

   (vii) x^{-10} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

6. Write and simplify

   (i) 6th term in $\left(\frac{2x}{3} + \frac{3y}{2}\right)^9$ (ii) 7th term in $(3x - 4y)^{10}$

   (iii) rth term in $\left(\frac{3a}{5} + \frac{5b}{7}\right)^8$, $(1 \le r \le 9)$

7. Find the middle term (s) in

(i) $\left(\frac{3}{a^3} + 5a^4\right)^{20}$ (ii) $(4x^2 + 5x^3)^{17}$ (iii) $\left(4a + \frac{3b}{2}\right)^{11}$ (iv) $\left(\frac{3x}{7} - 2y\right)^{10}$

8. Find the coefficients of x^{32} and x^{-18} in the expansion $\left(2x^3 - \frac{3}{x^2}\right)^{14}$.

9. Show that the coefficients of x^{11} and x^{12} in the expansion $\left(2 + \frac{8x}{3}\right)^{20}$ are equal.

4. PARTIAL FRACTIONS

Resolve the following into Partial fractions.

1) $\frac{5x+1}{(x-1)(x+2)}$ 2) $\frac{2x+3}{(x+2)(2x+1)}$ 3) $\frac{13x+43}{(2x+5)(x+6)}$

4) $\frac{x^2+5x+7}{(x-3)^3}$ 5) $\frac{1}{(x-1)^2(x-2)}$ 6) $\frac{x-1}{(x-2)^2(x+1)}$

7) $\frac{5x+6}{(x+2)(1-x)}$ 8) $\frac{x+4}{(x^2-4)(x+1)}$ 9) $\frac{2x+3}{(x-1)^3}$

10) $\frac{x^2-x+1}{(x+1)(x-1)^2}$ 11) $\frac{1}{(1-3x)(1-2x)^2}$ 12) $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$

13) $\frac{x^2-3}{(x+2)(x^2+1)}$ 14) $\frac{3x-1}{(x+2)(x^2-x+1)}$ 15) $\frac{3x^2+2x}{(x^2+2)(x-3)}$

16) $\frac{3x}{(x-1)(x-2)^2}$

5. MEASURES OF DISPERSION

1. Find the mean deviation about the mean for the following data:

(i) 38, 70, 48, 42, 55, 63, 46, 54, 44.

(ii) 3, 6, 10, 4, 10, 9

(iii) 6, 7, 10, 12, 13, 4, 12, 16.

2. Find the mean deviation from median for the following data:

   (i) 6, 7, 10, 12, 13, 4, 12, 16.

   (ii) 13, 17, 16, 11, 13, 10, 16, 11, 18, 12, 17.

   (iii) 4, 6, 9, 3, 10, 13, 2.

3. Find the mean deviation about the mean for the following distribution:

(i)

xi	10	11	12	13
fi	3	12	18	12

(ii)

xi	10	30	50	70	90
fi	4	24	28	16	8

(iii)

xi	2	5	7	8	10	35
fi	6	8	10	6	8	2

4. Find the mean deviation about median for the following frequency distribution:

(i)

xi	5	7	9	10	12	15
fi	8	6	2	2	2	6

(ii)

xi	6	9	3	12	15	13	21	22
fi	4	5	3	2	5	4	4	3

5. Find the Variance and standard deviation of the following data:

   (i) 5, 12, 3, 4, 18, 6, 8, 2, 10.

   (ii) 6, 7, 10, 12, 13, 4, 8, 12.

   (iii) 350, 361, 370, 373, 376, 379, 385, 387, 394, 395.

6. Find the variance and standard deviation of the following frequency distribution:

(i)

xi	6	10	14	18	24	28	30
fi	2	4	7	12	8	4	3

(ii)

xi	4	8	11	1	20	24	32
fi	3	5	9	5	4	3	1

7. Find the mean deviation about the mean for the following continuous distribution:

(i)

Hight (in cms)	95-105	105-115	115-125	125-135	135-145	145-155
Number of boys	9	13	26	30	12	10

(ii)

Sales (in Rs. thousands	40-50	50-60	60-70	70-80	80-90	90-100
Number of comapnies	5	15	25	30	20	5

(iii)

Marks obtained	0-10	10-20	20-30	30-40	40-50
Number of students	8	8	15	16	6

8. Find the mean deviation about the median for the following continuous distribution:

(i)

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60	
Number of boys	6	8	14	16	4	2	

(ii)

Class Interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	5	8	7	12	28	20	10	10

(iii)

Age (years)	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
Number of workers	120	125	175	160	150	140	100	30

9. Calculate the variance and standard deviation of the following continuous frequency distribution:

(i)

Class interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

(ii)

Age in years	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Number of members	3	61	32	153	140	51	2

10. Find the mean deviation from the mean of the following data, using step deviation method.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of Students	6	5	8	15	7	6	3

11. The coefficient of variation of two distributions are 60 and 70 and their standard deviations are 21 and 16 respectively. Find their arithmetic means.

12. From the prices of shares X and Y given below, for 10 days of trading, find out which share is more stable?

X	35	54	52	53	56	58	52	50	51	49
Y	108	107	105	105	106	107	104	103	104	101

13. An analysis of monthly wages paid to the workers of two firms A and B belonging to the same industry gives the following data:

	Firm A	Firm B
Number of workers	500	600
Average daily wages (Rs.)	186	175
Variance of distribution of wages	81	100

(i) Which firm A or B, has greater variability in individual wages?

(ii) Which firm has larger wage bill?

14. The scores of two cricketers A and B in 10 innings are given below. Find who is a better run getter and who is a more consistent player.

x_i Scores of A:	40	25	19	80	38	8	67	121	66	76
yi Scores of B:	28	70	31	0	14	111	66	31	25	4

6. THE CIRCLE

1. Find the equation of the circle whose centre (C) and radius (r) are as given below:

   (i) C = (1,4); $r = 5$ (ii) C = (–1, 2); $r = 4$ (iii) C = (a, $-b$); $r = a + b$

   (iv) C = (a, $-b$); $r = \sqrt{a^2 - b^2}$ ($|a| > |b|$) (v) C = (cos α, sin α); $r = 1$

   (vi) C = (–7, –3); $r = 4$ (vii) C = $\left(\frac{5}{2}, -\frac{4}{3}\right)$; $r = 6$ (viii) C = (1, 7); $r = \frac{5}{2}$

   (ix) C = (0, 0); $r = 9$ (x) $C = \left(-\frac{1}{2}, -9\right)$; $r = 5$

2. Find the centre and the radius for the following circles:

   (i) $x^2 + y^2 + 2x - 4y - 4 = 0$ (ii) $3x^2 + 3y^2 - 6x + 4y - 4 = 0$

   (ii) $x^2 + y^2 - 4x - 8y - 41 = 0$ (iv) $3x^2 + 3y^2 - 5x - 6y + 4 = 0$

   (v) $2x^2 + 2y^2 - 3x + 2y - 1 = 0$ (vi) $2x^2 + 2y^2 - 4x + 6y - 3 = 0$

   (vii) $x^2 + y^2 + 2ax - 2by + b^2 = 0$

3. Find the equation of the circle passing through the point (5, 6) and having the centre at the point (-1, 2).

4. (2, 3) is the centre of the circle represented by the equation $x^2 + y^2 + ax + by - 12 = 0$. Find the values of a and b. Also, find the radius of the circle.

5. Find 'a' if the radius of the circle $x^2 + y^2 - 4x + 6y + a = 0$ is 4.

6. Find the equations of the circles whose extremities of a diameter are given below:

   (i) (1, 2), (4, 5) (ii) (–4, 3), (3, –4) (iii) (8, 6), (1, 2) (iv) (4, 2) (1, 5) (v) (7, –3), (3, 5)

   (vi) (1,1), (2, –1) (vii) (3,1), (2, 7) (viii) (0, 0), (8, 5) (ix) (1, 2), (4, 6)

7. Find the equation of the circle passing through three points as given below:

   (i) (1,1), (–2, 2), (–6, 0) (ii) (1, 2), (3, –4), (19,8)

   (iii) (3, 4), (3, 2), (1, 4) (iv) (2,1), (5, 5), (–6, 7)

   (v) (5, 7), (8,1), (1, 3) (vi) (9,1), (7, 9), (6,10)

   (vii) (5, 2), (7, 0), (–1, –4) (viii) (2, 0), (0,1), (4, 5)

   (ix) (0, 0), (2, 0), (0, 2) (x) (1,1), (3, 2), (2, –1)

8. Find the positions of the following points with respect to the following circles. Also state the powers of the points:

(i) P = (2, 4);S ≡ $x^2 + y^2 - 4x - 6y + 11 = 0$

(ii) P = (3, 4);S ≡ $x^2 + y^2 - 4x - 6y + 12 = 0$

(iii) P = (1, 5);S ≡ $x^2 + y^2 - 2x - 4y + 3 = 0$

(iv) P = (2, −1);S ≡ $x^2 + y^2 - 2x - 4y + 3 = 0$

(v) P = (4, 2);S ≡ $2x^2 + 2y^2 - 5x - 4y - 3 = 0$

(vi) P = (1, 2);S ≡ $x^2 + y^2 + 6x + 8y - 96 = 0$

(vii) P = (5, −6);S ≡ $x^2 + y^2 + 8x + 12y + 15 = 0$

(viii) P = (2, 3);S ≡ $x^2 + y^2 - 2x + 8y - 23 = 0$

(ix) P = (0, 0);S ≡ $x^2 + y^2 - 14x + 2y + 25 = 0$

(x) P = (−2, 5);S ≡ $x^2 + y^2 - 25 = 0$

9. Find the length of the tangent from

P = (1,3) to the circle S ≡ $x^2 + y^2 - 2x + 4y - 11 = 0$

P = (12,17) to the circle S ≡ $x^2 + y^2 - 6x - 8y - 125 =$

0 P = (0,0) to the circle S ≡ $x^2 + y^2 - 14x + 2y + 25 = 0$

P = (−2, 5) to the circle S ≡ $x^2 + y^2 - 25 = 0$

P = (2,5) to the circle S ≡ $x^2 + y^2 - 5x + 4y - 5 = 0$

10. If the length of the tangent from the point (5,4) to the circle $x^2 + y^2 + 2ky = 0$ is 1, then find K.

11. If the length of the tangent from the point (2,5) to the circle $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$, then find K.

12. A point P moves such that the lengths of tangents from it to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ are in the ratio 2:3. Then find the equation to the locus of P.

13. A point P moves such that the lengths of tangents from it to the circles $x^2 + y^2 + 8x + 12y + 15 = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are equal. Then find the equation to the locus of P.

14. A point P moves such that the lengths of tangents from it to the circles $x^2 + y^2 - 2x + 4y - 20 = 0$ and $x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio 2:1. Then find the

equation to the locus of P.

15. Find the equation of tangent at $(-1,1)$ to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

16. Find the point of contact of the line $4x - 3y + 7 = 0$ with the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

17. Find the equation of tangent at:

(i) $(7, -5)$ to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

(ii) $(-1, 2)$ to the circle $x^2 + y^2 - 4x - 8y + 7 = 0$.

(iii) $(-6, -9)$ to the circle $x^2 + y^2 + 4x + 6y - 39 = 0$.

(iv) $(3, 4)$ to the circle $x^2 + y^2 - 4x - 6y + 11 = 0$.

(v) $(3, 2)$ to the circle $x^2 + y^2 - x - 3y - 4 = 0$.

(vi) $(1,1)$ to the circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$.

(vii) $(3, -2)$ to the circle $x^2 + y^2 = 13$.

18. Show that $x + y + 1 = 0$ towards the circle $x^2 + y^2 - 3x + 7y + 14 = 0$ and find its point of contact.

7. SYSTEM OF CIRCLES

1. State the relative positions of the following pairs of circles:

(i) $x^2 + y^2 - 14x + 6y + 33 = 0$; $x^2 + y^2 + 30x - 2y + 1 = 0$

(ii) $x^2 + y^2 - 8x - 6y + 21 = 0$; $x^2 + y^2 - 2y - 15 = 0$

(iii) $x^2 + y^2 + 6x + 18y + 26 = 0$; $x^2 + y^2 - 4x - 6y - 12 = 0$

(iv) $x^2 + y^2 - 4x - 6y - 12 = 0$; $5x^2 + 5y^2 - 8x - 14y - 32 = 0$

(v) $x^2 + y^2 + 6x + 6y + 14 = 0$; $x^2 + y^2 - 2x - 4y - 4 = 0$

(vi) $x^2 + y^2 - 2x + 4y - 4 = 0$; $x^2 + y^2 + 4x - 6y - 3 = 0$

(vii) $(x - 2)^2 + (y + 1)^2 = 9$; $(x + 1)^2 + (y - 3)^2 = 4$

2. Find the internal centre of similitude of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

3. Find the external centre of similitude of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x - 6y + 9 = 0$.

4. Find the internal centre of similitude of the circles $x^2 + y^2 - 4x - 10y + 28 = 0$ and $x^2 + y^2 + 4x - 6y + 4 = 0$

5. Find the external centre of similitude of the circles $x^2 + y^2 + 22x - 4y - 100 = 0$ and $x^2 + y^2 - 22x + 4y + 100 = 0$

6. Find the internal and external centre of similitude of the circles $x^2 + y^2 - 14x + 6y + 33 = 0$ and $x^2 + y^2 + 30x - 2y + 1 = 0$

7. Find the internal and external centres of similitude of the circles $x^2 + y^2 = 9$ and

$x^2 + y^2 - 16x + 2y + 49 = 0$

8. Find the internal and external centres of similitude of the circles $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 + 4x + 2y - 4 = 0$

9. Find the angle between the following pairs of circles:

   (i) $x^2 + y^2 + 4x - 14y + 28 = 0; x^2 + y^2 + 4x - 5 = 0$

   (ii) $x^2 + y^2 - 12x - 6y + 41 = 0; x^2 + y^2 + 4x + 6y - 59 = 0$

   (iii) $x^2 + y^2 + 2ax + 8 = 0; x^2 + y^2 + 2by - 8 = 0$

   (iv) $x^2 + y^2 + 4x + 8 = 0; x^2 + y^2 - 16y - 8 = 0$

   (v) $x^2 + y^2 - 6x - 8y + 12 = 0; x^2 + y^2 - 4x + 6y - 24 = 0$

   (vi) $x^2 + y^2 - 5x - 14y - 34 = 0; x^2 + y^2 + 2x + 4y + 1 = 0$

   (vii) $x^2 + y^2 - 4x + 14y - 116 = 0; x^2 + y^2 + 6x - 10y - 135 = 0$

10. Find the equations of radical axes of the following pairs of circles:

   (i) $x^2 + y^2 - 3x - 4y + 5 = 0; 3x^2 + 3y^2 - 7x + 8y + 11 = 0$

   (ii) $x^2 + y^2 + 2x + 4y + 1 = 0; x^2 + y^2 + 4x + y = 0$

   (iii) $x^2 + y^2 + 4x + 6y - 7 = 0; 4x^2 + 4y^2 + 8x + 12y - 9 = 0$

   (iv) $x^2 + y^2 - 2x - 4y - 1 = 0; x^2 + y^2 - 4x - 6y + 5 = 0$

11. Find the equations of common chords of the following pairs of circles:

(i) $x^2 + y^2 - 4x - 4y + 3 = 0$; $x^2 + y^2 - 5x - 6y + 4 = 0$

(ii) $x^2 + y^2 + 2x + 3y + 1 = 0$; $x^2 + y^2 + 4x + 3y + 2 = 0$

(iii) $(x - a)^2 + (y - b)^2 = c^2$; $(x - b)^2 + (y - a)^2 = c^2$ $(a \neq b)$

(iv) $x^2 + y^2 - 6x - 4y + 9 = 0$; $x^2 - y^2 - 8x - 6y + 23 = 0$

12. Find the equations of common tangents of the following pairs of cirles:

(i) $x^2 + y^2 + 10x - 2y + 22 = 0$; $x^2 + y^2 + 2x - 8y + 8 = 0$

(ii) $x^2 + y^2 - 2x - 4y = 0$; $x^2 + y^2 - 8y - 4 = 0$

(iii) $x^2 + y^2 - 8x - 2y + 8 = 0$; $x^2 + y^2 - 2x + 6y + 6 = 0$

(iv) $x^2 + y^2 - 2x = 0$; $x^2 + y^2 + 6x - 6y + 2 = 0$

(v) $x^2 + y^2 - 2x - 4y - 20 = 0$; $x^2 + y^2 + 6x + 2y - 90 = 0$

8. PARABOLA

1. Find the coordinates of the vertex and focus, and the equations of directrix and axis of the following parabolas:

(i) $y^2 = 16x$ (ii) $x^2 = -4y$ (iii) $y^2 = 16x$ (iv) $3x^2 - 9x + 5y - 2 = 0$

(v) $y^2 - x + 4y + 5 = 0$ (vi) $y^2 + 4x + 4y - 3 = 0$

(vii) $x^2 - 2x + 4y - 3 = 0$ (viii) $4y^2 + 12x - 20y + 67 = 0$ (ix) $x^2 - 6x - 6y + 6 = 0$

2. Find the equation of the parabola whose vertex is (3, −2) and focus is (3, 1).

3. Find the equation of the parabola whose vertex is (1, −7) and focus is (1, −2) .

4. Find the equation of the parabola whose focus is (3, 5) and vertex is (1, 3).

5. Find the coordinates of the points on the parabola $y^2 = 8x$ whose focal distance is 10.

6. Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is 5/2.

7. Find the equation of the parabola passing through the points (−1, 2), (1, −1), (2,1) and having its axis parallel to the X-axis.

8. Find the equation of the parabola passing through the points (−2,1), (1, 2), (−1, 3) and having its axis parallel to the X-axis.

9. Find the equation of the parabola passing through the points (4, 5), (−2,11), (−4, 21) and having its axis parallel to the Y-axis.

10. Show that the line $7x + 6y = 13$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$ and find the point of contact.

11. Show that the line $2x - y + 2 = 0$ is a tangent to the parabola $y^2 = 16x$ and find the point of contact also.

12. Find the value of k if the line $2y = 5x + k$ is a tangent to the parabola $y^2 = 6x$.

13. Find the equations of the tangent and normal to the parabola $y^2 = 6x$ at the positive end of the latusrectum.

14. Find the equation of the tangent and normal to the parabola $x^2 - 4x - 8y + 12 = 0$ at (4, 3/2) .

15. Find the equation of the normal to the parabola $y^2 = 4x$ which is parallel to $y - 2x + 5 = 0$

16. Find the equation of the normal to the parabola $y^2 = 4x$ inclined at an angle 60^0 with its axis and also find the point of contact.

17. Find the equation of the tangent to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$, also find the coordinates of their points of contact.

18. Find the position (exterior or interior or on) of the following points with respect to the parabola $y^2 = 6x$(i) (6, −6) (ii) (0,1) (iii) (2, 3) .

9. ELLIPSE

1. Find the eccentricity, coordinates of foci, Length of latusrectum and equations of directrices of the following ellipses:

   (i) $9x^2 + 16y^2 - 36x + 32y - 92 = 0$ (ii) $3x^2 + y^2 - 6x - 2y - 5 = 0$ (iii) $9x^2 + 16y^2 = 144$

   (iv) $4x^2 + y^2 - 8x + 2y + 1 = 0$ (v) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

2. Find the equation of the ellipse referred to its major and minor axes X, Y axes respectively with latusrectum of length 4 and distance between foci $4\sqrt{2}$.

3. Find the equation of the ellipse in the standard form whose distance between foci is 2 and the length of the latus rectum is 15/2.

4. Find the equation of the ellipse in the standard form whose distance between foci is 8 and the distance between directrices is 32.

5. Find the equation of ellipse in standard form, if it passes through the points $(-2, 2)$ and $(3, -1)$.

6. If the ends of major axis of an ellipse are $(5, 0)$, $(-5, 0)$. Find the equation of the ellipse in the standard form if its focus lies on the line $3x - 5y - 9 = 0$.

7. If the length of the major axis of an ellipse is three times the length of its major axis then find the eccentricity of the ellipse.

8. If the length of the latusrectum is equal to half of its minor axis of an ellipse in the standard from then find the eccentricity of the ellipse.

9. If the length of the latusrectum is equal to half of its major axis of an ellipse in the standard form then find the eccentricity of the ellipse.

10. Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(x-k)^2}{b^2} = 1$, given the following data.

    (i) centre $(2, -1)$, one end of major axis $(2, -5)$, $e = 1/3$.

    (ii) centre $(4, -1)$, one end of major axis $(-1, -1)$ and passes through $(8,0)$.

    (iii) centre $(0, -3)$, $e = 2/3$, semi-major axis $5.\pi$.

    (iv) centre $(2, -1)$, $e = 1/2$, length of latusrectum 4.

11. Find the equation of the ellipse whose focus is $(1, -1)$ eccentricity $\frac{2}{3}$ and directive $x + y + 2 = 0$

12. Find the equation of tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$.

13. Find the equation of tangent and normal to the ellipse $x^2 + 2y^2 - 4x + 12y + 14 = 0$ at $(2, -1)$.

14. Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ which makes equal intercepts on the coordinate axis.

15. Find the coordinates of the points on the ellipse $x^2 + 3y^2 = 37$ at which the normal is parallel to the line $6x - 5y = 2$.

16. Find the value of k if the line $4x + y + k = 0$ is tangent to the ellipse $x^2 + 3y^2 = 3$.

17. Find the equation of the tangents to the ellipse $2x^2 + y^2 = 8$ which are (i) parallel to $x - 2y - 4 = 0$ (ii) perpendicular to $x + y + 2 = 0$ (iii) which makes an angle $\frac{\pi}{4}$ with x-axis.

18. Find the equations of tangent and normal to the ellipse $2x^2 + 3y^2 = 11$ at the point whose ordinate is 1.

19. Find the equation of tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latusrectum in the first quadrant.

10. HYPERBOLA

1. Define rectangular hyperbola and find its eccentricity.

2. If e and e_1 are the eccentricities of a hyperbola and its conjugate hyperbola prove that $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.

3. One focus of a hyperbola is located at the point (1, –3) and the corresponding directrix is the line $y = 2$. Find the equation of the hyperbola if its eccentricity is 3/2.

4. Find the equations of the hyperbola whose foci are $(\pm 5, 0)$, the transverse axis is of length 8.

5. Find the equation of the hyperbola, whose asymptotes are the straight lines $x + 2y + 3 = 0$, $3x + 4y + 5 = 0$ and passing through $(1, -1)$.

6. If $3x - 4y + k = 0$ is a tangent to $x^2 - 4y^2 = 5$, find the value of k.

7. If the eccentricity of the hyperbola is 5/4, then find eccentricity of the conjugate hyperbola.

8. Find the equation of the hyperbola whose asymptotes are $3x = \pm 5y$ and the vertices are $(\pm 5, 0)$.

9. If the angle between the asymptotes is 30^0 then find its eccentricity.

10. Find the centre, foci, eccentricity, equation of the directrices, length of the latusrectum of the hyperbola (i) $16y^2 - 9x^2 = 144$ (ii) $x^2 - 4y^2 = 4$ (iii) $5x^2 - 4y^2 + 20x + 8y = 4$ (iv) $9x^2 - 16y^2 + 72x - 32y - 16 = 0$ (v) $4x^2 - 9y^2 - 8x - 32 = 0$ (vi) $4(y + 3)^2 - 9(x - 2)^2 = 1$

11. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel (ii) perpendicular to the line $x + 2y = 0$.

12. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are (i) parallel (ii) perpendicular to the line $y = x - 7$.

13. Find the equations of the tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through $(-2,1)$.

14. Show that the angle between the asymptotes of a standard hyperbola is $2\tan^{-1}\frac{b}{a}$ *or* $2\sec^{-1}(e)$.

15. Find the equation of Hyperbola passing through $(1, -1)$ and whose asymptotes are the lines $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$.

11. INTEGRATION

Evaluate the following:

1. $\int \frac{x^6 - 1}{1 + x^2}\,dx$

2. $\int \frac{2x^3 - 3x + 5}{2x^2}\,dx$

3. $\int \left(1 \sqrt{\sin 2x}\right) dx$

4. $\int \left(\sqrt[3]{2x^2}\right) dx$

5. $\int e^x - \frac{1}{x} + \frac{2}{\sqrt{1 - x^2}}\,dx$

6. $\int \frac{\sin^2 x}{1 + \cos 2x}\,dx$

7. $\int (sex^2 x - \cos x + x^2)\,dx$

8. $\int \sec^2 x \operatorname{cosec}^2 x\,dx$

9. $\int \frac{(a^x - b^x)^2}{a^x b^x}\,dx$

10. $\int \left(\cosh x + \frac{1}{\sqrt{1 + x^2}}\right) dx$

11. $\int \frac{1}{\cosh x + \sinh x}\,dx$

12. $\int x^3 - \cos x + \frac{4}{\sqrt{x^2 - 1}}\,dx$

13. $\int \frac{1 + \cos 2x}{1 - \cos 2x}\,dx$

14. $\int x^2 e^x\,dx$

15. $\int \frac{\sec^2 x}{(1 + \tan x)^3}\,dx$

16. $\int 2x \sin(x^2 + 1)\,dx$

17. $\int \frac{(\log x)^2}{x}\,dx$

18. $\int \frac{1}{8 + 2x}\,dx$

19. $\int \frac{2}{\sqrt{25 + 9x^2}}\,dx$

20. $\int \frac{3}{\sqrt{9x^2 - 1}}\,dx$

21. $\int \frac{\sin x}{\sin(a + x)}\,dx$

8

22. $\int \frac{dx}{\sqrt{1 + 5x}}$

23. $\int 2x\, e^{x^2}\,dx$

24. $\int \frac{x}{1 + x^{18}}\,dx$

25. $\int \frac{e^x (1 + x)}{\cos^2 (xe^x)}\,dx$

26. $\int \frac{2x + 1}{x^2 + x + 1}\,dx$

27. $\int \frac{\cos(\log x)}{x}\,dx$

28. $\int \frac{1}{x \log x[\log(\log x)]}\,dx$

29. $\int \frac{1}{(x + 3)\sqrt{x + 2}}\,dx$

30. $\int \frac{\sin(\tan^{-1} x)}{1 + x} dx$ 31. $\int \frac{1}{1 + \sin 2x} dx$ 32. $\int \frac{\cos x + \sin x}{\sqrt{1 + \sin 2x}} dx$ 33. $\int \sin^4 x \, dx$

34. $\int \tan^6 x \, dx$ 35. $\int \sec^5 x \, dx$ 36. $\int \operatorname{cosec}^5 x \, dx$ 37. $\int \cot^4 x \, dx$

38. $\int \cos^8 x \, dx$

12. DEFINITE INTE3GRATION

1. $\int_0^a (a^2 x - x^3) dx$ 2. $\int_2^3 \frac{2x}{1+x^2} dx$ 3. $\int_0^{\pi} \sqrt{2 + \cos\theta} \, d\theta$ 4. $\int_0^{\pi} \sin^3 x \cos^3 x \, dx$

5. $\int_0^2 |1 - x| \, dx$ 6. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$ 7. $\int_0^a (\sqrt{a} - \sqrt{x})^2 dx$ 8. $\int_0^1 x e^{-x^2} dx$

9. $\int_0^4 \frac{x^2}{1+x} dx$ 10. $\int_{-1}^2 \frac{x^2}{x^2 + 2} dx$ 11. $\int_0^1 \frac{1}{\sqrt{3 - 2x}} dx$ 12. $\int_0^1 \frac{1}{1 + x} dx$

13. $\int_0^{\pi/2} \frac{dx}{4 + 5 \cos x}$ 14. $\int_0^1 \frac{x^2}{x^2 + 1} dx$ 15. $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$

16. $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$ 17. $\int_0^a x (a - x)^n dx$ 18. $\int_0^2 x\sqrt{2 - x} \, dx$ 19. $\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx$

20. $\int_0^{\pi/2} \frac{\cos^{5/2} x}{\cos^{5/2} x + \sin^{5/2} x} dx$ 21. $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ 22. $\int_0^{\pi/2} x \sin x \, dx$ 23. $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

24. $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$ 25. $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$ 26. $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

27. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x \, dx$ 28. $\int_0^{\pi/2} \sin^{10} x \, dx$ 29. $\int_0^{\pi/2} \cos^{11} x \, dx$ 30. $\int_0^{\pi/2} \cos^7 x \sin^2 x \, dx$

31. $\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$ 32. $\int_0^2 x^{3/2}\sqrt{2 - x} \, dx$ 33. $\int_0^{\pi/2} \tan^5 x \cos^8 x \, dx$

34. $\int_0^{\pi} (1 + \cos)^3 \, dx$ 35. $\int_0^{2\pi} (1 + \cos x)^5 (1 - \cos x)^3 \, dx$.

13. DIFFERENTIAL EQUATIONS

1. Form differential equation of the following family of curves by eliminating parameters given in brackets.

   1) $xy = ae^{x} + be^{-x}$ (a, b)
   2) $y = (a + bx)e^{k}$; (a, b)
   3) $y = a\cos(nx + b)$ (a, b)
   4) $y = ae^{3x} + be^{4x}$ (a, b)
   5) $y = ax^{2} + bx$ (a, b)
   6) $ax^{2} + by^{2} = 1$ (a, b)

1. Solve the differential equation $\frac{dy}{dx} = e^{x+y}$

2. Solve the differential equation $y^{2} - x\frac{dy}{dx} = a\left(y + x\frac{dy}{dx}\right)$

3. Solve the differential equation $\frac{dy}{dx} = \frac{y^{2} + 2y}{x - 1}$

4. Solve the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

5. Solve the differential equation $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

6. Solve the differential equation $\frac{dy}{dx} = \frac{a^{2}}{(x - y)^{2}}$

7. Solve the differential equation $\frac{dy}{dx} = x^{2}e^{3y} - x^{2}$

8. Solve the differential equation $\frac{dy}{dx} = 2y\tanh x$

9. Solve the differential equation $\frac{dy}{dx} = \tan^{2}(x + y)$

10. Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^{2}e^{-y}$

1. Solve $\frac{dy}{dx} = \frac{y^{2} - 2xy}{x^{2} - xy}$

2. Solve $\frac{dy}{dx} = \frac{x^{2} + y^{2}}{2xy}$

3. Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

4. Solve $\frac{dy}{dx} = \frac{x - y}{x + y}$

5. Solve $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$

6. Solve $\frac{dy}{dx} = \frac{(x + y)^2}{2x^2}$

7. Solve $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$

8. Solve $(y^2 - 2xy)\,dx + (2xy - x^2)\,dy = 0$

9. Solve $(2x - y)\,dy = (2y - x)\,dx$

10. Solve $\frac{dy}{dx} = \frac{(x + y)^2}{2x^2}$

Find the order and Degree of the following:

1. $\frac{d_2 y}{y\,dx^2} = -p^2$

2. $\frac{d^3 y}{dx^3}^2 - 3\,\frac{dy}{dx}^2$

3. $\frac{d^2 y}{dx^2} + \frac{dy}{dx}^{3\ 6/5} = 6y$

4. $y = c(x - c)^2$

14. PREMUTATIONS & COMBINATIONS

1. If $np_4 = 1680$ find the value of n?

2. If $(n + 1)\,p_5 : np_5 = 3 : 2$ then find the value of n?

3. Find the number of ways of permuting the letters of the word, PICTURE so hat

   (i) all vowels come together

   (ii) no two vowels come together

   (iii) The relative position of vowels and consonants are not disturbed.

4. If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order. Find the rank of the word 'PRISON'.

5. Find the sum of all 4-digit numbers that can be formed using the digit 1, 3, 5, 7, 9.

6. Find the number of 4 letter words that can be formed using the letters of the word. PISTON in which at least one letter is repeated.

7. Find the number of ways of seating 5 Indians, 4 Americans and 3 Russians at a round table so that

   (i) all Indians sit together

   (ii) no two Russians sit together

   (iii) Persons of same nationality sit together.

8. Find the number of different chains that can be prepared using 7 different coloured beads.

9. Find the number of ways of arranging the letters of the words

   (i) INDEPENDENCE (ii) MATHEMATICS (iii) SINGING

   (iv) PERMUTATION (v) COMBINATION (vi) INTERMEDIATE

10. Find the number of ways of selecting 4 English 3 Telugu and 2 Hindi books out of 7 English, 6 Telugu and 5 Hindi books.

11. Find the number of ways of selecting 11 member cricket team from 7 batsmen 6 bowlers and 2 wicket keepers so that team contains 2 wicket keepers and atleast 4 bowlers.

12. Prove that ${}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3 = {}^{30}C_4$.

13. If $nc_{21} = nc_{27}$ find $50C_n$

14. Simplify ${}^{34}C_5 + \sum_{r=0}^{4} {}^{(38-r)}C_4$

15. Find the number of ways of forming a committee of 5 members out of 6 Indians, and 5 Americans so that always the Indians will be in majority in the committee.

16 If a set A has 12 elements find the number of subsets of A having

   (i) 4 elements (ii) Atleast 3 elements (iii) At most 3 elements.

15. PROBABILITY

1. Find the probability of throwing a total score of 7 with 2 dice.
2. A page is opened at random from a book containing 200 pages what is the probability that the number on the page is a perfect square.
3. If A and B are events with P(A) = 0.5, P(B) = 0.4 and $P(A \cap B) = 0.3$ find the probability that (i) A does not occur (ii) neither A nor B occurs.
4. Find the probability that a non-leap year contains (i0 53 Sundays (ii) 52 Sundays only.
5. Two dice are rolled. What is the probability that none of the dice shows the number '2'?
6. In an experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probability of A, B, $A \cap B$ and $A \cup B$.
7. A, B, C are 3 news papers from a city. 20% of the population read A, 16% read B, 14% read C, 8% both A and B, 5%, both A and C, 4% both B and C and 2% all the three. Find the percentage of the population who read at least one news paper.
8. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.
9. A bag contains 12 two rupee coins 7 one rupee coins and 4 half rupee coins. If three coins are selected at randam then find the probability that

   (i) The sum of three coins is Maximum

   (ii) The sum of three coins is minimum

   (iii) Each coin is of different value.
10. A pair of dice is thrown. Find the probability that either of the dice shows 2 when their sum is 6.
11. Let A and B be independent events with P(A) = 0.2, P(B) = 0.5.

    Find (i) P(A/B) (ii) P(B/A) (iii) $P(A \cap B)$ and $P(A \cup B)$.
12. If A, B, C are three independent events of an experiment such that $P(A \cap B^C \cap C^C) = \frac{1}{4}$,

    $P(A^C \cap B \cap C^C) = \frac{1}{8}$, $P(A^C \cap B^C \cap C^C) = \frac{1}{4}$. Then find P(A), P(B) and (P(C).

13. A pair of dice is rolled. What is the probability that neither dice shows a 2 given that they sum to 7.

14. If A, B are two events with $P(A \cup B) = 0.65$, $P(A \cap B) = 0.15$. Then find the value of $P(A^C) + P(B^C)$.

15. A pair dice is rolled. Consider the events A = {1,3,5}, B = {2,3} and C = {2,3,4,5} find

(i) $P(A \cap B)$, $P(A \cup B)$ (ii) $P(A / B)$, $P(B / A)$

(iii) $P(A / C)$, $P(C / A)$ (iv) $P(B / C)$, $P(C / A)$

16. Suppose A and B are independent events with P(A) = 0.6 , P(B) = 0.7 . Then compute

(i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A / B)$ (iv) $P(A^C \cap B^C)$

17. A problem in calculus is given to two students A and B whose chances of solving it are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability of the problem being solved if both of them try independently.

18. Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of event that all 3 screws are non-defective assuming that the drawing is

(i) with replacement (ii) without replacement.

19. The probability that a boy A will get a scholarship is 0.9 and that another boy B will get is 0.8. What is the probability that atleast one of them will get the scholarship.

20. If A, B are two events then show that

$P(A/B).\ P(B) + P(A/B^C).\ P(B^C) = P(A)$