MATHEMATICS

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First Year

Intermediate Vocational Bridge Course

Paper - I : Mathematics

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CONTENTS

UNIT - I
ALGEBRA

01. FUNCTIONS **01 - 21**
1.1 Types of functions – Definitions and Theorems (Without Proofs) 1
1.2 Inverse functions and Theorems (without proofs) 5
1.3 Domain, Range, Inverse of real valued functions. 10

02. MATHEMATICAL INDUCTION **22 - 32**
2.1 Principles of Mathematical Induction and Theorems (without proofs) 22
2.2 Applications of Mathematical Induction (Simple problems) 23

03. MATRICES **33 - 80**
3.1 Types of Matrices – Definitions. 33
3.2 Scalar multiple of a matrix and multiplication of matrices 37
3.3 Transpose of a matrix 40
3.4 Determinants – Properties of determinants (Simple problems) 48
3.5 Adjoint and Inverse of a matrix- Theorems without proof 63
3.6 Solving simultaneous linear equations – Cramer's rule and Matrix Inversion method. 68

UNIT – II
VECTOR ALGEBRA

04. ADDITION OF VECTORS **81 - 107**
4.1 Vectors as a triad of real numbers – some basic concepts. 81
4.2 Classification (types) of Vectors. 83
4.3 Addition of vectors 85
4.4 Scalar multiplication of a vector 86
4.5 Angle between two non- zero vectors 87
4.6 Vector equations of line and plane. 95

05. PRODUCT OF VECTORS **108 - 152**
5.1 Scalar Product of two vectors – Definition and simple problems 108
5.2 Properties of Scalar product 112
5.3 Expression for Scalar (dot) product, Angle between two vectors 114
5.4 Vector product (cross product) of two vectors and properties 115
5.5 Vector product in (i , j , k) system. 118
5.6 Vector Areas 121
5.7 Scalar Triple Product - Vector triple product Definitions and simple problems 125

UNIT – III
TRIGONOMETRY

05. TRIGONOMETRIC RATIOS UPTO TRANSFORMATIONS **153 - 221**

6.1 Trigonometric Ratios – Variation – Simple problems 153
6.2 Trigonometric Ratios of compound angles – Simple problems 167
6.3 Trigonometric Ratios of multiple and sub multiple angles – Simple problems 183
6.4 Sum and product – Transformations – Simple Problems – Problems on identities. 198

06. TRIGONOMETRIC EQUATIONS **222 - 237**

7.1 General solutions of trigonometric equations – Simple problems. (without simple trigonometric equations) 222

07. HYPERBOLIC FUNCTIONS **238 - 253**

8.1 Definitions of hyperbolic functions – Simple problems. 238
8.2 Definitions of Inverse hyperbolic functions. 241

UNIT – IV
CALCULUS

08. LIMITS AND CONTINUITY **254 - 276**

9.1 Definition of the limits – Standard Formulae (without proofs) – Simple problems. 254
9.2 Definition of Continuity and simple problems. 268

09. DIFFERENTIAON **277 - 312**

10.1 Derivative of function – Elementary Properties – Simple problems. 277
10.2 Trigonometric and Inverse Trigonometric functions – Derivatives – Simple problems. 290

10. APPLICATIONS OF DERIVATIVES **313 - 339**

11.1 Errors and approximations 313
11.2 Equations of Tangents and Normals 318
11.3 Lengths of Tangent, Normal, Sub tangent and sub normal- Simple problems. 319
11.4 Angle between two curves – condition of orthogonality 324
11.5 Derivative as rate of change. 327

UNIT – V
CO-ORDINATE GEOMETRY (2-D & 3-D)

11. LOCUS **340 - 344**

12.1 Definition of Locus – Problems. 340

12. TRANSFORMATION OF AXES	**345 - 349**
13.1 Translation of axes – Rules and simple problems	345
13.2 Rotation of axes – Rules and simple problems.	346
13. THE STRAIGHT LINE	**350 - 390**
13.1 Equation of Straight line – various forms – Illustrations - Simple problems	350
14.2 Intersection of two straight lines	357
14.3 Family of straight lines - Concurrent lines	360
14.4 Condition for Concurrent lines	363
14.5 Angle between two lines	367
14.6 Length of the perpendicular from a point to a line	370
14.7 Distance between two parallel lines	370
14.8 Concurrent lines - Theorems related to a triangle (without proof) – Simple problems.	376
14. PAIR OF STRAIGHT LINES	**391 - 415**
14.1 Introduction - Equations of a pair of lines passing through origin, angle between a pair of Lines.	391
15.2 Condition for perpendicular and coincident lines, bisector of angles.	395
15.3 Pair of bisectors of angles (Theorems without proofs) – Problems.	398
15.4 Pair of lines – second degree general equation (Theorems without proofs) – Problems.	404
15.5 Conditions for parallel lines – Distance between them, Point of intersection of pair of lines.	405
15. THREE DIMENSIONAL CO-ORDINATES	**416 - 433**
16.1 Co-ordinates – simple problems.	416
16.2 Section formula – Centroid of triangle – Centroid of tetrahedron.	423
16.3 Simple problems related to centroid.	429
16. DIRECTION CONSINES AND DIRECTION RATIOS	**434 - 444**
17.1 Definition of Direction Cosines – Simple Problems.	434
17.2 Definition of Direction ratios – Simple Problems.	439
REFERENCES	**445**

1. FUNCTIONS

Introduction:

All scientists use Mathematics essentially to study relationships. Physicists, Chemists, Engineers, Biologists and Social scientists, all seek to discern connection among the various elements of their chosen fields and so to arrive at a clear understanding of why these elements behave the way they do. A *function* is a special case of a relation.

The famous mathematician *Lejeune Dirichlet* (1805-1859) defined a function as follows. A variable is a symbol which represents any one of a set of numbers; if two variables

x and y are so related that whenever a value is assigned to x there is automatically assigned, by some rule or correspondence, a value to y, then we say y is a (single valued) function of x that permissible values that x may assume constitute the domain of definition of the function and the values taken on by y constitute the range of values of the function.

In this chapter we focus our attention on a special types of functions that play an important role in Mathematics and its many applications. Here we study its basic properties and then discuss several special types of functions.

1.1 Types of Functions-Definitions:

1.1.1 Definition (Function):

Let A and B be non-empty sets and f be a relation from A to B. If for each element $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in f$, then f is called a *function (or mapping)* from (or A into B) A to B. It is denoted by $f : A \to B$. The set A is called the *domain* of f and B is called the *co-domain* of f.

For example if $f : A \to B$ is a function defined as $f(x) = x+1$ and $A = \{1,2,3\}$, then $f(A) = \{2,3,4\}$.

1.1.2 Note:

A relation f from A to B $(i.e\ f \subseteq A \times B)$ is a function from A to B if for each element $a \in A$, there exists exactly one $b \in B$ such that $(a,b) \in f$ and this b will be denoted by $f(a)$. In other words for each element $a \in A$, there exists a unique $f(a) \in B$ such that $(a, f(a)) \in f$.

1.1.3 Definition (Image, Pre-Image):

If $f : A \to B$ is a function and if $f(a) = b$, then b is called the *image* of a under f or the f^{-} *image* of a. The element a is called the *pre-image* or *inverse image* of b under f and it is denoted by $f^{-1}(b)$

1.1.4 Examples:

1. Example: The relation $f = \{(x, x^2+1) / x \in R\}$ is a function from R to R^+, since every $x \in R$ has association with unique element x^2+1 in R^+. The function

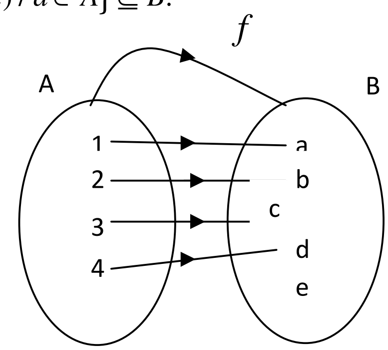
$f:R\to R^+$ is given by $f(x)=x^2+1$.

2. Example: The relation $f=\left\{(x,\frac{1}{x})/x\in R\right\}$ is not a function from R to R, since there is no $b\in R$ such that $(0,b)\in f$ But $f(x)=\frac{1}{x}$ is a function from $R-\{0\}\to R$ since every $x\in R-\{0\}$ has association with unique element in R

1.1.5 Definition (range):

If $f:A\to B$ is a function, then $f(A)$, the set of all f^- *images* of elements in A is called the *range* of f. Clearly $f(A)=\{f(a)/a\in A\}\subseteq B$.

Also $f(A)=\{b\in B/b=f(a)\ for\ some\ a\in A\}$.



1.1.6 Examples:

1. Example: Let $f:N\to N$ be defined by $f(n)=2n$.

The range of $f=f(N)=\{2n/n\in N\}$

Which is the set of all even natural numbers.

2. Example: Let $f:R\to R$ be defined by $f(x)=x^2$.

The range of $f=f(R)=\{x^2/x\in R\}=[0,\infty)$ $\left[\because x^2\geq 0\ for\ all\ x\in R\right]$

1.1.7 Definition (Injection or one-one function):

A function $f:A\to B$ is called an *injection* or a *one-one function* if distinct elements of A have distinct f^- *images* in B.

i.e $f:A\to B$ is called an *injection*

$\Leftrightarrow a_1,a_2\in A$ and $a_1\neq a_2$ implies that $f(a_1)\neq f(a_2)$

$\Leftrightarrow a_1,a_2\in A$ and $f(a_1)=f(a_2)$ implies that $a_1=a_2$

1.1.8 Examples:

1. Example: Let $A=\{a,b,c,d\}$ and $B=\{1,2,3,4,5\}$.

If $f=\{(a,3),(b,5),(c,1),(d,4)\}$ then f is a function from A into B and for different elements in A, there are different f^- *images* in B. Hence f is an *injection.*

2. Example: Let $A=\{a,b,c\}$ and $B=\{1,2,3,4\}$.

If $g=\{(a,2),(b,2),(c,3)\}$ then g is a function from A into B, but $g(a)=g(b)$. Hence g is not an *injection.*

3. Example: Let $f:R\to R$ be defined by $f(x)=2x+1$.

Then f is an *injection* since for any $a_1,a_2\in R$ and $f(a_1)=f(a_2)$

$\Rightarrow 2a_1 + 1 = 2a_2 + 1 \Rightarrow \; a_1 = a_2.$

4. Example: Let $f : R \to R$ be defined by $f(x) = x^2$.

Then f is not an *injection* because $f(-1) = 1 = f(1)$.

5. Example: Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

We can't define an injection from A into B because at least two different elements in A have the same f^- *images* in B.

1.1.9 Definition (Surjection or onto function):

A function $f : A \to B$ is called a *surjection* or an *onto function* if the *range* of f is equal to the *co-domain* of f

i.e $f : A \to B$ is called a *surjection* $\Leftrightarrow$ range $f = f(A) = B(co-domain)$

$\Leftrightarrow B = \{ f(a) / a \in A \}$

$\Leftrightarrow$ for every $b \in B$ there exists atleast one $a \in A$ such that $f(a) = b$.

1.1.10 Examples:

1. Example: Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

If $f = \{(a,1), (b,2), (c,1), (d,3)\}$ then f is a function from A into B and range $f = f(A) = \{1, 2, 3\} = B(co-domain)$ Hence f is a *surjection.* Note that f is not an *injection*

2. Example: Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

If $g = \{(a,1), (b,1), (c,2), (d,2)\}$ then g is a function from A into B and range $g = g(A) = \{1, 2\} \neq B(co-domain)$ Hence g is not a *surjection.* Note that g is not an *injection*

3. Example: Let $A = \{-3, -2, -1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$. Let $f : A \to B$ be defined by $f(x) = x^2 \forall x \in A$

range $f = f(A) = \{ f(-3), f(-2), f(-1), f(1), f(2), f(3) \}$

$= \{1, 4, 9\} = B(co-domain)$

Hence f is a *surjection.* Note that f is not an *injection*

4. Example: Let $f : R \to R$ be defined by $f(x) = 2x + 1$.

Then f is a *surjection* since for any $y \in R(co-domain)$ there exists

$x = \frac{y-1}{2} \in R(domain)$ such that $f(x) = 2x+1 = 2(\frac{y-1}{2})+1 = y$

i.e every element in the co-domain has a pre-image in the domain. Note that f is an *injection* too.

1.1.11 Definition (Bijection):

If a function $f: A \to B$ is both an *injection* and a *surjection* then f is said to be a *bijection* or *one-to-one* from A *onto* B.

i.e $f: A \to B$ is a *bijection* $\Leftrightarrow$ $f: A \to B$ is both an *injection* and a *surjection*

$\Leftrightarrow a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ implies that $a_1 = a_2$

$\Leftrightarrow$ for every $b \in B$ there exists atleast one $a \in A$ such that $f(a) = b$.

1.1.12 Examples:

1. Example: Let $f: R \to R$ be defined by $f(x) = 2x+1$, then from examples 3(1.1.8) and 4(1.1.10), f is a *bijection.*

2. Example: Let $f: N \to N$ be defined by $f(x) = 2x+1$.

Then f is an *injection* since for any $a_1, a_2 \in N$ and $f(a_1) = f(a_2)$

$\Rightarrow 2a_1 + 1 = 2a_2 + 1 \Rightarrow a_1 = a_2$.

range $f = f(N) = \{f(1), f(2), f(3), ...\} = \{3, 4, 5, ...\} \neq N(co-domain)$

Hence f is not a *surjection.* Observe that the natural numbers 1,2 in the co-domain N of f has no pre- image in the domain N.

1.1.13 Definition (Equality of functions):

Let f and g be functions. We say that f and g are *equal* and write $f = g$ if *domain of* f = *domain of* g and $f(x) = g(x)$ for all $x \in$ *domain of* f.

1.1.14 Example:

Let $f(x) = x^2 - 2x$ and $g(x) = -x+6$

$$f(x) = g(x) \Leftrightarrow x^2 - 2x = -x + 6 \Leftrightarrow x^2 - x - 6 = 0$$

$$\Leftrightarrow (x-3)(x+2) = 0 \Leftrightarrow x = 3, x = -2$$

$f(x)$ and $g(x)$ are *equal* on the *domain* $\{-2, 3\}$

1.1.15 Definition (Constant function):

A function $f: A \to B$ is said to be a *constant function*, if the *range of* f contains exactly one element, *i.e* $f(x) = c$ for all $x \in A$, for some fixed $c \in B$.

1.1.16 Example:

Let $A=\{a,b,c,d\}$ and $B=\{1,2,3,4,5\}$

If $f=\{(a,1),(b,1),(c,1),(d,1)\}$ then f is a *constant function* from A into B.

1.1.17 Definition (Identity function):

Let A be non-empty set. Then the function $f:A\to A$ defined by $f(x)=x$ for all $x\in A$ is called the *identity function* on A and is denoted by I_A.

1.1.18 Example:

If $A=\{a,b,c\}$, then $I_A=\{(a,a),(b,b),(c,c)\}$

1.2 Inverse Functions and Theorems:

If f is a relation from A to B, then the relation $\{(b,a)/(a,b)\in f\}$ is denoted by f^{-1}.

1.2.1 Definition (Inverse Function):

If $f:A\to B$ is a bijection, then the relation $f^{-1}=\{(b,a)/(a,b)\in f\}$ is a Function from B to A and is called the *inverse* of f.

1.2.2 Examples:

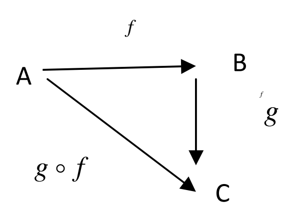
1. Example: If $A=\{1,2,3\}$ and $B=\{a,b,c\}$ then $f=\{(1,a),(2,c),(3,b)\}$ is a bijection from A to B and $f^{-1}=\{(a,1),(b,3),(c,2)\}$ is a bijection from B to A

2. Example: If $A=\{1,2,3\}$ and $B=\{a,b,c,d\}$ then $f=\{(1,a),(2,c),(3,b)\}$ is an injection but not a surjection, and $f^{-1}=\{(a,1),(b,3),(c,2)\}$ is a relation from B to A but not a function because $d\in B$ has no f^{-1} images in A.

3. Example: If $A=\{1,2,3\}$ and $B=\{a,b\}$ then $f=\{(1,a),(2,b),(3,a)\}$ is a surjection but not an injection, and $f^{-1}=\{(a,1),(b,2),(a,3)\}$ is a relation from B to A but not a function because $a\in B$ has two f^{-1} images in A.

1.2.3 Definition (Composite Function):

If $f:A\to B$, $g:B\to C$, then the relation $\{(a,g(f(a)))/a\in A\}$ is called *composite* of g with f and is denoted as $g\circ f$.



1.2.4 Theorem:

Let $f : A \to B$ and $g : B \to C$ be functions. Then $g \circ f$ is a function from A to C and $\left(g \circ f\right)(a) = g\left(f(a)\right)$ for all $a \in A.$

Proof: Given $f : A \to B$ and $g : B \to C$ are functions.

To prove $g \circ f$ is a function from A to C and $\left(g \circ f\right)(a) = g\left(f(a)\right)$ $\forall a \in A.$

Let $a \in A$

Since f is a function from A to B then $f(a) \in B$ for all $a \in A.$

Since g is a function from B to C then $g\left(f(a)\right) \in C.$

Hence $g \circ f$ is a relation from A to C. Further, given $a \in A$ there is one and only

element c in C, namely $g\left(f(a)\right)$ such that $(a,c) \in g \circ f.$

Hence $g \circ f$ is a function from A to C and $\left(g \circ f\right)(a) = g\left(f(a)\right)$ $\forall a \in A.$

1.2.5 Theorem:

Let $f : A \to B$ and $g : B \to C$ be injections. Then $g \circ f : A \to C$ is an injection.

Proof: Given $f : A \to B$ and $g : B \to C$ are injections.

To prove $g \circ f : A \to C$ is an injection.

Let $a_1, a_2 \in A$ such that $\left(g \circ f\right)(a_1) = \left(g \circ f\right)(a_2)$ $\Rightarrow g\left(f(a_1)\right) = g\left(f(a_2)\right)$

$\Rightarrow f(a_1) = f(a_2)$ [since g is an injection]

$\Rightarrow a_1 = a_2$ [since f is an injection]

$\therefore$ $g \circ f : A \to C$ is an injection.

1.2.6 Theorem:

Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f : A \to C$ is an injection. Then $f : A \to B$ is an injection.

Proof: Given $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is an injection

To prove $f : A \to B$ is an injection.

Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ then $g\left(f(a_1)\right) = g\left(f(a_2)\right)$

$\Rightarrow \left(g \circ f\right)(a_1) = \left(g \circ f\right)(a_2)$

$\Rightarrow a_1 = a_2$ [since $g \circ f$ is an injection]

$\therefore$ $f : A \to B$ is an injection.

1.2.7 Note:

Let $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is an injection. Then $g : B \to C$ need not be an injection.

For example, Let $A = \{1,2\}$, $B = \{a,b,c\}$, $C = \{d,e\}$, $f = \{(1,a),(2,b)\}$ and

$g = \{(a,d),(b,e),(c,e)\}$ then $g \circ f = \{(1,d),(2,e)\}$

Hence $g \circ f$ is an injection but g is not an injection. However if $g \circ f$ is an injection then necessarily f is an injection.

1.2.8 Theorem:

Let $f : A \to B$ and $g : B \to C$ be surjections. Then $g \circ f : A \to C$ is a surjection.

Proof: Given $f : A \to B$ and $g : B \to C$ are surjections.

To prove $g \circ f : A \to C$ is a surjection.

Let $c \in C,$ since $g : B \to C$ is a surjection then there exists $b \in B$ such that $g(b) = c.$

For $b \in B$ and $f : A \to B$ is a surjection then there exists $a \in A$ such that $f(a) = b.$

$\therefore c = g(b) = g\left(f(a)\right) = \left(g \circ f\right)(a)$

$\therefore$ For each $c \in C$ there exists $a \in A$ such that $\left(g \circ f\right)(a) = c.$

Hence $g \circ f : A \to C$ is a surjection.

1.2.9 Theorem:

Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f : A \to C$ is a surjection. Then $g : B \to C$ is a surjection.

Proof: Given $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is a surjection

To prove $f : A \to B$ is a surjection.

Let $c \in C,$ since $g \circ f : A \to C$ is a surjection then there exists $a \in A$ such that

$\left(g \circ f\right)(a) = c \Rightarrow g\left(f(a)\right) = c.$

Let $b = f(a).$ Then $f(a) = b \in B$ and $g\left(b\right) = c.$

$\therefore$ For each $c \in C$ there exists $b \in B$ such that $g\left(b\right) = c.$

$\therefore$ $g : B \to C$ is a surjection.

1.2.10 Note:

Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f : A \to C$ is a surjection. Then $f : A \to B$ need not be a surjection. In note 1.2.7 $g \circ f$ is a surjection but f is not a surjection. However if $g \circ f$ is a surjection then necessarily g is a surjection.

1.2.11 Theorem:

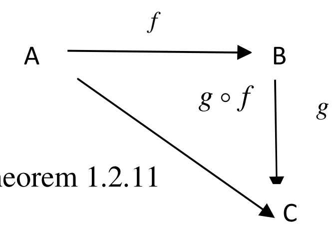
Let $f : A \to B$ and $g : B \to C$ be bijections. Then $g \circ f : A \to C$ is a bijection.

Proof: This is a consequence of Theorems 1.2.5 and 1.2.8.

1.2.12 Theorem:

Let $f : A \to B$ and $g : B \to C$ be bijections. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Given $f : A \to B$ and $g : B \to C$ are bijections.



To prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Since $f : A \to B$ and $g : B \to C$ are bijections, then by Theorem 1.2.11

$g \circ f : A \to C$ is a bijection. Hence $(g \circ f)^{-1} : C \to A$ is a bijection.

Further, $f^{-1} : B \to A$ and $g^{-1} : C \to B$ are also bijections.

Hence $f^{-1} \circ g^{-1} : C \to A$ is a bijection.

Therefore the functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ have same domain and co- domain.

Let $c \in C$, since $g : B \to C$ is a bijjection then there exists a unique $b \in B$ such that $g(b) = c \Rightarrow g^{-1}(c) = b$.

Let $b \in B$, since $f : A \to B$ is a bijjection then there exists a unique $a \in A$ such that $f(a) = b \Rightarrow f^{-1}(b) = a$.

Thus $c = g(b) = g(f(a)) = (g \circ f)(a) \Rightarrow (g \circ f)^{-1}(c) = a$

Now $(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$

Hence $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.2.13 Theorem:

The identity function $I_A : A \to A$ is a bijection and $I_A^{-1} = I_A$.

Proof: Given $I_A : A \to A$ is a function.

To prove $I_A : A \to A$ is a bijection and $I_A^{-1} = I_A$.

We have $I_A = \{(a,a) / a \in A\}$

Given $a \in A$ we have $I_A(a) = a$. Hence I_A is a surjection.

Let $a_1, a_2 \in A$ such that $I_A(a_1) = I_A(a_2)$ then $a_1 = a_2$. Hence I_A is an injection.

$\therefore$ $I_A : A \to A$ is a bijection and $I_A^{-1} = I_A$.

1.2.14 Theorem:

Let $f: A \rightarrow B$, I_A and I_B be identity functions on A and B respectively. Then $f \circ I_A = f = I_B \circ f$.

Proof: Given $f: A \rightarrow B$ is a function. Also given that I_A and I_B be identity functions on A and B respectively. $I_A : A \rightarrow A$ and $I_B : B \rightarrow B$ are identity functions.

To prove $f \circ I_A = f = I_B \circ f$.

Since $f: A \rightarrow B$ and $I_A : A \rightarrow A$ are functions, then $f \circ I_A : A \rightarrow B$ is a function.

Therefore the functions $f \circ I_A$ and f have same domain and co-domain.

Let $a \in A$, then that $(f \circ I_A)(a) = f(I_A(a)) = f(a) [\because I_A(a) = a \forall a \in A]$

$\therefore f \circ I_A = f$ (I)

Since $f: A \rightarrow B$ and $I_B : B \rightarrow B$ are functions, then $I_B \circ f : A \rightarrow B$ is a function.

Therefore the functions $I_B \circ f$ and f have same domain and co-domain.

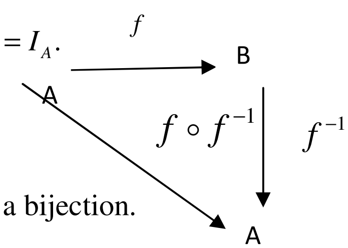
Let $a \in A$, then that $(I_B \circ f)(a) = I_B(f(a)) = f(a) [\because f: A \rightarrow B \Rightarrow f(a) \in B]$

$\therefore I_B \circ f = f$ (II)

From (I) and (II) we have $f \circ I_A = f = I_B \circ f$.

1.2.15 Theorem:

Let $f: A \rightarrow B$ be a bijection. Then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$.



Proof: Given $f: A \rightarrow B$ is a bijection.

To prove $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$.

Since $f: A \rightarrow B$ is a bijection then $f^{-1}: B \rightarrow A$ is also a bijection.

Hence $f \circ f^{-1}: B \rightarrow B$ and $f^{-1} \circ f : A \rightarrow A$ are bijections.

We have $I_B : B \rightarrow B$ and $I_A : A \rightarrow A$ are bijections.

Therefore the functions $f \circ f^{-1}$ and I_B have same domain and co-domain.

Let $b \in B$, since $f: A \rightarrow B$ is a bijjection then there exists a unique $a \in A$

such that $f(a) = b \Rightarrow f^{-1}(b) = a$.

Thus $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b = I_B(b)$.

$\therefore f \circ f^{-1} = I_B$

The functions $f^{-1} \circ f$ and I_A have same domain and co-domain.

We have $\left(f^{-1}\circ f\right)(a)=f^{-1}\left(f(a)\right)=f^{-1}(b)=a=I_A(a).$

$\therefore f^{-1}\circ f=I_A$

1.2.16 Theorem:

Let $f:A\to B, g:B\to C$ and $h:C\to D$.Then $h\circ\left(g\circ f\right)=\left(h\circ g\right)\circ f.$

Proof: Given $f:A\to B, g:B\to C$ and $h:C\to D$ are functions.

To prove $h\circ\left(g\circ f\right)=\left(h\circ g\right)\circ f.$

Since $f:A\to B, g:B\to C$ are functions then $g\circ f:A\to C$ is a function.

Since $g\circ f:A\to C, h:C\to D$ is a function then $h\circ\left(g\circ f\right):A\to D$ is a function.

Further $g:B\to C, h:C\to D \Rightarrow h\circ g:B\to D$ is a function.

Also $f:A\to B, h\circ g:B\to D \Rightarrow \left(h\circ g\right)\circ f:A\to D$ is a function.

Hence the functions $h\circ\left(g\circ f\right)$ and $\left(h\circ g\right)\circ f$ have same domain and co- domain.

Let $a\in A,$

Now $\left[h\circ\left(g\circ f\right)\right](a)=h\left[\left(g\circ f\right)(a)\right]=h\left[g\left(f(a)\right)\right]$

$$=\left(h\circ g\right)\left(f(a)\right)=\left[h\circ\left(g\circ f\right)\right](a).$$

$\therefore h\circ\left(g\circ f\right)=\left(h\circ g\right)\circ f$

1.3 Real valued Functions(Domain, Range and Inverse):

If X is any set and $f:X\to R$ then f is called a *real valued function.*

For example, Let $X=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \middle| a,b,c,d\in R\right\}$, define $f:X\to R$ by $f(A)=\det A\ \forall A\in X$, then f is a *real valued function.*

In this section a function f is defined through a formula, without mentioning the domain and the range explicitly. In such cases, the domain of f is taken to be the set of all real x for which the formula is meaningful. The range of f is the set $\left\{f(x)\middle| x \text{ is in the domain of } f\right\}$.

Definition (n^{th} root of a non-negative real number):

Let x be a non-negative real number and n be a positive integer. Then there exists a unique non-negative real number y such that $y^n=x.$ This number y is called n^{th} root of x and is denoted by $x^{1/n}$ or $\sqrt[n]{x}.$

When $n=2,$ $\sqrt[2]{x}$ is called the square root of $x.$ $\sqrt[2]{x}$ is simply written as $\sqrt{x}.$

1.3.1 Algebra of real valued functions:

If f and g are real valued functions with domains A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B \neq \phi$. defined through a formula

(i) If $f : A \to R$ and $g : B \to R$ are functions such that $A \cap B \neq \phi$. We define $f + g : A \cap B \to R$ as $(f + g)(x) = f(x) + g(x)\ \forall\, x \in A \cap B.$

(ii) If $f : A \to R$ and $g : B \to R$ are functions such that $A \cap B \neq \phi$. We define $f - g : A \cap B \to R$ as $(f - g)(x) = f(x) - g(x)\ \forall\, x \in A \cap B.$

(iii) If $f : A \to R$ and $g : B \to R$ are functions such that $A \cap B \neq \phi$. We define $fg : A \cap B \to R$ as $(fg)(x) = f(x).g(x)\ \forall\, x \in A \cap B.$

(iv) If $f : A \to R$ and $g : B \to R$ are functions such that $E = \{x \in A \cap B \,/\, g(x) \neq 0\} \neq \phi$. We define $\frac{f}{g} : E \to R$ as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \forall\, x \in E.$

(v) If $f : A \to R$ and $c \in R$. We define $cf : A \to R$ as $(cf)(x) = cf(x)\ \forall\, x \in A.$

(vi) If $f : A \to R$ and $n \in N$. We define $f^n : A \to R$ as $f^n(x) = \left(f(x)\right)^n\ \forall\, x \in A.$

(vii) If $f : A \to R$ and $E = \{x \in A \,/\, f(x) \geq 0\} \neq \phi$. We define $\sqrt{f} : E \to R$ as $\left(\sqrt{f}\right)(x) = \sqrt{f(x)}\ \forall\, x \in E.$

1.3.2 Some more types of functions:

1. Even and odd functions:

Let A be a nonempty subset of R such that $-x \in A$ for all $x \in A$ and $f : A \to R$

(i) If $f(-x) = f(x)$ for any $x \in A$ then f is called an *even function.*

(ii) If $f(-x) = -f(x)$ for any $x \in A$ then f is called an *odd function.*

Examples:

(i) $f(x) = x^2, g(x) = \cos x, h(x) = |x|$ for any $x \in R$ are all *even functions.*

(ii) $f(x) = x, g(x) = \sin x, h(x) = \tan x$ for any $x \in R$ are all *odd functions.*

(iii) $f(x) = x^2 + x^3, g(x) = \cos x + \sin x$ for any $x \in R$ are neither *even* nor *odd functions.*

2. Polynomial function:

If n be a nonnegative integer, $a_0, a_1, a_2, ...a_n$ are real numbers(at least one $a_i \neq 0$) then the function f defined on R by $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ for all $x \in R$ is called a *polynomial function.*

Examples:

(i) $f(x) = ax^2 + bx + c$ for any $a, b, c \in R$ is a *polynomial function.*

$(ii)\;\; g(x) = ax^2 + bx + c$ is a *polynomial function.*

$(iii)\;\; h(x) = k(0 \neq k \in R)$ is a *polynomial function*

3. Rational function:

If f and g are polynomial functions and $g(x) \neq 0$ for all $x \in R$ then the function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is called a *rational function.*

Examples:

$(i)\;\; f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 1}$ is a *rational function.*

$(ii)\;\; g(x) = \dfrac{1}{x}, x \in R - \{0\}$ is a *rational function.*

4. Algebraic function:

A function obtained by applying a finite number of algebraic operations on polynomial functions is called an *algebraic function*

Examples:

$(i)\;\; f(x) = \dfrac{x^2 + \sqrt{16 - x^2}}{x + 1}, \left(x \in [-4, 4] - \{-1\}\right)$ is an *algebraic function.*

$(ii)\;\; g(x) = 3x + \sqrt{x^2 - 5x + 6},\;\; x \in R - (2, 3)$ is an *algebraic function.*

5. Exponential function:

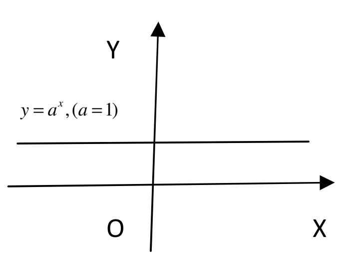
The function a^x when $1 \neq a > 0$ and x is rational is called an *exponential function.* The domain of a^x is R and range is R^+.

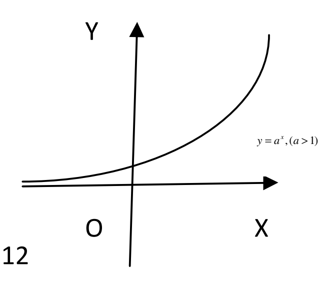
Examples:

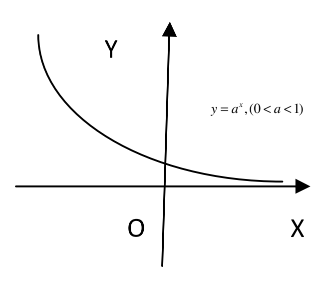
$(i)\;\; f(x) = 3^x, x \in R$ is an an *exponential function*

$(ii)\;\; g(x) = 2^{-x}, x \in R$ is an *exponential function*

Graph of Exponential function:



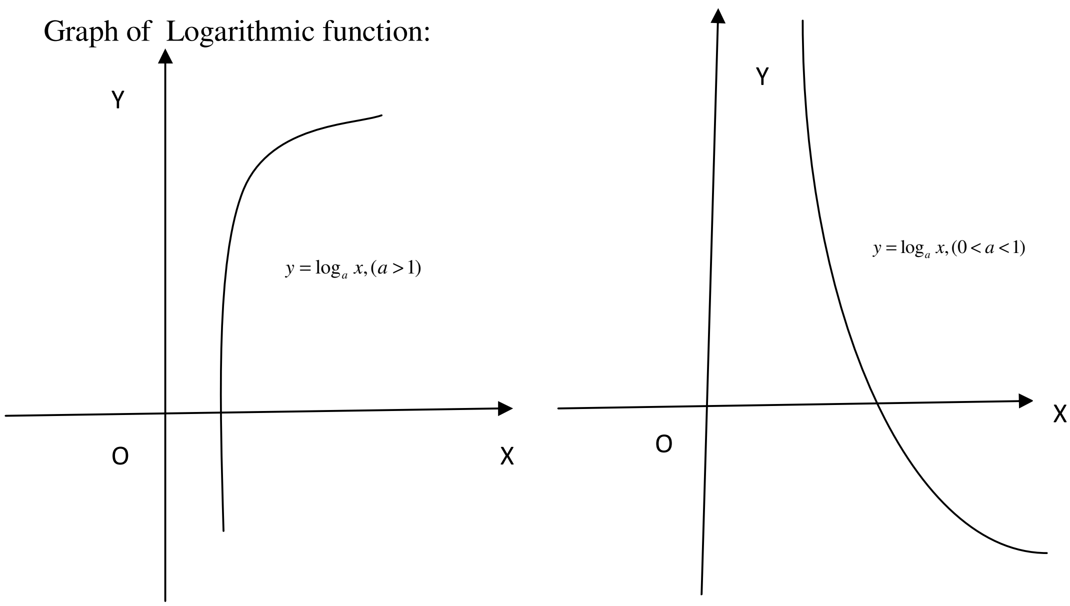




6. Logarithmic function:

If $a > 0, a \neq 1$ given $y > 0$ there is a unique $x \in R$ such that $a^x = y.$ This function defined on R^+ by $f(y) = x,$ where $a^x = y,$ is called the *logarithmic function* to the base a and is denoted by $\log_a$. Thus $\log_a y = x$ iff $a^x = y.$ The *logarithmic function* to the base e is called the *natural logarithmic function* and is denoted by $\log_e$ or $\ln$. Thus $\log_e y = x$ iff $e^x = y.$ The range is $R.$

Graph of Logarithmic function:

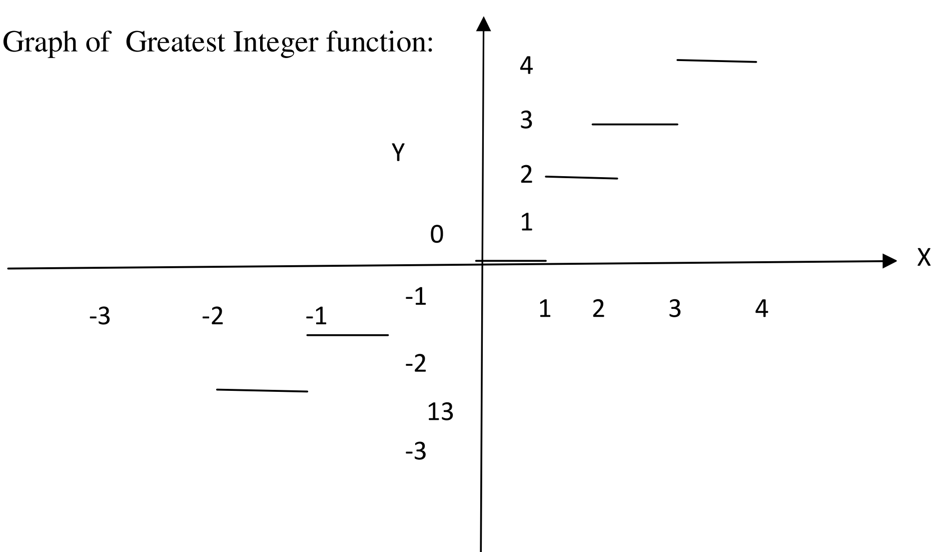


7. Greatest Integer function:

For any real number x, we denote by $[x]$, the greatest integer less than or equal to $x.$ For example $[2.45] = 2, [0.47] = 0, [-0.36] = -1, [-3.56] = -4.$

The function $f : R \rightarrow R$ defined by $f(x) = [x]$ for all $x \in R$ is called the *greatest integer function.* The domain of $[x]$ is R and range is Z.

Graph of Greatest Integer function:



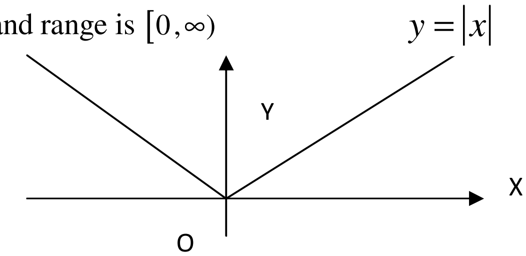
8. **Modulus function:**

The function $f: R \to R$ defined by $f(x) = |x|$ for all $x \in R$ is called the *modulus function.* For any non-negative real number x, $f(x)$ is equal to x. But for negative real number x, $f(x)$ is equal to $-x$. For example

$|2| = 2, |-4| = 4.$ $\quad$ *i.e* $f(x) = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}.$

The domain of $|x|$ is R and range is $[0, \infty)$

Graph of Modulus function:



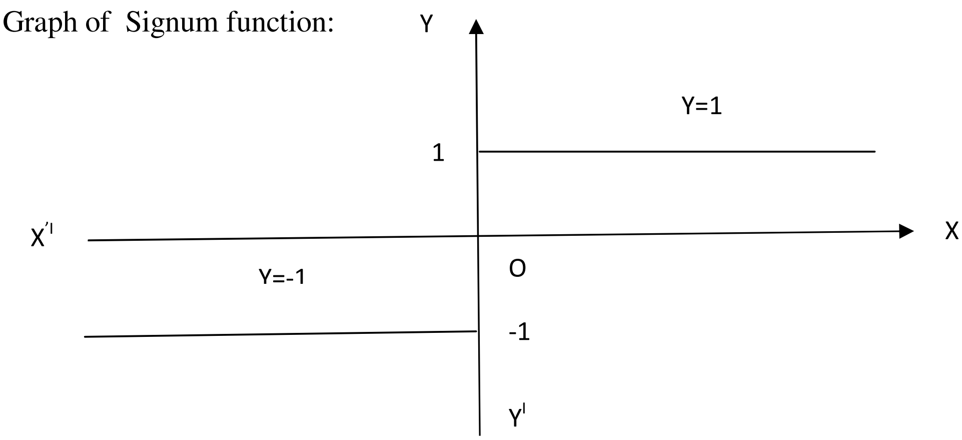
9. **Signum function:**

The function $f: R \to R$ defined by $f(x) = \frac{|x|}{x} = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$ for all $x \in R$

is called the *signum function.* It is denoted by $\text{sgn}(x)$.

The domain of $\text{sgn}(x)$ is R and range is $\{-1, 0, 1\}$.

Graph of Signum function:



1.3.4 Solved Problems:

1. **Problem:** If $A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and $f: A \to B$ is onto such that $f(x) = \cos x$ then

find B

Solution: Given $f: A \to B$ is onto such that $f(x) = \cos x$

Also given $A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$

We have $f(0) = \cos 0 = 1,\ f(\frac{\pi}{6}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\ f(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$

$f(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2},\ f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0.$

$\therefore B = \left\{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\right\}$

2. Problem: If $A = \{-2,-1,0,1,2\}$ and $f: A \to B$ is onto such that $f(x) = x^2 + x + 1$ then find B

Solution: Given $f: A \to B$ is onto such that $f(x) = x^2 + x + 1$

Also given $A = \{-2,-1,0,1,2\}$

We have $f(-2) = (-2)^2 + (-2) + 1 = 4 - 2 + 1 = 3,$

$f(-1) = (-1)^2 + (-1) + 1 = 1 - 1 + 1 = 1,\ f(0) = (0)^2 + (0) + 1 = 0 + 0 + 1 = 1,$

$f(1) = (1)^2 + (1) + 1 = 1 + 1 + 1 = 3,\ f(2) = (2)^2 + (2) + 1 = 4 + 2 + 1 = 7.$

$\therefore B = \{1,3,7\}$

3. Problem: If $f = \{(1,2),(2,-3),(3,1)\}$ then find $(i) 2f\ (ii) f^2\ (iii) f+2\ (iv) \sqrt{f}$

Solution: Given $f = \{(1,2),(2,-3),(3,1)\}$

Now $(i) 2f = \{(1,4),(2,-6),(3,2)\}$ $(ii) f^2 = \{(1,4),(2,9),(3,1)\}$

$(iii) f+2 = \{(1,4),(2,-1),(3,3)\}$

(iv) since $f(2) = -3$ $\sqrt{f}$ does not exists

4. Problem: If $f = \{(4,5),(5,6),(6,-4)\}$ and $g = \{(4,-4),(6,5),(8,5)\}$ then find

$(i) f+g\ (ii) f-g\ (iii) 2f+4g\ (iv) f+4\ (v) fg\ (vi) f/g\ (vii) |f|$
$(viii) \sqrt{f}\ (ix) f^2\ (x) f^3$

Solution: Given $f = \{(4,5),(5,6),(6,-4)\}$ and $g = \{(4,-4),(6,5),(8,5)\}$

Now $(i) f+g = \{(4,1)\}$ $(ii) f-g = \{(4,9)\}$ $(iii) 2f+4g = \{(4,-6)\}$

$(iv) f+4=\{(4,9),(5,10),(6,0)\}$ $(v) fg=\{(4,-20)\}$ $(vi) f/g=\{(4,-5/4)\}$

$(vii) |f|=\{(4,5),(5,6),(6,4)\}$ $(viii)$ since $f(6)=-4$, $\sqrt{f}$ does not exists

$(ix) f^2=\{(4,25),(5,36),(6,16)\}$ $(x) f^3=\{(4,125),(5,216),(6,-64)\}$

5. Problem: If $f(x)=2x-1$ and $g(x)=x^2$ then find $(i)(3f-2g)x$ $(ii)(fg)x$

$(iii)(f+g+2)x$ $(iv)(\sqrt{f}/g)x$

Solution: Given $f(x)=2x-1$ and $g(x)=x^2$

Now $(i)(3f-2g)x=3f(x)-2g(x)=3(2x-1)-2(x^2)=6x-3-2x^2$,

$(ii)(fg)(x)=f(x)g(x)=(2x-1)(x^2)=2x^3-x^2$

$(iii)(f+g+2)(x)=f(x)+g(x)+2=2x-1+x^2+2=x^2+2x+1$

$(iv)\ (\sqrt{f}/g)x=\sqrt{f(x)}\big/g(x)=\sqrt{2x-1}\big/x^2$

6. Problem: If $f(x)=\dfrac{1-x^2}{1+x^2}$ then show that $f(\tan\theta)=\cos 2\theta.$

Solution: Given $f(x)=\dfrac{1-x^2}{1+x^2}$

$$\text{Now L.H.S}=f(\tan\theta)=\frac{1-(\tan\theta)^2}{1+(\tan\theta)^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta=\text{R.H.S}$$

7. Problem: If $f(x)=\log\left|\dfrac{1-x}{1+x}\right|$ then show that $f(\dfrac{2x}{1+x^2})=2f(x).$

Solution: Given $f(x)=\log\left|\dfrac{1-x}{1+x}\right|$

$$\text{Now}\quad \text{L.H.S}=f(\frac{2x}{1+x^2})=\log\left|\frac{1-\dfrac{2x}{1+x^2}}{1+\dfrac{2x}{1+x^2}}\right|=\log\left|\frac{\dfrac{1+x^2-2x}{1+x^2}}{\dfrac{1+x^2+2x}{1+x^2}}\right|$$

$$=\log\left|\frac{1+x^2-2x}{1+x^2+2x}\right|=\log\left|\frac{(1-x)^2}{(1+x)^2}\right|=\log\left|\frac{1-x}{1+x}\right|^2$$

$$=2\log\left|\frac{1-x}{1+x}\right|=2f(x)=\text{R.H.S}$$

8. Problem: If $f(x)=4x-1$ and $g(x)=x^2+2$ then find

$$(i)(g\circ f)x \ (ii)(g\circ(f\circ f))x \ (iii)(g\circ f)\left(\frac{a+1}{4}\right) \ (iv)(f\circ f)x$$

Solution: Given $f(x)=4x-1$ and $g(x)=x^2+2$

Now $(i)(g\circ f)x=g\left(f(x)\right)=g(4x-1)=(4x-1)^2+2$

$$=16x^2-8x+1+2=16x^2-8x+3$$

$$(ii)(g\circ(f\circ f))x=g\left(f\left(f(x)\right)\right)=g\left(f\left(4x-1\right)\right)$$

$$=g\left(4\left(4x-1\right)-1\right)=g\left((16x-4)-1\right)=g\left(16x-5\right)$$

$$=\left(16x-5\right)^2+2=\left(256x^2-160x+25\right)+2=256x^2-160x+27$$

$$(iii)(g\circ f)\left(\frac{a+1}{4}\right)=g\left(f\left(\frac{a+1}{4}\right)\right)=g\left(4\left(\frac{a+1}{4}\right)-1\right)$$

$$=g\left((a+1)-1\right)=g\left(a\right)=a^2+2$$

$$(iv)\ (f\circ f)x=f\left(f(x)\right)=f\left(4x-1\right)=4\left(4x-1\right)-1$$

$$=16x-4-1=16x-5$$

9. Problem: If $f(x)=2,\ \ g(x)=x^2$ and $h(x)=2x$ then find $(f\circ(g\circ h))x$

Solution: Given $f(x)=2,\ g(x)=x^2$ and $h(x)=2x$

$$(f\circ(g\circ h))x=f((g\circ h))x=f((g(h(x)))=f((g(2x))=f((2x)^2)=f(4x^2)=2$$

10. Problem: If $f(x)=ax+b$ then find $f^{-1}(x)$

Solution: Given $f(x)=ax+b$

Put $f(x)=y\Rightarrow ax+b=y\ \Rightarrow ax=y-b$

$$\Rightarrow x=\frac{y-b}{a}\ \Rightarrow f^{-1}(y)=\frac{y-b}{a}\left[\because f(x)=y\Rightarrow x=f^{-1}(y)\right]$$

$$\therefore f^{-1}(x)=\frac{x-b}{a}$$

11. Problem: If $f(x)=5^x$ then find $f^{-1}(x)$

Solution: Given $f(x)=5^x$

Put $f(x)=y\Rightarrow 5^x=y\ \Rightarrow x=\log_5 y$

$$\Rightarrow f^{-1}(y)=\log_5 y\left[\because f(x)=y\Rightarrow x=f^{-1}(y)\right]$$

$$\therefore f^{-1}(x)=\log_5 x$$

12. Problem: If $f(x)=4x-1$ and $g(x)=x^2+2$ then find $(f\circ g)^{-1}(x)$

Solution: Given $f(x)=4x-1$ and $g(x)=x^2+2$

Now $(f\circ g)x=f\left(g(x)\right)=f(x^2+2)=\ 4(x^2+2)-1$

$$=4x^2+8-1=4x^2+7$$

Put $(f\circ g)(x)=y\Rightarrow 4x^2+7=y\ \Rightarrow 4x^2=y-7\Rightarrow x^2=\dfrac{y-7}{4}$

$$\Rightarrow x=\sqrt{\frac{y-7}{4}}$$

$$\Rightarrow (f\circ g)^{-1}(y)=\sqrt{\frac{y-7}{4}}\left[\because (f\circ g)(x)=y\Rightarrow x=(f\circ g)^{-1}(y)\right]$$

$$\therefore (f\circ g)^{-1}(x)=\sqrt{\frac{x-7}{4}}$$

13. Problem: If $f(x)=\dfrac{x+1}{x-1}$ and $g(x)=x^2+2$ then find $(f\circ g)(x)$

Solution: Given $f(x)=\dfrac{x+1}{x-1}$ and $g(x)=x^2+2$

Now $(f\circ g)x=f\left(g(x)\right)=f(x^2+2)=\ \dfrac{x^2+2+1}{x^2+2-1}=\dfrac{x^2+3}{x^2+1}$

14. Problem: If $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x}$ then find $(g\circ f)(x)$ and $(g\sqrt{f})(x)$

Solution: Given $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x}$

Now $(g\circ f)x=g\left(f(x)\right)=g(\dfrac{1}{x})=\ \sqrt{\dfrac{1}{x}}=\dfrac{1}{\sqrt{x}}$

Also $(g\sqrt{f})(x)=g(x)\sqrt{f}(x)=\sqrt{x}(\dfrac{1}{\sqrt{x}})=1$

15. Problem: If $A=\{1,2,3,4\}$ and $f:A\to B$ is defined by $f(x)=\dfrac{x^2-x+1}{x+1}$ then Find range of f

Solution: Given $f : A \to B$ is onto such that $f(x) = \dfrac{x^2 - x + 1}{x+1}$

Also given $A = \{1, 2, 3, 4\}$

We have $f(1) = \dfrac{1^2 - 1 + 1}{1+1} = \dfrac{1-1+1}{2} = \dfrac{1}{2}$, $f(2) = \dfrac{2^2 - 2 + 1}{2+1} = \dfrac{4-2+1}{3} = \dfrac{3}{3} = 1$,

$f(3) = \dfrac{3^2 - 3 + 1}{3+1} = \dfrac{9-3+1}{4} = \dfrac{7}{4}$, $f(4) = \dfrac{4^2 - 4 + 1}{4+1} = \dfrac{16-4+1}{5} = \dfrac{13}{5}$

$\therefore$ Range of $f = f(A) = \left\{ \dfrac{1}{2}, 1, \dfrac{13}{5}, \dfrac{7}{4} \right\}$

Exercise 1

1. If $f(x) = e^x$ and $g(x) = \log_e x$, then show that $g \circ f = f \circ g$ and find f^{-1} and g^{-1}.

2. If $f(x) = 2x - 1$ and $g(x) = \dfrac{x+1}{2} \forall x \in R$, then find $(g \circ f)(x)$.

3. If $f : R \to R$, $g : R \to R$ are defined by $f(x) = 3x - 1$ and $g(x) = x^2 + 1$ then find

   $(i)(g \circ f)x$ $\quad (ii)(g \circ f)(2)$ $\quad (iii)(f \circ f)(x^2 + 1)$

4. If $f : R \to R$, $g : R \to R$ are defined by $f(x) = 3x - 2$ and $g(x) = x^2 + 1$ then find

   $(i)(g \circ f^{-1})(2)$ $\quad (ii)(g \circ f)(x-1)$ $\quad (iii)(g \circ f)(2a - 3)$

5. If $f : R \to R$, $g : R \to R$ are defined by $f(x) = 2x - 3$ and $g(x) = x^3 + 5$ then find

   $(i)(g \circ f)(1)$ $\quad (ii)(g \circ f^{-1})(2)$ $\quad (iii)(f \circ g)(x)$

6. If $f = \{(1,a), (2,c), (3,b), (4,d)\}$ and $g^{-1} = \{(1,c), (2,a), (3,d), (4,b)\}$ then find

   $(i) g^{-1} \circ f^{-1}$ $\quad (ii)(g \circ f)^{-1}$ $\quad (iii)(f \circ g)^{-1}$ $\quad (iv) f^{-1} \circ g^{-1}$

7. If $f(x) = 3^x$ then find $f^{-1}(x)$

8. If $f(x) = 3x + 5$ then find $f^{-1}(x)$

9. If $A = \left\{0, \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{2}\right\}$ and $f : A \to B$ is onto such that $f(x) = \sin x$ then find B

10. If $f : R - \{0\} \to R$ is defined by $f(x) = x + \dfrac{1}{x}$ then prove that $(f(x))^2 = f(x^2) + f(1)$

Key Concepts

Let A and B be non-empty sets and f be a relation from A to B. If for each element $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in f$, then f is called a *function (or mapping)* from (or A into B) A to B. It is denoted by $f: A \to B$. The set A is called the *domain* of f and B is called the *co-domain* of f. If $f: A \to B$ is a function and if $f(a) = b$, then b is called the *image* of a under f The element a is called the *pre-image* or *inverse image* of b under f and it is denoted by $f^{-1}(b)$. If $f: A \to B$ is a function, then $f(A)$, the set of all f^{-} *images* of elements in A is called the *range* of f. Clearly $f(A) = \{f(a) / a \in A\} \subseteq B$.

1. $f: A \to B$ is called an *injection* $\Leftrightarrow a_1, a_2 \in A$ and $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

2. $f: A \to B$ is called a *surjection* $\Leftrightarrow$ range $f = f(A) = B(co-domain)$
3. $f: A \to B$ is a *bijection* $\Leftrightarrow$ $f: A \to B$ is both an *injection* and a *surjection*
4. Let f and g be functions. We say that f and g are *equal* and write $f = g$ if *domain of* $f =$ *domain of* g and $f(x) = g(x)$ for all $x \in$ *domain of* f.

5. $f: A \to A$ defined by $f(x) = x$ for all $x \in A$ is called the *identity function* on A and is denoted by I_A.

6. If $f: A \to B,\ g: B \to C$, then the relation $\{(a, g(f(a))) / a \in A\}$ is called *composite* of g with f and is denoted as $g \circ f$.
7. Let $f: A \to B$ and $g: B \to C$ be injections. Then $g \circ f: A \to C$ is an injection
8. Let $f: A \to B$ and $g: B \to C$ be surjections. Then $g \circ f: A \to C$ is a surjection
9. Let $f: A \to B$ and $g: B \to C$ be bijections. Then $g \circ f: A \to C$ is a bijection
10. Let $f: A \to B$ and $g: B \to C$ be bijections. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
11. The identity function $I_A: A \to A$ is a bijection and $I_A^{-1} = I_A$.
12. Let $f: A \to B$, I_A and I_B be identity functions on A and B respectively. Then $f \circ I_A = f = I_B \circ f$.
13. Let $f: A \to B$ be a bijection. Then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$.
14. Let $f: A \to B, g: B \to C$ and $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

15. If f and g are real valued functions with domains A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B \neq \phi$. defined through a formula. If $f: A \to R$ and $g: B \to R$ are functions such that $A \cap B \neq \phi$.

   $(i) f + g: A \cap B \to R$ as $(f+g)(x) = f(x) + g(x)\ \forall x \in A \cap B$.
   $(ii) f - g: A \cap B \to R$ as $(f-g)(x) = f(x) - g(x)\ \forall x \in A \cap B$.

$(iii) fg : A\cap B \to R$ as $(fg)(x) = f(x).g(x)\ \forall\, x\in A\cap B.$

$(iv)\dfrac{f}{g} : E \to R$ as $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}\forall\, x\in E.$ Where

$E = \{x\in A\cap B\,/\,g(x)\neq 0\}\neq\phi.$

$(v) cf : A \to R$ as $(cf)(x) = cf(x)\ \forall\, x\in A.$

$(vii) f^{n} : A \to R$ as $f^{n}(x) = (f(x))^{n}\ \forall\, x\in A.$

$(viii)\sqrt{f} : E \to R$ as $(\sqrt{f})(x) = \sqrt{f(x)}\,\forall\, x\in E.$ Where

$E = \{x\in A\,/\,f(x)\geq 0\}\neq\phi.$

16. Let A be a nonempty subset of R such that $-x\in A$ for all $x\in A$ and $f : A\to R$ is called an *even function* if $f(-x) = f(x)\forall x\in A$ and *odd function* if $f(-x) = f(x)\forall x\in A.$

17. The function $f : R\to R$ defined by $f(x) = |x|$ for all $x\in R$ is called the *modulus function. i.e* $f(x) = \begin{cases} x \ if\ x\geq 0 \\ -x\ if\ x<0 \end{cases}.$ The domain of $|x|$ is R and range is $[0,\infty)$

18. The function $f : R\to R$ defined by $f(x) = \dfrac{|x|}{x} = \begin{cases} 1\ if\ x>0 \\ 0\ if\ x=0 \\ -1\ if\ x<0 \end{cases}$ for all $x\in R$ is called the *signum function.* It is denoted by $\operatorname{sgn}(x).$ The domain of $\operatorname{sgn}(x)$ is R and range is $\{-1,0,1\}$.

Answers

Exercise 1

(1) $f^{-1} = \log_2 x,\ \ g^{-1} = e^{x}$ (2) $(g\circ f)(x) = I(x)$ (3) $(i) 9x^2 - 6x + 2\,(ii) 26\,(iii) 9x^2 + 5$

(4) $(i)\dfrac{25}{9}\,(ii) 9x^2 - 30x + 26\,(iii) 36a^2 - 132a + 122$ (5) $(i) 4\,(ii)\dfrac{165}{8}\ (iii) 2x^3 + 7$

(6) $(i)\{(a,c),(b,d),(c,a),(d,b)\}\,(ii)\{(1,2),(2,1),(3,4),(4,1)\}$
$(iii)\{(a,c),(b,d),(c,a),(d,b)\}\,(iv)\{(1,2),(2,1),(3,4),(4,1)\}$

(7) $\log_3 x$ (8) $\dfrac{x-5}{3}$ (9) $\left\{0,\dfrac{1}{2},\dfrac{1}{\sqrt{2}},\dfrac{\sqrt{3}}{2},1\right\}$

2. MATHEMATICAL INDUCTION

One key basis for mathematical thinking is deductive reasoning. An informal example of deductive reasoning, borrowed from the study of logic, is an argument expressed in three statements:

(a) Ramesh is a man.

(b) All men are mortal, therefore,

(c) Ramesh is mortal.

If statements (a) and (b) are true, then the truth of (c) is established. To make this simple mathematical example, we could write:

(i) Six is divisible by two.

(ii) Any number divisible by two is an even number, therefore,

(iii) Six is an even number.

Thus, deduction in a nutshell is *given a statement to be proven, often called a conjecture or a theorem in Mathematics, valid deductive steps are derived and a proof may or may not be established,* i.e., *deduction is the application of a general case to a particular case.*

2.1 Principles of finite Mathematical Induction & Theorems:

In contrast to deduction, inductive reasoning depends on working with each case and developing a conjecture by observing incidences till we have observed each and every case. It is frequently used in Mathematics and is a key aspect of scientific reasoning, where collecting and analysing data is the norm. Thus, in simple language, we can say the word induction means the generalisation from particular cases or facts.

2.1.1 Definition: A subset *S* of *R* is said to be an *inductive set* if $i) 1 \in S,\ ii) k \in S \Rightarrow k+1 \in S.$

Example: i) *R* is an *inductive set*.

ii) $A = \{x \in R \,/\, x > 0\}$ is an *inductive set*.

iii) $B = \{x \in R \,/\, x > 3\}$ is not an *inductive set.*

Note: i) The intersection of all inductive sets in *R* is called the set of *natural numbers* or the *set of positive integers.* It is denoted by N or Z^+.

ii) $N = \bigcap \{A \,/\, A \text{ is an inductive set in } R\}$.

iii) The set of natural numbers is an inductive set in *R*

2.1.2 Induction Theorem:

If S is a subset of N such that $i) 1 \in S$, $ii) k \in S \Rightarrow k+1 \in S$ then $S = N$

2.1.3 The Principle of finite Mathematical Induction:

Suppose there is a given statement P(n) involving the natural number n such that

(i) *The statement is true for* $n = 1$, i.e., $P(1)$ *is true, and*

(ii) *If the statement is true for* $n = k$ (*where k is some positive integer*), *then the statement is also true for* $n = k + 1$, i.e., *truth of* $P(k)$ *implies the truth of* $P(k + 1)$.

Then, P(n) is true for all natural numbers n.

2.1.4 Steps involved in Mathematical Induction:

Let $S(n)$ be a statement for each $n \in N$.

If $i) S(1)$ is true,

$ii) S(k)$ is true $\Rightarrow S(k+1)$ is true then $S(n)$ is true for all $n \in N$.

2.1.5 The Principle of a complete Mathematical Induction:

Let $S(n)$ be a statement for each $n \in N$.

If $i) S(1)$ is true,

$ii) S(1), S(2), S(3) \ldots S(k)$ is true $\Rightarrow S(k+1)$ is true then $S(n)$ is true for all $n \in N$.

2.2 Applications of Mathematical Induction:

Mathematical Induction is very useful in proving many theorems and statements. For example, it is useful in Binomial theorem, Leibnitz's theorem etc.

2.2.1 Solved Problems:

1. Problem: *Show that* $1+2+3+\ldots+n = n(n+1)/2$ *for all* $n \in N$.

Solution: Let $S(n)$ be a statement that $1+2+3+\ldots+n = n(n+1)/2$.

If $n = 1$ then L.H.S $= 1$

$$\text{R.H.S} = \frac{1(1+1)}{2} = 1$$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1+2+3+\ldots+k = k(k+1)/2.$

Adding both sides $(k+1)$ we get $1+2+3+...+k+k+1=\frac{k(k+1)}{2}+k+1$

$$=(k+1)\left(\frac{k}{2}+1\right)=(k+1)\left(\frac{k+2}{2}\right)=\frac{(k+1)(k+2)}{2}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$\therefore 1+2+3+...+n=n(n+1)/2 \; for\, all\, n\in N.$

2. Problem: *Show that* $1^2+2^2+3^2+...+n^2=n(n+1)(2n+1)/6 \; for\, all\, n\in N.$

Solution: Let $S(n)$ be a statement that $1^2+2^2+3^2+...+n^2=n(n+1)(2n+1)/6.$

If $n=1$ then $\text{L.H.S}=1^2=1$

$$\text{R.H.S}=\frac{1(1+1)(2.1+1)}{6}=\frac{1.2.3}{6}=1$$

$\therefore \text{L.H.S}=\text{R.H.S}$

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1^2+2^2+3^2+...+k^2=k(k+1)(2k+1)/6.$

Adding both sides $(k+1)^2$ we get

$$1^2+2^2+3^2+...+k^2+(k+1)^2=k(k+1)(2k+1)/6+(k+1)^2$$

$$=(k+1)\left(\frac{k(2k+1)}{6}+k+1\right)=(k+1)\left(\frac{2k^2+k+6k+6}{6}\right)=(k+1)\left(\frac{2k^2+7k+6}{6}\right)$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$\therefore 1^2+2^2+3^2+...+n^2=n(n+1)(2n+1)/6 \; for\, all\, n\in N.$

3. Problem: *Show that* $1^3+2^3+3^3+...+n^3=n^2(n+1)^2/4 \; for\, all\, n\in N.$

Solution: Let $S(n)$ be a statement that $1^3+2^3+3^3+...+n^3=n^2(n+1)^2/4.$

If $n=1$ then $\text{L.H.S}=1^3=1$

$\text{R.H.S} = \dfrac{1^2(1+1)^2}{4} = \dfrac{1.4}{4} = 1$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore \quad 1^3 + 2^3 + 3^3 + ... + k^3 = k^2(k+1)^2 / 4.$

Adding both sides $(k+1)^3$ we get

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = k^2(k+1)^2 / 4 + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n \in N$.

$\therefore 1^3 + 2^3 + 3^3 + ... + n^3 = n^2(n+1)^2 / 4$ *for all* $n \in N$.

4. Problem: *Show that* $4^3 + 8^3 + 12^3 + ... + 64n^3 = 16n^2(n+1)^2$ *for all* $n \in N$.

Solution: Let $S(n)$ be a statement that $4^3 + 8^3 + 12^3 + ... + 64n^3 = 16n^2(n+1)^2$

If $n = 1$ then $\text{L.H.S} = 64.1^3 = 64.1 = 64$

$\text{R.H.S} = 16.1^2.(1+1)^2 = 16.1.4 = 64$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore \quad 4^3 + 8^3 + 12^3 + ... + 64k^3 = 16k^2(k+1)^2$

Adding both sides $64(k+1)^3$ we get

$$4^3 + 8^3 + 12^3 + ... + 64k^3 + 64(k+1)^3 = 16k^2(k+1)^2 + 64(k+1)^3$$

$$= 16(k+1)^2\left(k^2 + 4(k+1)\right) = 16(k+1)^2\left(k^2 + 4k + 4\right) = 16(k+1)^2(k+2)^2$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n \in N$.

$\therefore \; 4^3 + 8^3 + 12^3 + ... + 64n^3 = 16n^2(n+1)^2 \quad \textit{for all } n \in N.$

5. Problem: *Show that* $1.3 + 2.4 + 3.5 + ... + n(n+2) = n(n+1)(2n+7)/6$ *for all* $n \in N$.

Solution: Let $S(n)$ be a statement that $1.3 + 2.4 + 3.5 + ... + n(n+2) = n(n+1)(2n+7)/6$

If $n = 1$ then L.H.S $= 1.3 = 3$

$$\text{R.H.S} = \frac{1(1+1)(2.1+7)}{6} = \frac{1.2.9}{6} = 3$$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1.3 + 2.4 + 3.5 + ... + k(k+2) = k(k+1)(2k+7)/6$

Adding both sides $(k+1)(k+3)$ we get

$$1.3 + 2.4 + 3.5 + ... + k(k+2) + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

$$= (k+1)\left(\frac{k(2k+7)}{6} + k + 3\right) = (k+1)\left(\frac{2k^2 + 7k + 6k + 18}{6}\right)$$

$$= (k+1)\left(\frac{2k^2 + 13k + 18}{6}\right) = (k+1)\frac{(k+2)(2k+9)}{6} \; = \frac{(k+1)(k+2)(2k+9)}{6}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n \in N$.

$\therefore 1.3 + 2.4 + 3.5 + ... + n(n+2) = n(n+1)(2n+7)/6$ *for all* $n \in N$.

6. Problem: *Show that* $1.6 + 2.9 + 3.12 + ... + n(3n+3) = n(n+1)(n+2)$ *for all* $n \in N$.

Solution: Let $S(n)$ be a statement that $1.6 + 2.9 + 3.12 + ... + n(3n+3) = n(n+1)(n+2)$

If $n = 1$ then L.H.S $= 1.6 = 6$

R.H.S $= 1(1+1)(1+2) = 1.2.3 = 6$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1.6+2.9+3.12+...+k(3k+3)=k(k+1)(k+2)$

Adding both sides $(k+1)(3k+6)$ we get

$1.6+2.9+3.12+...+k(3k+3)+(k+1)(3k+6)=k(k+1)(k+2)+(k+1)(3k+6)$

$=(k+1)(k+2)(k+3)$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$\therefore 1.6+2.9+3.12+...+n(3n+3)=n(n+1)(n+2)\, for\, all\, n\in N.$

7. Problem: *Show that* $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$ *for all* $n\in N.$

Solution: Let $S(n)$ be a statement that $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$

If $n=1$ then L.H.S $=\frac{1}{1.2}=\frac{1}{2}$

R.H.S $=\frac{1}{1+1}=\frac{1}{2}$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Adding both sides $\frac{1}{(k+1)(k+2)}$ we get

$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

$=\frac{1}{k+1}\left(k+\frac{1}{k+2}\right)=\frac{1}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)=\frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$

$=\frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right)=\frac{k+1}{k+2}$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$\therefore \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$ *for all* $n\in N$.

8. Problem: *Show that* $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ *for all* $n\in N$.

Solution: Let $S(n)$ be a statement that $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

If $n=1$ then $\text{L.H.S}=\frac{1}{1.3}=\frac{1}{3}$

$\text{R.H.S}=\frac{1}{2.1+1}=\frac{1}{3}$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$$\therefore \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Adding both sides $\frac{1}{(2k+1)(2k+3)}$ we get

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{1}{2k+1}\left(k+\frac{1}{2k+3}\right)=\frac{1}{2k+1}\left(\frac{k(2k+3)+1}{2k+3}\right)=\frac{1}{2k+1}\left(\frac{2k^2+3k+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\frac{(k+1)(2k+1)}{2k+3}=\frac{k+1}{2k+3}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$$\therefore \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad \textit{for all } n\in N.$$

9. Problem:

Show that $a+(a+d)+(a+2d)+...+\textit{upto } n \textit{ terms}=\frac{n}{2}\left(2a+(n-1)d\right)$ *for all* $n\in N$.

Solution: n^{th} term $=a+(n-1)d$

Let $S(n)$ be a statement that

$$a+(a+d)+(a+2d)+...+\left(a+(n-1)d\right)=\frac{n}{2}\left(2a+(n-1)d\right)$$

If $n=1$ then L.H.S $=a$

R.H.S $=\frac{1}{2}\left(2a+(1-1)d\right)=\frac{1}{2}\left(2a\right)=a$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$$\therefore\ a+(a+d)+(a+2d)+...+\left(a+(k-1)d\right)=\frac{k}{2}\left(2a+(k-1)d\right)$$

Adding both sides $a+kd$ we get

$$a+(a+d)+(a+2d)+...+\left(a+(k-1)d\right)+\left(a+kd\right)=\frac{k}{2}\left(2a+(k-1)d\right)+\left(a+kd\right)$$

$$=ak+\frac{k(k-1)d}{2}+a+kd=a(k+1)+kd\left(\frac{k-1}{2}+1\right)=a(k+1)+kd\left(\frac{k+1}{2}\right)$$

$$=\frac{k+1}{2}(2a+kd)$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$$\therefore\ a+(a+d)+(a+2d)+...+\left(a+(n-1)d\right)=\frac{n}{2}\left(2a+(n-1)d\right)\ for\ all\ n\in N.$$

10. Problem: *Show that* $a+ar+ar^2+...+upto\ n\ terms=\frac{a(r^n-1)}{r-1}$ *for all* $n\in N$.

Solution: n^{th} term $=ar^{n-1}$

Let $S(n)$ be a statement that $a+ar+ar^2+...+ar^{n-1}=\frac{a(r^n-1)}{r-1}$

If $n=1$ then L.H.S $=a$

R.H.S $=\frac{a(r^1-1)}{r-1}=\frac{a(r-1)}{r-1}=a$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$$\therefore \quad a+ar+ar^2+...+ar^{k-1}=\frac{a(r^k-1)}{r-1}$$

Adding both sides ar^k we get

$$a+ar+ar^2+...+ar^{k-1}+ar^k=\frac{a(r^k-1)}{r-1}+ar^k=\frac{ar^k-a}{r-1}+ar^k=\frac{ar^k-a+ar^k(r-1)}{r-1}$$

$$=\frac{ar^k-a+ar^{k+1}-ar^k}{r-1}=\frac{a(r^{k+1}-1)}{r-1}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n\in N$.

$$\therefore \quad a+ar+ar^2+...+ar^{n-1}=\frac{a(r^n-1)}{r-1} \textit{ for all } n\in N.$$

11. Problem:

Show that $1.2.3+2.3.4+3.4.5+...+$ *upto n terms* $=n(n+1)(n+2)(n+3)/4$ *for all* $n\in N$.

Solution: n^{th} term $=n(n+1)(n+2)$

Let $S(n)$ be a statement that

$1.2.3+2.3.4+3.4.5+...+n(n+1)(n+2)=n(n+1)(n+2)(n+3)/4$

If $n=1$ then L.H.S $=1.2.3=6$

$$\text{R.H.S}=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1.2.3.4}{4}=6$$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1.2.3+2.3.4+3.4.5+...+k(k+1)(k+2)=k(k+1)(k+2)(k+3)/4$

Adding both sides $(k+1)(k+2)(k+3)$ we get

$$1.2.3+2.3.4+3.4.5+...+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$
$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$= (k+1)(k+2)(k+3)\left(\frac{k+4}{4}\right) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n \in N$.

$$\therefore 1.2.3 + 2.3.4 + 3.4.5 + ... + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \; for\, all\, n \in N.$$

12. Problem:

Show that $1 + (1+2) + (1+2+3) + ... + upto\, n\, brackets = n(n+1)(n+2)/6 \; for\, all\, n \in N.$

Solution: n^{th} bracket is $1+2+3+...+n$

Let $S(n)$ be a statement that

$1 + (1+2) + (1+2+3) + ... + (1+2+3+...+n) = n(n+1)(n+2)/6 \; for\, all\, n \in N.$

If $n = 1$ then L.H.S $= 1$

$$\text{R.H.S} = \frac{1(1+1)(1+2)}{6} = \frac{1.2.3}{6} = 1$$

$\therefore$ L.H.S = R.H.S

$\therefore S(1)$ is true.

Assume that $S(k)$ is true.

$\therefore 1 + (1+2) + (1+2+3) + ... + (1+2+3+...+k) = k(k+1)(k+2)/6$

Adding both sides $\left(1+2+3+...+k+(k+1)\right)$ we get

$$1 + (1+2) + (1+2+3) + ... + (1+2+3+...+k) + \left(1+2+3+...+k+(k+1)\right)$$
$$= \frac{k(k+1)(k+2)}{6} + \left(1+2+3+...+k+(k+1)\right) = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+2)}{2}\left(\frac{k}{3}+1\right) = \frac{(k+1)(k+2)}{2}\left(\frac{k+3}{3}\right) = \frac{(k+1)(k+2)(k+3)}{6}$$

$\therefore S(k+1)$ is true.

$\therefore$ By principle of Mathematical Induction $S(n)$ is true for all $n \in N$.

$$\therefore 1 + (1+2) + (1+2+3) + ... + (1+2+3+...+n) = \frac{n(n+1)(n+2)}{6} \; for\, all\, n \in N.$$

Exercise 2

By principle of Mathematical Induction prove the following:

1. $2+7+12+...+(5n-3)=n(5n-1)/2 \; for\; all\; n\in N.$
2. $1.2+2.3+3.4+...+n(n+1)=n(n+1)(n+2)/3 \; for\; all\; n\in N.$
3. $1.3+3.5+5.7+...+n(n+1)=\dfrac{n(4n^2+6n-1)}{3} \; for\; all\; n\in N.$
4. $2.3+3.4+4.5+...+(n+1)(n+2)=\dfrac{n(n^2+6n+11)}{3} \; for\; all\; n\in N.$
5. $2+3.2+4.2^2+...+(n+1)2^{n-1}=n\,2^n \; for\; all\; n\in N.$
6. $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+...+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1} \; for\; all\; n\in N.$
7. $\dfrac{1}{2.5}+\dfrac{1}{5.8}+\dfrac{1}{8.11}+...+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{2(3n+2)} \; for\; all\; n\in N.$
8. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+...+upto\; n\; brackets=\dfrac{n(n+1)^2(n+2)}{12} \; for\; all\; n\in N.$
9. $\dfrac{1^3}{1}+\dfrac{1^3+2^3}{1+3}+\dfrac{1^3+2^3+3^3}{1+3+5}+...+upto\; n\; brackets=\dfrac{n(2n^2+9n+13)}{24} \; for\; all\; n\in N.$
10. $\cos\theta\cos 2\theta\cos 4\theta...\cos 2^{n-1}\theta=\dfrac{\sin 2^n\theta}{2^n\sin\theta} \; for\; all\; n\in N.$

Key Concepts

1. A subset S of R is said to be an *inductive set* if $i)1\in S,\; ii)k\in S \Rightarrow k+1\in S.$
2. If S is a subset of N such that $i)1\in S,\; ii)k\in S \Rightarrow k+1\in S$ then $S=N$
3. Suppose there is a given statement $P(n)$ involving the natural number n such that

   (*i*) *The statement is true for n* = 1, i.e., $P(1)$ *is true, and*

   (*ii*) *If the statement is true for n* = *k* (*where k is some positive integer*), *then the statement is also true for n* = *k* + 1, i.e., *truth of* $P(k)$ *implies the truth of* $P(k+1)$. *Then, P(n) is true for all natural numbers n.*
4. Let $S(n)$ be a statement for each $n\in N.$

   If $i)S(1)$ is true, $ii)S(k)$ is true $\Rightarrow S(k+1)$ is true then $S(n)$ is true for all $n\in N.$
5. Let $S(n)$ be a statement for each $n\in N.$ If $i)S(1)$ is true,

   $ii)S(1), S(2), S(3)...S(k)$ is true $\Rightarrow S(k+1)$ is true then $S(n)$ is true for all $n\in N.$

3. MATRICES

3.1 Types of Matrices:

In this section we define a matrix, its order and various types of matrices.

3.1.1 Definition (Matrix):

An ordered rectangular array of elements is called a *matrix.*

The elements of matrices are real or complex numbers (functions). Matrices are generally enclosed by brackets. We denote matrices by capital letters $A, B, C...$

The following are some examples of matrices.

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} -1 & 4 \\ 3 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ -2 & 5 & 9 \end{pmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute the *rows* of the matrix and the vertical lines of elements are said to constitute the *columns* of the matrix. Thus A has 3 rows and 2 columns, B has 2 rows and 2 columns, C has 3 rows and 3 columns.

3.1.2 Definition (Order of matrix):

A matrix having m rows and n columns is said to be of *order* $m \times n,$ read as of m cross n or m by $n.$

In the above examples, A is of order $2 \times 3,$ B is of order $2 \times 2,$ C is of order $3 \times 3.$

In general a matrix having m rows and n columns is represented as follows.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1j} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2j} \dots a_{2n} \\ a_{31} & a_{32} & a_{33} \dots & a_{3j} \dots a_{3n} \\ \dots & \dots & \dots & \dots \quad \dots \\ a_{i1} & a_{i2} & a_{i3} \dots & a_{ij} \dots \quad a_{in} \\ \dots & \dots & \dots & \dots \quad \dots \\ a_{m1} & a_{m2} & a_{m3} \dots & a_{mj} \dots a_{mn} \end{pmatrix}$$

In compact form it is denoted by $A = \left(a_{ij}\right)_{m \times n}$ where $1 \le i \le m$ and $1 \le j \le n.$

3.1.3 Definition (Rectangular matrix):

In a matrix if the number of rows is not equal to the number of columns then that matrix is called a *rectangular matrix*. For example the matrix $\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 0 \end{pmatrix}$ is a rectangular matrix.

Since the number of rows is 2 and the number of columns is 3.

3.1.4 Definition (Square matrix):

A matrix in which the number of rows is equal to the number of columns, is called a *square matrix*.

$A = \left(a_{ij}\right)_{m\times n}$ is a square matrix if $m = n.$ In this case A is a square matrix of order $n.$ $1 \le j \le n.$

For example the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ is a *square matrix* of order 2

and $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ -2 & 5 & 9 \end{pmatrix}$ is a *square matrix* of order $3.$

3.1.5 Definition (Row matrix):

A matrix having only one row is called a *row matrix*.

For example the matrix $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a row matrix.

3.1.6 Definition (Column matrix):

A matrix having only one column is called a *column matrix*.

For example the matrix $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a column matrix.

3.1.7 Definition (Null matrix or Zero matrix):

A matrix consisting of all zero elements is called a *null matrix* or *zero matrix*.

For example the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a zero matrix. It is denoted by O.

3.1.8 Definition (Principal diagonal elements of a matrix):

In a square matrix the elements in first row first column, second row second column, third row third column...n^{th} row n^{th} column are called *principal diagonal elements of a matrix.* If $A=\left(a_{ij}\right)_{n\times n}$ where $1\le i, j\le n$ then the principal diagonal elements are $a_{11}, a_{22}, a_{33}...a_{nn}$.

For example 1,4,9 are the principal diagonal elements of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ -2 & 5 & 9 \end{pmatrix}$

3.1.9 Definition (Trace of a matrix):

In a square matrix the sum of the principal diagonal elements of a matrix is called the *trace of a matrix*. Trace of a square matrix A is denoted by $Tr(A)$. If $A=\left(a_{ij}\right)_{n\times n}$ where $1\le i, j\le n$ then the trace of a square matrix A is denoted by

$$Tr(A)=\sum_{i=1}^{n}a_{ii}=a_{11}+a_{22}+a_{33}+...+a_{nn}.$$

For example the trace of the matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ -2 & 5 & 9 \end{pmatrix}$ is $Tr(A)=1+4+9=14.$

3.1.10 Definition (Triangular matrices):

A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is said to be an *upper triangular matrix* if $a_{ij}=0\ for\ all\ i>j$

A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is said to be a *lower triangular matrix* if $a_{ij}=0\ for\ all\ i<j$

For example $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}$ are upper triangular matrices and

$$\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 9 \end{pmatrix} \text{ are lower triangular matrices.}$$

3.1.11 Definition (Scalar matrix):

If each non-diagonal element of a square matrix is equal to zero and each diagonal elements are equal to each other, then it is called a *scalar matrix.*

For example $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ are scalar matrices.

3.1.12 Definition (Unit matrix or Identity matrix):

If each non-diagonal element of a square matrix is equal to zero and each diagonal elements are equal to 1, then that matrix is called a *unit matrix* or *identity matrix.*

For example $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrices.

3.1.13 Definition (Equality of matrices):

Matrices A and B are said to be equal if A and B are of same order and the corresponding elements of A and B are the same.

Thus $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$ are equal iff $a_{ij} = b_{ij}$

$for\ i = 1, 2$ and $j = 1, 2, 3.$

3.1.14 Definition (Sum of two matrices):

Let A and B be matrices of same order. Then the sum of A and B denoted by $A + B$ is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then $A + B = (c_{ij})_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$

$for\, 1 \le i \le m, 1 \le j \le n$

For example, if $A=\begin{pmatrix}1 & 1 & -2\\ 3 & 4 & -1\end{pmatrix}$ and $B=\begin{pmatrix}4 & 2 & 2\\ 7 & 3 & 4\end{pmatrix}$ then

$$A+B=\begin{pmatrix}1 & 1 & -2\\ 3 & 4 & -1\end{pmatrix}+\begin{pmatrix}4 & 2 & 2\\ 7 & 3 & 4\end{pmatrix}=\begin{pmatrix}1+4 & 1+2 & -2+2\\ 3+7 & 4+3 & -1+4\end{pmatrix}=\begin{pmatrix}5 & 3 & 0\\ 10 & 7 & 3\end{pmatrix}$$

3.1.15 Properties of Addition of matrices:

If $A=\left(a_{ij}\right)_{m\times n}$, $B=\left(b_{ij}\right)_{m\times n}$ and $C=\left(c_{ij}\right)_{m\times n}$ be matrices of the same order. then the addition of matrices satisfies the following properties.

(i) *Commutative Property*: $A+B=B+A$

Now $A+B=\left(a_{ij}\right)_{m\times n}+\left(b_{ij}\right)_{m\times n}=\left(a_{ij}+b_{ij}\right)_{m\times n}=\left(b_{ij}\right)_{m\times n}+\left(a_{ij}\right)_{m\times n}=B+A.$

(ii) *Associative Property*: $A+(B+C)=(A+B)+C$

Now

$$A+(B+C)=\left(a_{ij}\right)_{m\times n}+\left[\left(b_{ij}\right)_{m\times n}+\left(c_{ij}\right)_{m\times n}\right]=\left(a_{ij}\right)_{m\times n}+\left[\left(b_{ij}+c_{ij}\right)_{m\times n}\right]$$
$$=\left[\left(a_{ij}\right)_{m\times n}+\left(b_{ij}+c_{ij}\right)_{m\times n}\right]=\left[\left(a_{ij}+b_{ij}\right)_{m\times n}+\left(c_{ij}\right)_{m\times n}\right]$$
$$=\left(a_{ij}+b_{ij}\right)_{m\times n}+\left(c_{ij}\right)_{m\times n}=(A+B)+C.$$

(iii) *Additive identity*: $A+O=O+A=A$

If A is a $m\times n$ matrix and O is the $m\times n$ null matrix. Then we call O is the additive identity matrix.

(iv) *Additive inverse*: $A+B=B+A=O$

If A is a $m\times n$ matrix then there exists a unique $m\times n$ matrix B such that $A+B=B+A=O$, O being the $m\times n$ null matrix. Then we call B is the additive inverse of A denoted by $-A$.

3.2 Scalar multiple of a matrix and multiplication of matrices:

This section is devoted to the study of multiplication of a matrix (i) by a scalar and (ii) by a matrix. We also study the properties of multiplication.

3.2.1 Definition (Scalar multiple of a matrix):

Let A be a matrix of order $m\times n$ and k be a scalar (i.e real or complex number). Then the $m\times n$ matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by kA.

If $A=\left(a_{ij}\right)_{m\times n}$ then $kA=k\left(a_{ij}\right)_{m\times n}=\left(k\,a_{ij}\right)_{m\times n}$

For example, if $k=3$ and $A=\begin{pmatrix}1 & 1 & -2\\ 3 & 4 & -1\end{pmatrix}$ then

$$kA=3A=3\begin{pmatrix}1 & 1 & -2\\ 3 & 4 & -1\end{pmatrix}=\begin{pmatrix}3 & 3 & -6\\ 9 & 12 & -3\end{pmatrix}.$$

3.2.2 Properties of scalar multiple of a matrix:

Let A and B be matrices of same order and α and β be scalars. Then

$(i)\,\alpha(\beta A)=\alpha\beta(A)=\beta(\alpha A)$ $\quad (ii)\,(\alpha+\beta)A=\alpha A+\beta A$

$(iii)\alpha(A+B)=\alpha A+\alpha B$ $\quad (iv)\,\alpha O=O\alpha=O$

$(v)\,0A=A0=O$

3.2.3 Multiplication of matrices:

We say that the matrices A and B are conformable for multiplication in that order (giving the product AB) if the number of columns of A is equal to the number of rows of B.

3.2.4 Definition (Product of two matrices):

Let $A=\left(a_{ij}\right)_{m\times p}$, $B=\left(b_{ij}\right)_{p\times n}$ be two matrices. Then the matrix $C=\left(c_{ij}\right)_{m\times n}$ where $c_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$ is called the product of the matrices A and B denoted by AB.

For example, if $A=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}_{2\times 2}$ and $B=\begin{pmatrix}4 & 2 & 2\\ 7 & 3 & 4\end{pmatrix}_{2\times 3}$

Let the rows A be R_1, R_2 and the columns of B be C_1, C_2, C_3. When $A_{2\times 2}$ is multiplied with $B_{2\times 3}$ then the order of the product matrix $C=AB$ is 2×3

$$C=AB=\begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\end{pmatrix}=\begin{pmatrix}R_1C_1 & R_1C_2 & R_1C_3\\ R_2C_1 & R_2C_2 & R_2C_3\end{pmatrix}$$

$c_{11}=R_1C_1=$ the sum of the products of the first row elements of A with the corresponding elements of the first column of B $=1\times 4+1\times 7=11$

$c_{12}=R_1C_2=$ the sum of the products of the first row elements of A with the corresponding elements of the second column of B $=1\times 2+1\times 3=5$

Similarly we can get $c_{13}=6, c_{21}=40, c_{22}=18, c_{23}=22.$

$$\therefore C = AB = \begin{pmatrix} 11 & 5 & 6 \\ 40 & 18 & 22 \end{pmatrix}$$

3.2.5 Properties of multiplication of matrices:

If $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ be matrices conformable for multiplication. Then

(i) *Associative Law*: $A(BC) = (AB)C$

(ii) *Distributive Law*: $A(B+C) = AB + AC \,(Left\ Distibutive\ Law)$

$$(A+B)C = AC + BC \,(Right\ Distibutive\ Law)$$

(iii) *Existence of multiplicative identity*:

If I is the identity matrix of order n, then for any square matrix A of order n,

$$AI = IA = A$$

3.2.6 Note:

(i) Matrix multiplication need not be commutative. If A and B are two matrices conformable for multiplication, AB exists, but BA may not exist, even if BA exists, AB and BA may not equal.

If the orders of A and B are 2×3 and 3×4 respectively then the order of AB is 2×4 but BA does not exist.

If the orders of A and B are 2×3 and 3×2 respectively then the order of AB is 2×2 and the order of BA is 3×3. Hence AB and BA can not be equal.

(ii) If $O \neq A$ and $O \neq B$ are two matrices conformable for multiplication, AB exists and $AB = O$

For example, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}_{2\times2}$ and $B = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}_{2\times2}$ then $AB = BA = O$

(iii) If $AB = AC$ and $O \neq A$, then it is not necessary that $B = C$

For example, if $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}_{2\times2}$, $B = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}_{2\times2}$ and $C = \begin{pmatrix} 0 & 0 \\ 5 & 6 \end{pmatrix}_{2\times2}$ then

$AB = AC = O$ but $B \neq C$.

(iv) For any positive integer n, $(A)^n = A.A.A....A(n\,times)$

(v) If is α a scalar and A is any square matrix and is n a positive integer, then

$$(\alpha A)^n = (\alpha A)(\alpha A)(\alpha A)....(\alpha A)(n\,times) = \alpha^n A^n$$

3.3 Transpose of a matrix:

In this section we define the transpose of a matrix and study its properties. We also define symmetric and skew-symmetric matrices.

3.3.1 Definition (Transpose of a matrix):

If A is a matrix of order $m \times n$, then the matrix obtained by interchanging the rows into columns or columns into rows of A is called the *transpose of* A.
The transpose of the matrix A is denoted by $A^T (or) A'$

If $A = \left(a_{ij}\right)_{m\times n}$ then $A^T = \left(a_{ji}\right)_{n\times m}$

For example, if $A = \begin{pmatrix} 1 & 1 & -2 \\ 3 & 4 & -1 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 1 & -2 \\ 3 & 4 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ -2 & -1 \end{pmatrix}$

3.3.2 Properties of transpose of a matrix:

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

Let A and B be matrices of suitable order. Then

$(i)\,(A^T)^T = A$ $\quad$ $(ii)\,(kA)^T = kA^T$

$(iii)\,(A+B)^T = A^T + B^T$ $\quad$ $(iv)(AB)^T = B^T A^T$

3.3.3 Definition (Symmetric matrix):

A square matrix A is said to be *symmetric matrix* if $A^T = A.$

For example $\begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & -1 & 5 \\ -1 & 7 & -3 \\ 5 & -3 & 2 \end{pmatrix}$ are symmetric matrices.

3.3.4 Note:

(i) Let $A = \left(a_{ij}\right)_{n\times n}$ be a symmetric matrix if $a_{ij} = a_{ji} \forall 1 \le i, j \le n.$

(ii) We have $O_{n\times n}, I_{n\times n}$ are symmetric matrices.

(iii) If A is a square matrix then $A + A^T$ is a symmetric matrix.

3.3.5 Definition (Skew-symmetric matrix):

A square matrix A is said to be *skew-symmetric matrix* if $A^T = -A$.

For example $\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -5 \\ -1 & 0 & -3 \\ 5 & 3 & 0 \end{pmatrix}$ are skew-symmetric matrices.

3.3.6 Note:

(i) $A = (a_{ij})_{n\times n}$ is a skew-symmetric matrix if

$$a_{ij} = -a_{ji} \forall 1 < i, j < n \text{ and} a_{ii} = 0 \forall 1 \le i \le n.$$

(ii) We have $O_{n\times n}$ is a skew-symmetric matrix.

(iii) If A is a square matrix then $A - A^T$ is a skew-symmetric matrix.

3.3.7 Solved Problems:

1. Problem: If $A = \begin{pmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \end{pmatrix}$ then find $A + B$.

Solution: Given $A = \begin{pmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \end{pmatrix}$

$$A + B = \begin{pmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4 & 9+0 & 0+2 \\ 1+7 & 8+1 & -2+4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \end{pmatrix}$$

$$\therefore A + B = \begin{pmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \end{pmatrix}.$$

2. Problem: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ then find $3B - 2A$.

Solution: Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$$3B - 2A = 3\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow 3B-2A=\begin{pmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{pmatrix}-\begin{pmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{pmatrix}$$

$$\Rightarrow 3B-2A=\begin{pmatrix} 9-2 & 6-4 & 3-6 \\ 3-6 & 6-4 & 9-2 \end{pmatrix}$$

$$\Rightarrow 3B-2A=\begin{pmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{pmatrix}$$

$$\therefore 3B-2A=\begin{pmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{pmatrix}.$$

3. Problem: If $A=\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}$ then find $A-B$ and $4B-3A$.

Solution: Given $A=\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}$

$$(i) A-B=\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}-\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow A-B=\begin{pmatrix} 0-1 & 1+2 & 2-0 \\ 2-0 & 3-1 & 4+1 \\ 4+1 & 5-0 & 6-3 \end{pmatrix}$$

$$\Rightarrow A-B=\begin{pmatrix} -1 & 3 & 2 \\ 2 & 2 & 5 \\ 5 & 5 & 3 \end{pmatrix}$$

$$\therefore A-B=\begin{pmatrix} -1 & 3 & 2 \\ 2 & 2 & 5 \\ 5 & 5 & 3 \end{pmatrix}$$

$$(ii) 4B-3A=4\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}-3\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\Rightarrow 4B-3A=\begin{pmatrix}4 & -8 & 0\\ 0 & 4 & -4\\ -4 & 0 & 12\end{pmatrix}-\begin{pmatrix}0 & 3 & 6\\ 6 & 9 & 12\\ 12 & 15 & 18\end{pmatrix}\Rightarrow 4B-3A=\begin{pmatrix}4-0 & -8-3 & 0-6\\ 0-6 & 4-9 & -4-12\\ -4-12 & 0-15 & 12-18\end{pmatrix}$$

$$\therefore 4B-3A=\begin{pmatrix}4 & -11 & -6\\ -6 & -5 & -16\\ -16 & -15 & -6\end{pmatrix}$$

4. Problem: If $A=\begin{pmatrix}2 & 3 & 1\\ 6 & -1 & 5\end{pmatrix}$ $B=\begin{pmatrix}1 & 2 & -1\\ 0 & -1 & 3\end{pmatrix}$ and $A+B-X=0$ then find X

Solution: Given $A=\begin{pmatrix}2 & 3 & 1\\ 6 & -1 & 5\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2 & -1\\ 0 & -1 & 3\end{pmatrix}$

$$A+B-X=0\Rightarrow X=A+B$$

$$\Rightarrow X=\begin{pmatrix}2 & 3 & 1\\ 6 & -1 & 5\end{pmatrix}+\begin{pmatrix}1 & 2 & -1\\ 0 & -1 & 3\end{pmatrix}$$

$$\Rightarrow X=\begin{pmatrix}2+1 & 3+2 & 1-1\\ 6+0 & -1-1 & 5+3\end{pmatrix}$$

$$\Rightarrow X=\begin{pmatrix}3 & 5 & 0\\ 6 & -2 & 8\end{pmatrix}$$

$$\therefore X=\begin{pmatrix}3 & 5 & 0\\ 6 & -2 & 8\end{pmatrix}.$$

5. Problem: Find the trace of the matrix $\begin{pmatrix}1 & 3 & -5\\ 2 & -1 & 5\\ 1 & 0 & 1\end{pmatrix}$

Solution: Let $A=\begin{pmatrix}1 & 3 & -5\\ 2 & -1 & 5\\ 1 & 0 & 1\end{pmatrix}$

The elements of the principle diagonal elements of A are $1,-1,1$

Hence the trace of $A=Tr(A)=1-1+1=1$

6. Problem: If $A=\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}0 & 4\\ -1 & 2\end{pmatrix}$ then find AB and BA.

Solution: Given $A=\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}0 & 4\\ -1 & 2\end{pmatrix}$

$$(i) AB=\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 4\\ -1 & 2\end{pmatrix}$$

$$\Rightarrow AB=\begin{pmatrix}2\times0+3\times-1 & 2\times4+3\times2\\ 1\times0+2\times-1 & 1\times4+2\times2\end{pmatrix}$$

$$\Rightarrow AB=\begin{pmatrix}2-3 & 8+6\\ 0-2 & 4+4\end{pmatrix}$$

$$\therefore AB=\begin{pmatrix}-1 & 14\\ -2 & 8\end{pmatrix}$$

$$(ii) BA=\begin{pmatrix}0 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$$

$$\Rightarrow BA=\begin{pmatrix}0\times2+4\times1 & 0\times3+4\times2\\ -1\times2+2\times1 & -1\times3+2\times2\end{pmatrix}\Rightarrow BA=\begin{pmatrix}2+4 & 0+8\\ -2+2 & -3+4\end{pmatrix}$$

$$\therefore BA=\begin{pmatrix}6 & 8\\ 0 & 1\end{pmatrix}$$

7. Problem: If $A=\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}$ then find A^2

Solution: Given $A=\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}$

$$A^2=AA=\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}$$

$$\Rightarrow A^2=\begin{pmatrix}4\times4+2\times-1 & 4\times2+2\times1\\ -1\times4+1\times-1 & -1\times2+1\times1\end{pmatrix}$$

$$\Rightarrow A^2=\begin{pmatrix}16-2 & 8+2\\ -4-1 & -2+1\end{pmatrix}$$

$$\therefore A^2=\begin{pmatrix}14 & 10\\ -5 & -1\end{pmatrix}$$

8. Problem: If $A=\begin{pmatrix}2 & 4\\ -1 & K\end{pmatrix}$ and $A^2=O$ then find K

Solution: Given $A = \begin{pmatrix} 2 & 4 \\ -1 & K \end{pmatrix}$

$$A^2 = O \Rightarrow AA = O \Rightarrow \begin{pmatrix} 2 & 4 \\ -1 & K \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & K \end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix} 2\times2+4\times-1 & 2\times4+4\times K \\ -1\times2+K\times-1 & -1\times4+K\times K \end{pmatrix} = O \Rightarrow \begin{pmatrix} 4-4 & 8+4K \\ -2-K & -4+K^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 8+4K \\ -2-K & -4+K^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow 8+4K = 0, -2-K = 0, -4+K^2 = 0$$

$\Rightarrow K = -2, K = -2, K = \pm 2$

$\therefore K = \pm 2$

9. Problem: If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ then show that $A^2 - 4A - 5I = O$

Solution: Given $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$$A^2 = AA = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow A^2 = \begin{pmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{pmatrix}$$

$$\Rightarrow A^2 = \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$\therefore A^2 = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$A^2-4A-5I=\begin{pmatrix}9&8&8\\8&9&8\\8&8&9\end{pmatrix}-4\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\Rightarrow A^2-4A-5I=\begin{pmatrix}9&8&8\\8&9&8\\8&8&9\end{pmatrix}-\begin{pmatrix}4&8&8\\8&4&8\\8&8&4\end{pmatrix}-\begin{pmatrix}5&0&0\\0&5&0\\0&0&5\end{pmatrix}$$

$$\Rightarrow A^2-4A-5I=\begin{pmatrix}9-4-5&8-8-0&8-8-0\\8-8-0&9-4-5&8-8-0\\8-8-0&8-8-0&9-4-5\end{pmatrix}\Rightarrow A^2-4A-5I=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O$$

$\therefore A^2-4A-5I=O$

10. Problem: If $A=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ then show that $A^2=-I$

Solution: Given $A=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$

$$A^2=AA\Rightarrow A^2=\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}i&0\\0&-i\end{pmatrix}$$

$$\Rightarrow A^2=\begin{pmatrix}i\times i+0\times 0&i\times 0+0\times -i\\0\times i+-i\times 0&0\times 0+-i\times -i\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}i^2&0\\0&i^2\end{pmatrix}$$

$$\Rightarrow A^2=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\qquad\left[\because i^2=-1\right]$$

$$\Rightarrow A^2=-\begin{pmatrix}1&0\\0&1\end{pmatrix}=-I\qquad\left[\because\begin{pmatrix}1&0\\0&1\end{pmatrix}=I\right]$$

$\therefore A^2=-I$

11. Problem: If $A=\begin{pmatrix}2&-4\\-5&3\end{pmatrix}$ then find $A+A^T$

Solution: Given $A=\begin{pmatrix}2&-4\\-5&3\end{pmatrix}$

$$A^T=\begin{pmatrix}2&-4\\-5&3\end{pmatrix}^T=\begin{pmatrix}2&-5\\-4&3\end{pmatrix}$$

$$A + A^T = \begin{pmatrix} 2 & -4 \\ -5 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -5 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 2+2 & -4-5 \\ -5-4 & 3+3 \end{pmatrix} = \begin{pmatrix} 4 & -9 \\ -9 & 6 \end{pmatrix}$$

$$\therefore A + A^T = \begin{pmatrix} 4 & -9 \\ -9 & 6 \end{pmatrix}$$

12. Problem: If $A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{pmatrix}$ is a symmetric matrix, find the value of x

Solution: Given $A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{pmatrix}$

$$A^T = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{pmatrix}$$

Since A is a symmetric matrix we have by definition $A = A^T$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{pmatrix}$$

$\therefore x = 6$

Exercise 3(a)

1. If $A = \begin{pmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -4 & -1 \end{pmatrix}$ then find $A + B$.
2. If $A = \begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 3 & -5 \end{pmatrix}$ and $A + B - X = 0$ then find X
3. If $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{pmatrix}$ and $X = A + B$ then find X
4. If $\begin{pmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+w \end{pmatrix} = \begin{pmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ then find the values of x, y, z, w.
5. If $\begin{pmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & w-5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{pmatrix}$ then find the values of x, y, z, w.

6. Find the trace of the matrix $\begin{pmatrix} 1 & 2 & -\frac{1}{2} \\ 0 & -1 & 2 \\ -\frac{1}{2} & 2 & 1 \end{pmatrix}$

7. If $A = \begin{pmatrix} 2 & -4 \\ -5 & 3 \end{pmatrix}$ then find AA^T

8. If $A = \begin{pmatrix} 0 & 4 \\ -1 & 2 \end{pmatrix}$ then find $A^T A$

9. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 4 \\ -1 & 2 \end{pmatrix}$ then find BA^T

10. If $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ verify that $(A+B)^T = A^T + B^T$

11. If $A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & 0 \\ 3 & -1 & -5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{pmatrix}$ then find $3A - 4B^T$

12. If $A = \begin{pmatrix} -2 & 5 \\ 5 & 0 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{pmatrix}$ then find $2A + B^T$

3.4 Determinants of a Matrix:

We have learnt in lower classes that the value $a_1b_2 - a_2b_1$ is called the *determinant* of the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

The determinant of 1×1 matrix is defined as its element.

In this section, we define the determinant of a 3×3 matrix, study its propertiesand the methods of evaluation of certain determinants.

3.4.1 Definition (Minor of an element):

Consider a square matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. The *minor* of an element in this matrix is defined as the determinant of the 2×2 matrix, obtained after deleting the row and column in which the element is present.

For example the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$.

3.4.2 Definition (Cofactor of an element):

The cofactor of an element in the i^{th} row and the j^{th} column of a 3×3 matrix is defined as its minor multiplied by $(-1)^{i+j}$

We denote the cofactor of a_{ij} by A_{ij}

For example consider the matrix in 3.4.1

Since b_2 is in 2^{nd} row and 2^{nd} column, we have the cofactor of b_2 is

$$B_2 = (-1)^{2+2}\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = (-1)^4(a_1c_3 - a_3c_1) = a_1c_3 - a_3c_1.$$

3.4.3 Definition (Determinant):

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. The sum of the products of elements of the first row with their corresponding cofactors is called the *determinant* of A. The determinant of the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. We also denote the determinant of the matrix A by $\det A$ or $|A|$.

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$

So far we have defined the concept of the determinant for square matrix of order n for $n = 1, 2, 3$. The concept can be extended to the case $n \geq 4$ by using the principle of Mathematical Induction. Let $A = (a_{ij})_{m\times n}$. Then the determinant of A is defined as $\sum_{j=1}^{n} a_{ij}A_{ij}$, where A_{ij} is the cofactor of a_{ij}

3.4.4 Example:

Let us find the determinant of $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{pmatrix}$

$\det A =$ The sum of the products of elements of the first row with their corresponding cofactors

$= 1(\text{cofactor of } 1) + 0(\text{cofactor of } 0) + (-2)(\text{cofactor of } (-2))$

$= 1(-16) + (-2)(19) = -16 - 38 = -54.$

3.4.5 Note:

The definition of the determinant is formulated by using the elements of the first row and the corresponding cofactors only. However the process can be adopted for the elements of any row or column and the corresponding cofactors. We thus have $\det A = \sum_{j=1}^{n} a_{ij} A_{ij}$ for $1 \le i \le n$.

Here the sum on the right hand side is independent of i.

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $\det A = a_1A_1 + b_1B_1 + c_1C_1$ expansion along first row.

Similarly $\det A = a_2A_2 + b_2B_2 + c_2C_2$ expansion along second row

$= a_3A_3 + b_3B_3 + c_3C_3$ expansion along third row

$= a_1A_1 + a_2A_2 + a_3A_3$ expansion along first column

$= b_1B_1 + b_2B_2 + b_3B_3$ expansion along second column

$= c_1C_1 + c_2C_2 + c_3C_3$ expansion along third column

For instance, consider

$$a_1A_1 + a_2A_2 + a_3A_3 = a_1(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2(-1)^{2+1}\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3(-1)^{3+1}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1A_1 + b_1B_1 + c_1C_1 = \det A$$

3.4.6 Properties of determinants:

(i) If each element of a row (or column) of a square matrix is zero, then the determinant of that matrix is zero.

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$

First row consists of all zero elements. Then

$\det A = 0.A_1 + 0.A_2 + 0.A_3 \ = 0$

(ii) If two rows (or columns) of a square matrix are interchanged, then the sign of the determinant changes.

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix}$

B is obtained by interchanging of first and second rows of A.

$$\det B = a_1(-1)^{2+1}(b_2c_3 - b_3c_2) + b_1(-1)^{2+2}(a_2c_3 - a_3c_2) + c_1(-1)^{2+3}(a_2b_3 - a_3b_2)$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -\left[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\right]$$

$$= -\det A$$

(iii) If each element of a row (or column) of a square matrix is multiplied by a number k, then the determinant of the matrix obtained is k times the determinant of the given matrix.

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ and $B = \begin{pmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{pmatrix}$

B is obtained by multiplying the elements of first row of A by k. The cofactors of a_1, a_2, a_3 in A are A_1, A_2, A_3 then the cofactors of ka_1, ka_2, ka_3 in B are also A_1, A_2, A_3 respectively. Hence

$$\det B = ka_1A_1 + ka_2A_2 + ka_3A_3$$

$$= k(a_1A_1 + a_2A_2 + a_3A_3)$$

$$= k \det A$$

(iv) If A is a square matrix of order 3 and k is a scalar, then $|kA| = k^3|A|$.

(v) If two rows (or columns) of a square matrix are identical, then the value of the determinant is zero.

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$

First and second rows are identical. Then

$$\det A = a_2.0 + b_2.0 + c_2.0 \;\; = 0$$

(vi) If the matrix is a diagonal matrix then the determinant of the matrix is product of the diagonal elements.

Let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$\det A = abc$

(vii) If the matrix is a triangular (upper or lower) matrix then the determinant of the matrix is product of the diagonal elements.

Let $A = \begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{pmatrix} \Rightarrow \det A = abc$

Let $B = \begin{pmatrix} a & d & e \\ d & b & f \\ 0 & 0 & c \end{pmatrix} \Rightarrow \det B = abc$

(viii) If the corresponding elements of two rows (or columns) of a square matrix are in the same ratio, then the value of the determinant is zero.

Let $A = \begin{pmatrix} ka_1 & kb_1 & kc_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$. Then

$$\det A = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = k(0) = 0 \ = 0$$

(ix) If each element in a row (or column) of a square matrix is the sum of two numbers then its determinant can be expressed as the sum of the determinants of two square matrices as shown below.

Let $A = \begin{pmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{pmatrix}, B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, C = \begin{pmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{pmatrix}$

If in A the cofactors of $a_1 + x_1, a_2 + x_2, a_3 + x_3$ are A_1, A_2, A_3 then the cofactors of a_1, a_2, a_3 in B and of x_1, x_2, x_3 in C are also A_1, A_2, A_3 respectively.

Now, $\det A = (a_1 + x_1)A_1 + (a_2 + x_2)A_2 + (a_3 + x_3)A_3$

$$= (a_1A_1 + a_2A_2 + a_3A_3) + (x_1A_1 + x_2A_2 + x_3A_3)$$

$$= \det B + \det C$$

$$\therefore \begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

(x) If each element in a row (or column) of a square matrix is multiplied by a number k and added to the corresponding element of another row (or column) of the matrix then the determinant of the resulting matrix is equal to the determinant of the given matrix.

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 + ka_1 & b_2 + ka_1 & c_2 + ka_1 \\ a_3 & b_3 & c_3 \end{pmatrix}$

B is obtained from A by multiplying each element of the first row of A by k then adding them to the corresponding elements of the second row of A

Now, $\det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det A + 0 = \det A$$

(xi) For any square matrix A, $\det A = \det A^T$.

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, A^T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

The values of the cofactors of a_1, b_1, c_1 are same in A, A^T.

Hence $\det A = a_1A_1 + b_1B_1 + c_1C_1 = \det A^T$

(xi) For any square matrix A, B of same order $\det(AB) = \det A \det B$.

(xii) For any positive integer n, $\det(A^n) = (\det A)^n$.

3.4.7 Notation:

While evaluating the determinant we use the following notations.

(i) $R_1 \leftrightarrow R_2$, to mean that the rows R_1 and R_2 are interchanged.

(ii) $R_1 \to kR_1$, to mean that the elements of row R_1 are multiplied by k.

(iii) $R_1 \to R_1 + kR_2$, to mean that the elements of row R_1 are added with k the corresponding elements of row R_2.

Similar notation is used for other rows and columns.

3.4.8 Solved Problems:

1. Problem: Find the determinant of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} = 2(-5) - 1.1 = -10 - 1 = -11.$

2. Problem: Find the determinant of the matrix $\begin{pmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{vmatrix}$

$$= 2\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 5 \\ -3 & 3 \end{vmatrix} + 4\begin{vmatrix} 0 & -2 \\ -3 & 1 \end{vmatrix}$$

$$= 2[(-2)3 - 1.5] + 1[0.3 - (-3).5] + 4[0.1 - (-2).(-3)]$$

$$= 2[-6 - 5] + 1[0 + 15] + 4[0 - 6]$$

$= 2[-11]+1[15]+4[-6] = -22+15-24 = 15-46 = -31.$

3. Problem: Find the determinant of the matrix $\begin{pmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{pmatrix}$$

We have $\det(A) = |A| = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

$$= 1\begin{vmatrix} 9 & 16 \\ 16 & 25 \end{vmatrix} - 4\begin{vmatrix} 4 & 16 \\ 9 & 25 \end{vmatrix} + 9\begin{vmatrix} 4 & 9 \\ 9 & 16 \end{vmatrix}$$

$$= 1[9.25-16.16]-4[4.25-9.16]+9[4.16-9.9]$$

$$= 1[225-256]-4[100-144]+9[64-81]$$

$$= 1[-31]-4[-44]+9[-17] = -31+176-153 = 176-184 = -8.$$

4. Problem: Find the determinant of the matrix $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$$= a\left[c.b - a.a\right] - b\left[b.b - a.c\right] + c\left[b.a - c.c\right]$$

$$= a\left[bc - a^2\right] - b\left[b^2 - ac\right] + c\left[ba - c^2\right]$$

$$= abc - a^3 - b^3 + abc + abc - c^3 \quad = 3abc - a^3 - b^3 - c^3$$

5. Problem: Find the determinant of the matrix $\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$

where $1, \omega, \omega^2$ are cube roots of unity.

Solution: Let $A = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$$= 1\begin{vmatrix} \omega^2 & 1 \\ 1 & \omega \end{vmatrix} - \omega\begin{vmatrix} \omega & 1 \\ \omega^2 & \omega \end{vmatrix} + \omega^2\begin{vmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{vmatrix}$$

$$= 1\left[\omega^2.\omega - 1.1\right] - \omega\left[\omega.\omega - 1.\omega^2\right] + \omega^2\left[\omega.1 - \omega^2.\omega^2\right]$$

$$= \left[\omega^3 - 1\right] - \omega\left[\omega^2 - \omega^2\right] + \omega^2\left[\omega - \omega^4\right]$$

$$= [1 - 1] - \omega[0] + \omega^2[\omega - \omega]$$

$$\left(\because \omega^3 = 1, \omega^4 = \omega^3.\omega = 1.\omega = \omega\right)$$

$$= 0 - 0 + 0 = 0$$

6. Problem: Find the value of x if $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{vmatrix} = 45$

Solution: We have $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{vmatrix} = 45$

$$\Rightarrow 1\begin{vmatrix} 3 & 4 \\ -6 & x \end{vmatrix} - 0.\begin{vmatrix} 2 & 4 \\ 5 & x \end{vmatrix} + 0.\begin{vmatrix} 3 & 4 \\ -6 & x \end{vmatrix} = 45$$

$$\Rightarrow 1[3x+24] - 0[2x-20] + 0.[3x+24] = 45$$

$$\Rightarrow 3x + 24 - 0 + 0 = 45$$

$$\Rightarrow 3x = 21 \quad \Rightarrow x = 7$$

7. Problem: Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution: L.H.S $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

On applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ we get

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2 - a^2 \\ 0 & c-a & c^2 - a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 1 & (c+a) \end{vmatrix}$$

On applying $R_3 \to R_3 - R_2$ we get

$$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 0 & c-b\end{vmatrix}$$

On expanding along the first column we get

$$=(b-a)(c-a)\begin{vmatrix}1 & b+a\\0 & c-b\end{vmatrix}$$

$$=(b-a)(c-a)(c-b-0)$$

$$=(a-b)(b-c)(c-a)=\text{R.H.S}$$

$$\therefore\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$$

8. Problem: Show that $\begin{vmatrix}1 & a^2 & a^3\\1 & b^2 & b^3\\1 & c^2 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$

Solution: $\text{L.H.S}=\begin{vmatrix}1 & a^2 & a^3\\1 & b^2 & b^3\\1 & c^2 & c^3\end{vmatrix}$

On applying $R_2\to R_2-R_1, R_3\to R_3-R_1$ we get

$$\text{L.H.S}=\begin{vmatrix}1 & a^2 & a^3\\0 & b^2-a^2 & b^3-a^3\\0 & c^2-a^2 & c^3-a^3\end{vmatrix}$$

$$=\begin{vmatrix}1 & a^2 & a^3\\0 & (b-a)(b+a) & (b-a)(b^2+a^2+ba)\\0 & (c-a)(c+a) & (c-a)(c^2+a^2+ca)\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}1 & a^2 & a^3\\0 & (b+a) & (b^2+a^2+ba)\\0 & (c+a) & (c^2+a^2+ca)\end{vmatrix}$$

On applying $R_3 \to R_3 - R_2$ we get

$$=(b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b+a) & (b^2+a^2+ba) \\ 0 & (c-b) & (c^2-b^2+ca-ba) \end{vmatrix}$$

$$=(b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & (b+a) & (b^2+a^2+ba) \\ 0 & 1 & (c+b+a) \end{vmatrix}$$

On expanding along the first column we get

$$=(b-a)(c-a)(c-b)\begin{vmatrix} (b+a) & (b^2+a^2+ba) \\ 1 & (c+b+a) \end{vmatrix}$$

$$=(b-a)(c-a)(c-b)\left[(a+b+c)(a+b)-(b^2+a^2+ba)\right]$$

$$=(a-b)(b-c)(c-a)(ab+bc+ca)=\text{R.H.S}$$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$$

9. Problem: Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution: $\text{L.H.S}=\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

On applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\text{L.H.S}=\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$=2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

On applying $R_1 \to R_1 - R_2$ we get

$$= 2\begin{vmatrix} b & c & a \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

On applying $R_3 \to R_3 - R_1$ we get

$$= 2\begin{vmatrix} b & c & a \\ c+a & a+b & b+c \\ a & b & c \end{vmatrix}$$

On applying $R_2 \to R_2 - R_3$ we get

$$= 2\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S}$$

$$\therefore \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

10. Problem: Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution: $\text{L.H.S} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

On applying $R_1 \to R_1 + R_2 + R_3$ we get

$$\text{L.H.S} = \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= 0 = \text{R.H.S}$$

$$\therefore \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

11. Problem: Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\left(a+b+c\right)^3$

Solution: $\text{L.H.S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

On applying $C_1 \to C_1 + C_2 + C_3$ we get

$$\text{L.H.S} = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

On expanding along the first column we get

$$= 2(a+b+c)\begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$$= 2(a+b+c)\left[(a+b+c)^2 - 0\right]$$

$$= 2(a+b+c)^3 = \text{R.H.S}$$

$$\therefore \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\left(a+b+c\right)^3$$

12. Problem: Show that $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

Solution: $\text{L.H.S} = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$

On applying $R_3 \to \frac{1}{xyz} R_3$ we get

$$\text{L.H.S} = \frac{1}{xyz} \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

$$= \frac{1}{xyz}(xyz) \begin{vmatrix} a & b & c \\ x & y & z \\ yz & xz & xy \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \text{R.H.S}$$

$$\therefore \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

Exercise 3(b)

I Find the determinants of the following matrices:

1. $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
2. $\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ -3 & 7 & 6 \end{pmatrix}$
5. $\begin{pmatrix} 2 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 2 & 1 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{pmatrix}$
7. $\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -4 & -2 & 5 \end{pmatrix}$
8. $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

II Find the determinants of the following matrices:

9. Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

10. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

11. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

12. Show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

13. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a & b & c \end{vmatrix} = a^3+b^3+c^3-3abc$

14. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3.5 Adjoint and Inverse of a Matrix:

In this Section, we discuss adjoint and inverse of a matrix.

3.5.1 Definition (Singular and Non-singular matrices):

A square matrix is said to be *singular* if its determinant is zero. Otherwise it is said to be *non-singular*.

For example $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a singular matrix while $\begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$ is non-singular.

3.5.2 Definition (Adjoint of a matrix):

The transpose of the matrix formed by replacing the elements of a square matrix A with the corresponding cofactors is called the Adjoint of A and it is denoted by $Adj\ A$.

Let $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ and A_i, B_i, C_i be the cofactors of a_i, b_i, c_i respectively.

Then $Adj\,A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}^T = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

3.5.3 Definition (Invertible matrix):

Let A be a non-singular matrix. We say that A is invertible if a matrix B exists such that $AB = BA = I$ where I is the unit matrix of the same order as A and B

3.5.4 Note:

(i) If B exists such that $AB = BA = I$ then such a unique B is denoted by A^{-1} and is called the multiplicative inverse of A

(ii) If A is invertible then A is non-singular matrix, hence $\det A \neq 0$

3.5.5 Theorem:

If $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is a non-singular matrix then A is invertible and

$$A^{-1} = \frac{\text{Adj}\,A}{\det A}$$

Proof: Given $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

By definition $Adj\,A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}^T = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

Now $A(Adj\,A) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

$$= \begin{pmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{pmatrix}$$

$$= \begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix} = \det A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\det A)I$$

Now, since $\det A \neq 0$, we have $A\left(\frac{\text{Adj}\,A}{\det A}\right) = I$

Similarly we can get $\left(\frac{\text{Adj}\,A}{\det A}\right)A = I$

Hence A is invertible and $A^{-1} = \frac{\text{Adj}\,A}{\det A}$

3.5.6 Corollary:

Let A and B are invertible matrices. Then A^{-1}, A and AB are invertible matrices. $(i)\,(A^{-1})^{-1} = A$ $(ii)\,(A^T)^{-1} = (A^{-1})^T$ $(iii)\,(AB)^{-1} = B^{-1}A^{-1}$

3.5.7 Solved Problems:

1. Problem: Find the adjoint and inverse of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} = 2(-5) - 1.1 = -10 - 1 = -11 \neq 0$

Hence A is invertible.

Adjoint matrix of A is $\text{Adj}\,A = \begin{pmatrix} -5 & -1 \\ -1 & 2 \end{pmatrix}$

$$A^{-1} = \frac{\text{Adj}\,A}{\det A} = -\frac{1}{11}\begin{pmatrix} -5 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{-2}{11} \end{pmatrix}$$

2. Problem: Find the adjoint and inverse of the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = (\cos\alpha).(\cos\alpha) - (\sin\alpha).(-\sin\alpha)$

$$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0$$

Hence A is invertible.

Adjoint matrix of A is $\text{Adj}\, A = A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$$\text{A}^{-1} = \frac{\text{Adj}\, A}{\det A} = \frac{1}{1}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

3. Problem: Find the adjoint and inverse of the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

We have $\det(A) = |A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$

$$= 1(16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1 \neq 0$$

Hence A is invertible.

Adjoint matrix of A is $\text{Adj}\, A = \begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$$\text{A}^{-1} = \frac{\text{Adj}\, A}{\det A} = \frac{1}{1}\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

4. Problem: If $A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ then show that $\text{Adj}\, A = 3A^T$

Solution: Given $A=\begin{pmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$

Adjoint matrix of A is $\operatorname{Adj} A=\begin{pmatrix}-3 & -6 & -6\\ 6 & 3 & -6\\ 6 & -6 & 3\end{pmatrix}^T=\begin{pmatrix}-3 & 6 & 6\\ -6 & 3 & -6\\ -6 & -6 & 3\end{pmatrix}$

$$3A^T=3\begin{pmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}^T=3\begin{pmatrix}-1 & 2 & 2\\ -2 & 1 & -2\\ -2 & -2 & 1\end{pmatrix}=\begin{pmatrix}-3 & 6 & 6\\ -6 & 3 & -6\\ -6 & -6 & 3\end{pmatrix}$$

$\therefore \operatorname{Adj} A=3A^T$

5. Problem: If $A=\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}$ then find A^3 and A^{-1}

Solution: Given $A=\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}$

We have $\det(A)=|A|=\begin{vmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{vmatrix}=3\begin{vmatrix}-3 & 4\\ -1 & 1\end{vmatrix}+3\begin{vmatrix}2 & 4\\ 0 & 1\end{vmatrix}+4\begin{vmatrix}2 & -3\\ 0 & -1\end{vmatrix}$

$$=3(-3+4)+3(2-0)+4(-2+0)=3+6-8=1\neq 0$$

Hence A is invertible.

Adjoint matrix of A is $\operatorname{Adj} A=\begin{pmatrix}1 & -2 & -2\\ -1 & 3 & 3\\ 0 & -4 & -3\end{pmatrix}^T=\begin{pmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{pmatrix}$

$$A^{-1}=\frac{\operatorname{Adj} A}{\det A}=\frac{1}{1}\begin{pmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{pmatrix}=\begin{pmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{pmatrix}$$

$$A^2=AA=\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}3 & -4 & 4\\ 0 & -1 & 0\\ -2 & 2 & -3\end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow A^3 = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

Exercise 3(c)

I Find the inverse of the following matrices:

1. $\begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$ **2.** $\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$ **3.** $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ **4.** $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

I I 5.

If $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$ then find $(A^T)^{-1}$

3.6 Solution of Simultaneous Linear Equations:

In this Section, we discuss some methods of solving system of simultaneous linear equations.

3.6.1 Crammer's Rule:

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is non-singular.

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the solution of the equation $AX = B$ Where $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then $x\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

On applying $C_1 \to C_1 + yC_2 + zC_3$ we get

$$x\Delta = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1$$

$$\therefore x = \frac{\Delta_1}{\Delta} \text{ where } \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Similarly we get $y = \frac{\Delta_2}{\Delta}$ where $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$$z = \frac{\Delta_3}{\Delta} \text{ where } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

This is known as Crammer's Rule.

3.6.2 Matrix Inversion Method:

Consider the system of equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is non-singular. Then we can find A^{-1}

$$AX = B \Leftrightarrow A^{-1}(AX) = A^{-1}B \Leftrightarrow (A^{-1}A)X = A^{-1}B \Leftrightarrow I\,X = A^{-1}B \left[\because I = A^{-1}A\right]$$

$X = A^{-1}B$ From this x, y and z are known.

3.6.3 Solved Problems:

1. **Problem:** Solve the following system of equations using crammer's rule $x-y+3z=5, 4x+2y-z=0, -x+3y+z=5$.

Solution: Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$

Then we can rewritten the given equations in the form of matrix equation as $AX = B$

We have $\Delta = \det(A) = |A| = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}$

$= 1(2+3) + 1(4-1) + 3(12+2) = 5+3+42 = 50 \neq 0$ We have

$\Delta_1 = \det(A_1) = |A_1| = \begin{vmatrix} 5 & -1 & 3 \\ 0 & 2 & -1 \\ 5 & 3 & 1 \end{vmatrix} = 5\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 5 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 5 & 3 \end{vmatrix}$

$= 5(2+3) + 1(0+5) + 3(0-10) = 25+5-30 = 0$

We have $\Delta_2 = \det(A_2) = |A_2| = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 0 & -1 \\ -1 & 5 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & -1 \\ 5 & 1 \end{vmatrix} - 5\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ -1 & 5 \end{vmatrix}$

$= 1(0+5) - 5(4-1) + 3(20-0) = 5-15+60 = 50$

We have $\Delta_3 = \det(A_3) = |A_3| = \begin{vmatrix} 1 & -1 & 5 \\ 4 & 2 & 0 \\ -1 & 3 & 5 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 4 & 0 \\ -1 & 5 \end{vmatrix} + 5\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}$

$= 1(10-0) + 1(20+0) + 5(12+2) = 10+20+70 = 100$

Hence by Crammer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{50} = 0,\ y = \frac{\Delta_2}{\Delta} = \frac{50}{50} = 1,\ z = \frac{\Delta_3}{\Delta} = \frac{100}{50} = 2$$

$\therefore$ The solution of the given system of equations is $x = 0, y = 1, z = 2.$

2. Problem: Solve the following system of equations using crammer's rule $2x - y + 3z = 9, x + y + z = 6, x - y + z = 2.$

Solution: Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$

Then we can rewritten the given equations in the form of matrix equation as $AX = B$

We have $\Delta = \det(A) = |A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$$= 2(1+1)+1(1-1)+3(-1-1) = 4+0-6 = -2 \neq 0$$ We have

$$\Delta_1 = \det(A_1) = |A_1| = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= 9(1+1)+1(6-2)+3(-6-2) = 18+4-24 = -2$$

We have $\Delta_2 = \det(A_2) = |A_2| = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} - 9\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix}$

$$= 2(6-2)-9(1-1)+3(2-6) = 8-0-12 = -4$$

We have $\Delta_3 = \det(A_3) = |A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix} + 9\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$$= 2(2+6)+1(2-6)+9(-1-1) = 16-4-18 = -6$$

Hence by Crammer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1,\ y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2,\ z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

$\therefore$ The solution of the given system of equations is $x = 1, y = 2, z = 3.$

3. Problem: Solve the following system of equations using crammer's rule
$2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0.$

Solution: Let $A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$

Then we can rewritten the given equations in the form of matrix equation as $AX = B$

We have $\Delta = \det(A) = |A| = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$

$$= 2(-8-1)+1(4-3)+3(-1-6) = -18+1-21 = -38 \neq 0$$

We have $\Delta_1 = \det(A_1) = |A_1| = \begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix} = 8\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix}$

$$= 8(-8-1) + 1(-16-0) + 3(4-0) = -72 - 16 + 12 = -76$$

We have $\Delta_2 = \det(A_2) = |A_2| = \begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} - 8\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3\begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix}$

$$= 2(-16-0) - 8(4-3) + 3(0-12) = -32 - 8 - 36 = -76$$

We have $\Delta_3 = \det(A_3) = |A_3| = \begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix} + 8\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$

$$= 2(0-4) + 1(0-12) + 8(-1-6) = -8 - 12 - 56 = -76$$

Hence by Crammer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-76}{-38} = 2,\ y = \frac{\Delta_2}{\Delta} = \frac{-76}{-38} = 2,\ z = \frac{\Delta_3}{\Delta} = \frac{-76}{-38} = 2$$

$\therefore$ The solution of the given system of equations is $x = 2, y = 2, z = 2$.

4. Problem: Solve the following system of equations using matrix inversion method $x - y + 3z = 5, 4x + 2y - z = 0, -x + 3y + z = 5$.

Solution: Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$

Then we can rewritten the given equations in the form of matrix equation as $AX = B$

We have $\det(A) = |A| = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}$

$$= 1(2+3) + 1(4-1) + 3(12+2) = 5 + 3 + 42 = 50 \neq 0$$

Hence A is invertible.

Adjoint matrix of A is $\operatorname{Adj} A = \begin{pmatrix} 5 & -3 & 14 \\ 10 & 4 & -2 \\ -5 & 13 & 6 \end{pmatrix}^T = \begin{pmatrix} 5 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{pmatrix}$

$$A^{-1} = \frac{\operatorname{Adj} A}{\det A} = \frac{1}{50}\begin{pmatrix} 5 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{pmatrix}$$

Hence by matrix inversion method

$$X = A^{-1}B$$

$$X = \frac{1}{50}\begin{pmatrix} 4 & 10 & -5 \\ -3 & 4 & 13 \\ 14 & -2 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$$

$$= \frac{1}{50}\begin{pmatrix} 25+0-25 \\ -15+0+65 \\ 70+0+30 \end{pmatrix}$$

$$= \frac{1}{50}\begin{pmatrix} 0 \\ 50 \\ 100 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ The solution of the given system of equations is $x = 0, y = 1, z = 2.$

Exercise 3(d)

1. Solve the following system of equations using crammer's rule
   $x - y + 3z = 5, 4x + 2y - z = 0, -x + 3y + z = 5.$
2. Solve the following system of equations using crammer's rule
   $3x + 4y + 5z = 18, 2x - y + 8z = 13, 5x - 2y + 7z = 20.$
3. Solve the system following of equations using crammer's rule
   $x + y + z = 3, 2x + 2y - z = 3, x + y - z = 1.$
4. Solve the following system of equations using matrix inversion method
   $2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0.$
5. Solve the following system of equations using matrix inversion method
   $3x + 4y + 5z = 18, 2x - y + 8z = 13, 5x - 2y + 7z = 20.$
6. Solve the following system of equations using matrix inversion method
   $2x + 4y - z = 0, x + 2y + 2z = 5, 3x + 6y - 7z = 2.$

Key Concepts

1. An ordered rectangular array of elements is called a *matrix.* The elements of matrices are real or complex numbers.

2. A matrix having m rows and n columns is said to be of *order* $m\times n,$ read as of m cross n or m by $n.$ In compact form it is denoted by $A=\left(a_{ij}\right)_{m\times n}$ where $1\le i\le m$ and $1\le j\le n.$

3. In a matrix if the number of rows is not equal to the number of columns then that matrix is called a *rectangular matrix*.

4. A matrix in which the number of rows is equal to the number of columns, is called a *square matrix*.

5. $A=\left(a_{ij}\right)_{m\times n}$ is a square matrix if $m=n.$ In this case A is a square matrix of order $n.$ $1\le j\le n.$

6. A matrix having only one row is called a *row matrix*.

7. A matrix having only one column is called a *column matrix*.

8. A matrix consisting of all zero elements is called a *null matrix* or *zero matrix*.

9. If $A=\left(a_{ij}\right)_{n\times n}$ where $1\le i,j\le n$ then the principal diagonal elements are $a_{11},a_{22},a_{33}...a_{nn}.$ Trace of a square matrix A is denoted by $Tr(A)$ and is denoted by $Tr(A)=\sum_{i=1}^{n}a_{ii}=a_{11}+a_{22}+a_{33}+...+a_{nn}.$

10. A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is said to be an *upper triangular matrix* if $a_{ij}=0\ for\ all\ i>j$

11. A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is said to be a *lower triangular matrix* if $a_{ij}=0\ for\ all\ i<j$

12. A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is said to be a *unit* or *identity matrix* if $a_{ij}=0\ for\ all\ i\ne j$

13. If $A=\left(a_{ij}\right)_{m\times n}$ and $B=\left(b_{ij}\right)_{m\times n}$ then $A+B=\left(c_{ij}\right)_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$ $for\ 1\le i\le m, 1\le j\le n$

14. If $A=\left(a_{ij}\right)_{m\times n}, B=\left(b_{ij}\right)_{m\times n}$ and $C=\left(c_{ij}\right)_{m\times n}$ be matrices of the same order.

(v) *Commutative Property*: $A+B=B+A$

(vi) *Associative Property*: $A+(B+C)=(A+B)+C$

(vii) *Additive identity*: $A+O=O+A=A$

(viii) *Additive inverse*: $A+B=B+A=O$

15. If $A=\left(a_{ij}\right)_{m\times n}$ then $kA=k\left(a_{ij}\right)_{m\times n}=\left(k\,a_{ij}\right)_{m\times n}$

16. Let $A=\left(a_{ij}\right)_{m\times p}$, $B=\left(b_{ij}\right)_{p\times n}$ be two matrices. Then the matrix $C=\left(c_{ij}\right)_{m\times n}$ where $c_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$ is called the product of the matrices A and B denoted by AB.

17. If $A=\left(a_{ij}\right), B=\left(b_{ij}\right)$ and $C=\left(c_{ij}\right)$ be matrices conformable for multiplication. Then

(iii) *Associative Law*: $A(BC)=(AB)C$

(iv) *Distributive Law*: $A(B+C)=AB+AC\,(Left\ Distibutive\ Law)$
$(A+B)C=AC+BC\,(Right\ Distibutive\ Law)$

(iii) *Existence of multiplicative identity*:

If I is the identity matrix of order n, then for any square matrix A of order n,

$$AI=IA=A$$

18. Matrix multiplication need not be commutative.

19. If A and B are two matrices conformable for multiplication, AB exists, but BA may not exist, even if BA exists, AB and BA may not equal.

20. If $O\neq A$ and $O\neq B$ are two matrices conformable for multiplication, AB exists and $AB=O$ If $AB=AC$ and $O\neq A$, then it is not necessary that $B=C$

21. For any positive integer n, $(A)^n=A.A.A....A(n\,times)$

22. If is α a scalar and A is any square matrix and is n a positive integer, then

$$(\alpha A)^n=(\alpha A)(\alpha A)(\alpha A)....(\alpha A)(n\,times)$$

23. If A is a matrix of order $m\times n$, then the matrix obtained by interchanging the rows into columns or columns into rows of A is called the *transpose of* A. The transpose of the matrix A is denoted by $A^T(or)A'$

24. If $A=\left(a_{ij}\right)_{m\times n}$ then $A^{T}=\left(a_{ji}\right)_{n\times m}$

25. Let A and B be matrices of suitable order. Then

$$(i)\,(A^{T})^{T}=A \qquad (ii)\,(kA)^{T}=kA^{T}$$
$$(iii)\,(A+B)^{T}=A^{T}+B^{T} \qquad (iv)(AB)^{T}=B^{T}A^{T}$$

26. A square matrix A is said to be *symmetric matrix* if $A^{T}=A.$

(i) Let $A=\left(a_{ij}\right)_{n\times n}$ be a symmetric matrix if $a_{ij}=a_{ji}\forall 1\le i,j\le n.$

(ii) We have $O_{n\times n}, I_{n\times n}$ are symmetric matrices.

(iii) If A is a square matrix then $A+A^{T}$ is a symmetric matrix.

27. A square matrix A is said to be *skew-symmetric matrix* if $A^{T}=-A.$

(i) $A=\left(a_{ij}\right)_{n\times n}$ is a skew-symmetric matrix if

$$a_{ij}=-a_{ji}\forall 1<i,j<n \text{ and} a_{ii}=0\forall 1\le i\le n.$$

(ii) We have $O_{n\times n}$ is a skew-symmetric matrix.

(iii) If A is a square matrix then $A-A^{T}$ is a skew-symmetric matrix.

28. We have learnt in lower classes that the value $a_1b_2-a_2b_1$ is called the *determinant* of the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ Consider a square matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. The *minor* of an element in this matrix is defined as the determinant of the 2×2 matrix, obtained after deleting the row and column in which the element is present.

29. Let $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. The sum of the products of elements of the first row with their corresponding cofactors is called the *determinant* of $A.$ The determinant of the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. We also denote the determinant of the matrix A by $\det A$ or $|A|$.

$$\det A=a_1A_1+b_1B_1+c_1C_1$$

Let $A = \left(a_{ij}\right)_{m\times n}$. Then the determinant of A is defined as $\sum_{j=1}^{n} a_{ij}A_{ij}$, where A_{ij} is the cofactor of a_{ij}

We thus have $\det A = \sum_{j=1}^{n} a_{ij}A_{ij}$ for $1 \le i \le n.$

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $\det A = a_1A_1 + b_1B_1 + c_1C_1$ expansion along first row.

$$a_1A_1 + a_2A_2 + a_3A_3 = a_1(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2(-1)^{2+1}\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3(-1)^{3+1}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1A_1 + b_1B_1 + c_1C_1 = \det A$$

(i) If each element of a row (or column) of a square matrix is zero, then the determinant of that matrix is zero.

(ii) If two rows (or columns) of a square matrix are interchanged, then the sign of the determinant changes.

(iii) If each element of a row (or column) of a square matrix is multiplied by a number k, then the determinant of the matrix obtained is k times the determinant of the given matrix.

(iv) If A is a square matrix of order 3 and k is a scalar, then $|kA| = k^3|A|$.

(v) If two rows (or columns) of a square matrix are identical, then the value of the determinant is zero.

(vi) If the matrix is a diagonal matrix then the determinant of the matrix is product of the diagonal elements.

(vii) If the matrix is a triangular (upper or lower) matrix then the determinant of the matrix is product of the diagonal elements.

(viii) If the corresponding elements of two rows (or columns) of a square matrix are in the same ratio, then the value of the determinant is zero.

(ix) If each element in a row (or column) of a square matrix is the sum of two numbers then its determinant can be expressed as the sum of the determinants of two square matrices as shown below.

(x) If each element in a row (or column) of a square matrix is multiplied by a number k and added to the corresponding element of another row (or column) of the matrix then the determinant of the resulting matrix is equal to the determinant of the given matrix.

(xi) For any square matrix A, $\det A = \det A^T$.

(xi) For any square matrix A, B of same order $\det(AB) = \det A \det B$.

(xii) For any positive integer n, $\det(A^n) = (\det A)^n$.

30. While evaluating the determinant we use the following notations.

(i) $R_1 \leftrightarrow R_2$, to mean that the rows R_1 and R_2 are interchanged.

(ii) $R_1 \to kR_1$, to mean that the elements of row R_1 are multiplied by k.

(iii) $R_1 \to R_1 + kR_2$, to mean that the elements of row R_1 are added with k the corresponding elements of row R_2.

31. A square matrix is said to be *singular* if its determinant is zero. Otherwise it is said to be *non-singular*.

32. The transpose of the matrix formed by replacing the elements of a square matrix A with the corresponding cofactors is called the Adjoint of A and it is denoted by $Adj\,A$.

33. Let A be a non-singular matrix. We say that A is invertible if a matrix B exists such that $AB = BA = I$ where I is the unit matrix of the same order as A and B

(i) If B exists such that $AB = BA = I$ then such a unique B is denoted by A^{-1} and is called the multiplicative inverse of A

(ii) If A is invertible then A is non-singular matrix, hence $\det A \neq 0$

34. If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is a non-singular matrix then A is invertible and $A^{-1} = \dfrac{\text{Adj}\,A}{\det A}$

35. Let A and B are invertible matrices. Then A^{-1}, A and AB are invertible matrices.
$(i)\,(A^{-1})^{-1} = A$ $\quad (ii)\,(A^T)^{-1} = (A^{-1})^T$ $\quad (iii)\,(AB)^{-1} = B^{-1}A^{-1}$

36. Consider the system of equations

$$a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$$

Let $AX = B$, Where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is non-singular and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$\therefore x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}$ *This is known as Crammer's Rule*

.

37. Consider the system of equations

$$a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$$

Where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is non-singular. Then we can find A^{-1}

$$AX = B \Leftrightarrow A^{-1}(AX) = A^{-1}B \Leftrightarrow (A^{-1}A)X = A^{-1}B \Leftrightarrow I\,X = A^{-1}B\left[\because I = A^{-1}A\right]$$

$X = A^{-1}B$ From this x, y and z are known.

Answers

Exercise 3(a)

(1) $\begin{pmatrix} 3 & 3 & 0 \\ 9 & 4 & 4 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 4 \\ 7 & -3 \end{pmatrix}$ (3) $\begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 3 \\ 5 & 2 & 3 \end{pmatrix}$ (4) $x = 1, y = \frac{5}{2}, z = 2, w = 0.$

(5) $x = 2, y = 2, z = 5, w = 5.$ (6) 1 (7) $\begin{pmatrix} 20 & -22 \\ -22 & 34 \end{pmatrix}$ (8) $\begin{pmatrix} 1 & -2 \\ -2 & 20 \end{pmatrix}$ (9) $\begin{pmatrix} 12 & 8 \\ 4 & 3 \end{pmatrix}$

(11) $\begin{pmatrix} -5 & 15 & 5 \\ 10 & 20 & -8 \\ 9 & -23 & -15 \end{pmatrix}$ (12) $\begin{pmatrix} -6 & 14 \\ 13 & 0 \\ -1 & 10 \end{pmatrix}$

Exercise 3(b)

(1) 1 (2) 4 (3) 2 (4) -108 (5) 37

(6) -54 (7) 27 (8) $abc+2fgh-af^2-bg^2-ch^2$

Exercise 3(c)

(1) $\frac{-1}{11}\begin{pmatrix}-5 & -1\\ -1 & 2\end{pmatrix}$ (2) $\begin{pmatrix}\frac{1}{4} & \frac{1}{8}\\ -\frac{1}{6} & \frac{1}{12}\end{pmatrix}$ (3) $\frac{-1}{4}\begin{pmatrix}1 & -3 & 4\\ -3 & 1 & 0\\ 1 & 1 & -4\end{pmatrix}$ (4) $\frac{1}{3}\begin{pmatrix}1 & 4 & -2\\ -2 & -5 & 4\\ 1 & -2 & 1\end{pmatrix}$

(5) $\begin{pmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{pmatrix}$

Exercise 3(d)

(1) $x=0, y=1, z=2$ (2) $x=3, y=1, z=1$ (3) $x=1, y=1, z=1$ (4) $x=2, y=2, z=2$
(5) $x=3, y=1, z=1$ (6) no solution

4. ADDITION OF VECTORS

Introduction:

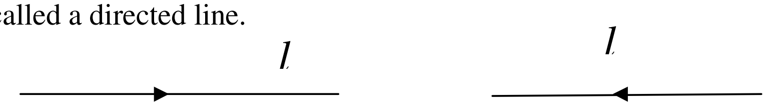
In our day to day life we come across many queries such as What is your height? How should a foot ball player hit the ball, to give a pass to one another player of his team? Observe that one possible answer to the first query is 1.75 meters, a quantity that specifies a value (magnitude) which is a real number. Such quantities are called scalars. However, the answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and also direction (in which another player is positioned). Such quantities are called vectors. In Physics, Engineering and Mathematics, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, volume, temperature, density, area, work, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum etc.

Vector methods have revolutionised Mechanics, Engineering, Physics and Mathematics. *Rene Descarte* (1596-1660) after whom the Cartesian coordinate system is named, *G.W. Leibnitz* (1646-1716), a famous mathematician of 17th century and *R.H. Hamilton* (1835-1865), a well known theoretical physicist are the trio who laid the seeds to this branch of Mathematics. *J.W. Gibb's* (1839-1903) work on vector analysis was of major importance in Mathematics.

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors and their algebraic and geometric properties. Angle between two non-zero vectors, vector equations of a line and a plane are discussed to give a full realisation of the applicability of vectors in various areas as mentioned above.

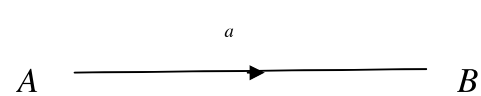
4.1 Vectors as a triad of real numbers, some basic concepts:

Let l be any straight line in a plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed, is called a directed line.



4.1.1 Definition (Directed line segment):

If A and B are two distinct points in the space, the ordered pair (A, B), denoted by $\overline{AB}$ is called a directed line segment with initial point A and terminal point B. The magnitude of $\overline{AB}$, denoted by $\left|\overline{AB}\right| = a\,(say)$, is the length of $\overline{AB}$ or distance between A and B.



4.1.2 Definition:

A line segment with a specified magnitude and direction is called a *vector*. Notice that the directed line segment $\overline{AB}$ is a vector.

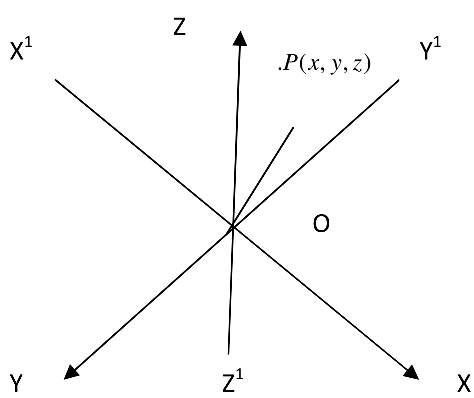
The *zero vector*, denoted by $\bar{0}$, is the collection of $\overline{PP}$ where P is any point in the space. The zero vector, also known as *null vector*, has neither support nor any direction. Observe that, for the zero vector, the initial and terminal points coincide and its magnitude is the scalar 0.

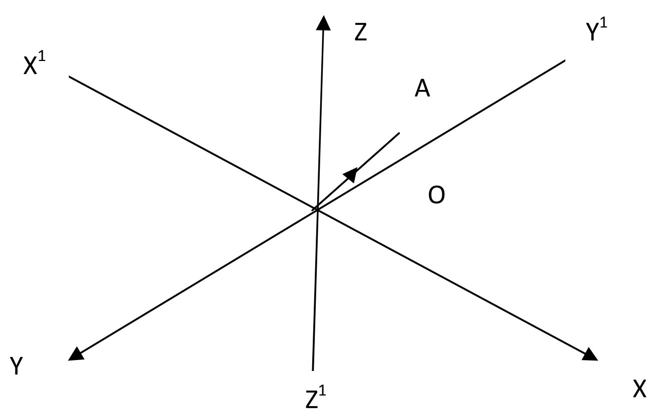
Let a, b and c be real numbers (not necessarily distinct). A set formed with a, b, c in which the order of occurance is also preassigned is called an *ordered triad* or *a triple*. If a, b, c are distinct real numbers, then we get six ordered triads, namely $(a,b,c),(a,c,b),(b,c,a),(b,a,c),(c,a,b),(c,b,a)$. For the ordered triad $(a,b,c), a,b,c$ are called *the first*, *the second* and *the third* components respectively.

The set of all ordered triads (a,b,c) of real numbers is denoted by R^3.

4.1.3 Position vector:

Consider a three-dimensional rectangular coordinate system $\overline{OX}, \overline{OY}, \overline{OZ}$ and a point P in the space having coordinates (x, y, z) with respect to the origin $O(0,0,0)$ as shown in the Fig given below. Then the vector $\overline{OP}$ having O and P as its initial and terminal points respectively, is called the position vector of the point with respect to O. This is denoted by $\bar{r}$. The magnitude of $\overline{OP}$, using the distance formula is given by $|\overline{OP}| = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$. It is customary that the position vector of A with respect to the origin O is denoted by $\bar{a}$.



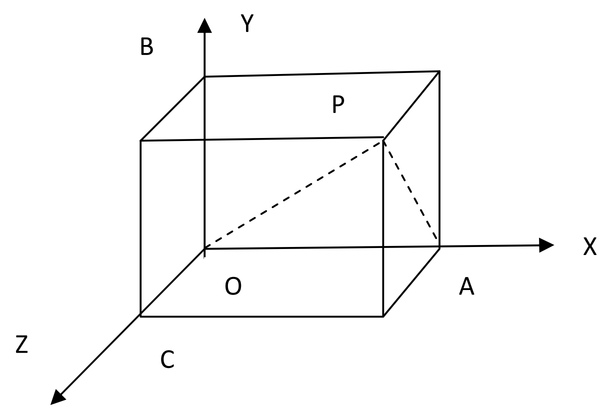


4.1.4 Direction cosines and Direction ratios:

Consider the position vector $\overline{OP} = \overline{r}$ of a point $P = (x, y, z)$. Let α, β, γ be the angles made by the vector $\overline{r}$ with the positive direction (counter clockwise direction) of X, Y, Z axes respectively. Then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the direction cosines of the vector $\overline{r}$. These direction cosines are usually denoted by l, m, n respectively.

Draw perpendicular from P to the X, Y and Z axes and let A, B, C be the feet of the perpendiculars respectively.

We observe that ΔOAP is right angled and hence $\cos\alpha = \frac{x}{|\overline{r}|} = \frac{x}{r}$. Similarly from the right angled triangles ΔOBP and ΔOCP, we may write $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus the coordinates x, y, z of the point P may also be expressed as (lr, mr, nr). The numbers lr, mr, nr which are proportional to the *direction cosines* are called the *direction ratios* of the vector $\overline{r}$. These are usually denoted by a, b, c respectively.



We observe that

$$r^2 = x^2 + y^2 + z^2$$

$$= l^2r^2 + m^2r^2 + n^2r^2$$

$$= r^2(l^2 + m^2 + n^2)$$

so that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$.

4.2 Classification (Types) of vectors:

4.2.1 Definition (Unit vector):

A vector whose magnitude is unity (*i.e* 1 unit) is called a *unit vector*. It is denoted by $\overline{e}$. The unit vector in the direction of a given vector $\overline{a}$ is usually denoted by $\hat{a}$.

4.2.2 Definition (Equal vectors):

Two vectors $\overline{a}$ and $\overline{b}$ are said to be *equal vectors* and written as $\overline{a} = \overline{b}$, if they have the same magnitude and direction, regardless of the positions of their initial points.

4.2.3 Definition (Collinear vectors, like and unlike vectors):

Two or more vectors are said to be *collinear vectors* if they are parallel to the same line, irrespective of their magnitudes and direction. Such vectors have the same support or parallel support.

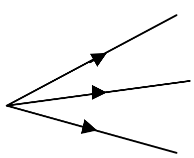
Two vectors are called *like vectors* or *unlike vectors* according as they have the same direction or opposite direction. In the following Fig $\bar{a}$ and $\bar{b}$ are *like vectors,* where as $\bar{a}$ and $\bar{c}$ are *unlike vectors.*

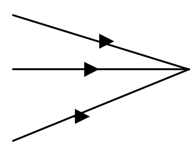
4.2.4 Definition (Negative of a vector):

Let $\bar{a}$ be a vector. The vector having the same magnitude as $\bar{a}$ but having the opposite direction is called the *negative vector* of $\bar{a}$ and denoted by $-\bar{a}$. Note that if $\bar{a} = \overline{AB}$ then $-\bar{a} = \overline{BA}$.

4.2.5 Definition (Co-initial vectors and Co-terminal vectors):

Two or more vectors having same initial point are called *co-initial vectors* and same final point are called *co-terminal vectors.*

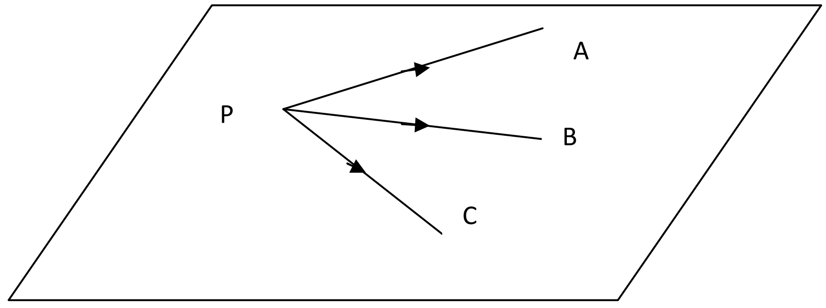




4.2.5 Definition (Co-initial vectors and Co-terminal vectors):

Vectors whose supports are in the same plane or parallel to the same plane are called *coplanar vectors*. Vectors which are not coplanar are called *non-coplanar vectors.*

Note that the vectors $\bar{a} = \overline{PA}, \bar{b} = \overline{PB}$ and $\bar{c} = \overline{PC}$ are coplanar vectors if and only if the four points P, A, B, C lie in the same plane. Coplanarity or non-coplanarity of vectors arises only when there are three or more non-zero vectors, since any two vectors are always coplanar.



4.3 Sum (addition) of vectors:

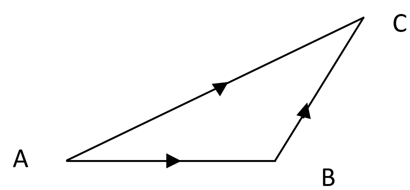
We shall now introduce the concept of addition (sum) of vectors, derive the commutative law, associative law and a few other properties.

4.3.1 Triangle law of vector addition:

A vector $\overline{AB}$ simply means the displacement from a point A to the point B along the line AB. Now consider a situation that a person moves from A to B and then from B to C. The net displacement made by the person from point A to the point C, is given by the vector $\overline{AC}$ and expressed as

$$\overline{AC} = \overline{AB} + \overline{BC}$$

This is known as the *triangle law of vector addition*.

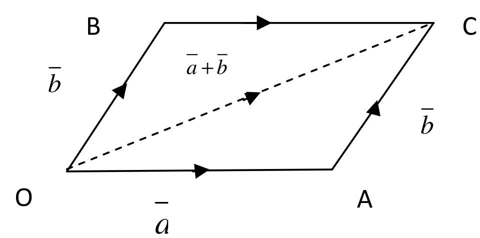


4.3.2 Parallelogram law of vector addition:

If we have two vectors $\overline{a}$ and $\overline{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction, then their sum $\overline{a}+\overline{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point known as the *parallelogram law of vector addition*.

We have $\overline{OA} + \overline{AC} = \overline{OC}$

or $\overline{OA} + \overline{OB} = \overline{OC} (\because \overline{OB} = \overline{AC})$



4.3.3 Properties of vector addition:

1. *Commutative property*: For any two vectors $\overline{a}$ and $\overline{b}$, $\overline{a}+\overline{b}=\overline{b}+\overline{a}$.

2. *Associative property*: For any three vectors $\overline{a}, \overline{b}$ and $\overline{c}$, $(\overline{a}+\overline{b})+\overline{c}=\overline{a}+(\overline{b}+\overline{c})$.

3. *Identity property*: For any vector $\overline{a}$, $\overline{a}+\overline{0}=\overline{0}+\overline{a}=\overline{a}$. Here the zero vector is the additive identity vector.

4. *Inverse property*: For any vector $\overline{a}$, $\overline{a}+\overline{b}=\overline{b}+\overline{a}=\overline{0}$. Here the vector $\overline{b}$ is the additive inverse of the vector $\overline{a}$.

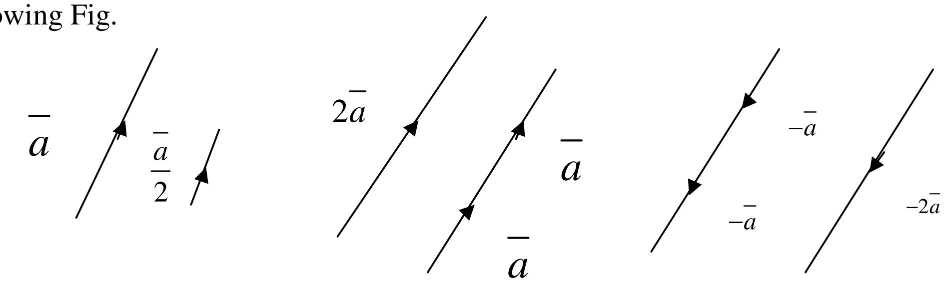
4.4 Scalar Multiplication of a vector:

We shall now introduce the operation of scalar multiplication of a vector, initially through a geometric visualisation and later state some laws of scalar multiplication.

4.4.1 Scalar multiplication:

Let $\bar{a}$ be a given non-zero vector and λ a scalar. Then the product of the vector $\bar{a}$ by a scalar $\lambda,$ denoted as $\lambda\bar{a},$ is defined as a vector $\lambda\bar{a}$ collinear with $\bar{a}.$ The vector $\lambda\bar{a}$ is called the multiplication of vector $\bar{a}$ by the scalar λ and $\lambda\bar{a}$ has the direction same (or opposite) to that of vector $\bar{a}$ according as the value of λ is a positive (or negative).

The geometric visualisation of multiplication of a vector by a scalar is given in the following Fig.



When $\lambda = -1$ then $\lambda\bar{a} = -\bar{a},$ which is a vector having magnitude equal to the magnitude of $\bar{a}$ and direction opposite to that of $\bar{a}.$ The vector $-\bar{a}$ is called the negative of a vector $\bar{a},$ we always have

$$\bar{a} + (-\bar{a}) = (-\bar{a}) + \bar{a} = \bar{0}.$$

Also, if $\lambda = \frac{1}{|\bar{a}|},$ provided $\bar{a} \neq \bar{0}$ then

$$|\lambda\bar{a}| = |\lambda||\bar{a}| = \frac{1}{|\bar{a}|}|\bar{a}| = 1.$$

So $\lambda\bar{a}$ represents the unit vector $\hat{a}$ in the direction of $\bar{a}.$

Hence $\hat{a} = \frac{1}{|\bar{a}|}\bar{a}.$

4.4.2 Definition:

Let $\overline{a}$ be a vector and λ be a scalar. Then we define the vector $\lambda\overline{a}$ to be the vector $\overline{0}$ if either the vector $\overline{a}$ is a zero vector or λ is the zero scalar; otherwise $\lambda\overline{a}$ is the direction of $\overline{a}$ with magnitude $\lambda|\overline{a}|$ if $\lambda > 0$, and $\lambda\overline{a} = (-\lambda)(-\overline{a})$, if $\lambda < 0$.

4.4.3 Some laws of scalar multiplication of a vector:

We now state some laws of scalar multiplication of a vector which are useful for further discussion.

1. If $\overline{a}$ is a vector and λ is a scalar, then $(-\lambda)\overline{a} = \lambda(-\overline{a}) = -(\lambda\overline{a})$.

2. If $\overline{a}$ is a vector and m, n are scalars, then $m(n\overline{a}) = (mn)\overline{a} = (nm)\overline{a} = n(m\overline{a})$.

3. If $\overline{a}$ is a vector and m, n are scalars, then $(m+n)\overline{a} = m\overline{a} + n\overline{a}$.

4. If m is a scalar and $\overline{a}, \overline{b}$ are any two vectors, then $m(\overline{a}+\overline{b}) = m\overline{a} + m\overline{b}$.

4.4.4 Note:

1. Two vectors are collinear (parallel) if and only if one is a scalar multiple of the other.

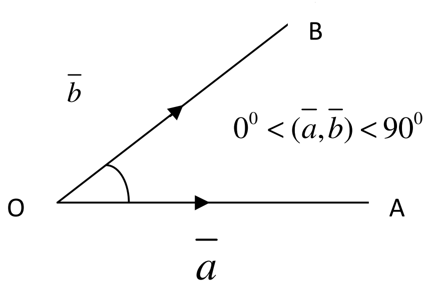
2. Three points A, B and C are collinear if and only if $\overline{AB}, \overline{BC}$ are collinear vectors.

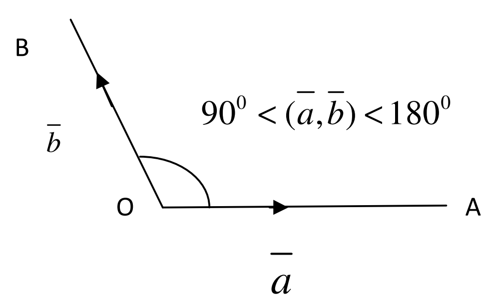
4.5 Angle between two non-zero vectors:

We have learnt about angle between two lines in plane geometry. We now introduce the concept of the angle between two non-zero vectors, which is slightly different from the angle between two lines. The concept of angle between two vectors is largely useful in Chapter 5, which deals with dot and cross product of two vectors.

4.5.1 Definition:

Let $\overline{a}$ and $\overline{b}$ be two non-zero vectors. Let O, A and B be points such that $\overline{OA} = \overline{a}$ and $\overline{OB} = \overline{b}$. Then the measure of $\angle AOB$ which lies between 0^0 and 180^0 is called the angle between $\overline{a}$ and $\overline{b}$ and is denoted by $(\overline{a}, \overline{b})$.





4.5.2 Note:

1. Let $\overline{a}$ and $\overline{b}$ be non-zero vectors. Then

   $(i)\,(\overline{a},\overline{b}) = 0^0 \Leftrightarrow \overline{a}$ and $\overline{b}$ are like vectors.

   $(ii)\,(\overline{a},\overline{b}) = 180^0 \Leftrightarrow \overline{a}$ and $\overline{b}$ are unlike vectors.

   $(iii)\,(\overline{a},\overline{b}) = 0^0$ or $180^0 \Leftrightarrow \overline{a}$ and $\overline{b}$ are collinear vectors.

   (iv) If $(\overline{a},\overline{b}) = 90^0$ then $\overline{a}$ and $\overline{b}$ are perpendicular vectors.

2. Let $\overline{a}$ and $\overline{b}$ be non-zero vectors and m,n be positive scalars. Then

   $(i)\,(\overline{a},\overline{b}) = (\overline{b},\overline{a}).$

   $(ii)\,(\overline{a},\overline{b}) = (-\overline{a},-\overline{b}).$

   $(iii)\,(\overline{a},-\overline{b}) = (-\overline{a},\overline{b}) = 180^0 - (\overline{a},\overline{b}).$

   $(iv)\,(\overline{a},\overline{b}) = (m\overline{a},n\overline{b}).$

   $(v)\,(m\overline{a},-n\overline{b}) = (-m\overline{a},n\overline{b}) = 180^0 - (m\overline{a},n\overline{b}).$

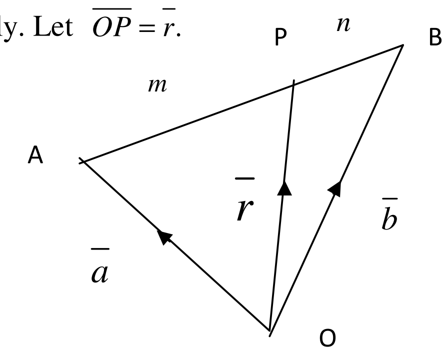
4.5.3 Definition:

Let A and B be two points and P a point on the straight line $AB.$ We say that P divides the line segment AB in the ratio $m:n\,(m+n \neq 0),$ if $n\,AP = m\,PB.$

4.5.4 Theorem:

Let $\overline{a}$ and $\overline{b}$ be position vectors of the points A and B with respect to the origin $O.$ If a point P divides the line segment AB in the ratio $m:n\,(m+n \neq 0),$ then the position vector of P is $\dfrac{m\overline{b}+n\overline{a}}{m+n}.$

Proof: Let P be a point on the straight line AB lying between A and $B,$ in which case P is said to divide the line segment AB internally. Let $\overline{OP} = \overline{r}.$



By definition $n\,\overline{AP} = m\,\overline{PB}$

$\Rightarrow n\,(\overline{AO}+\overline{OP}) = m\,(\overline{PO}+\overline{OB})$

$\Rightarrow n\,(\overline{OP}-\overline{OA}) = m\,(\overline{OB}-\overline{OP})$

$\Rightarrow n(\bar{r}-\bar{a}) = m(\bar{b}-\bar{r})$

$\Rightarrow (m+n)\bar{r} = m\bar{b}+n\bar{a}$

$\therefore \bar{r} = \dfrac{m\bar{b}+n\bar{a}}{m+n}$

4.5.5 Note:

The above formula is called (division) *section formula* and it holds weather P divides AB internally or externally. The position vector of the point P which divides the line segment AB in the ratio $m:n\,(m-n \neq 0),$ is given by $\dfrac{m\bar{b}-n\bar{a}}{m-n}.$

4.5.6 Corollary:

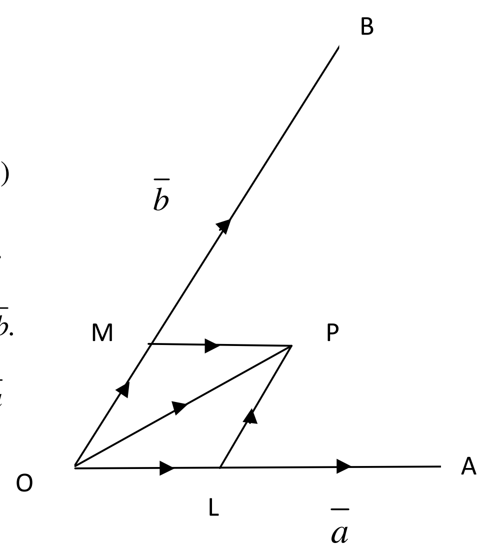
If P is the mid point of AB then $m=n$, and hence the position vector of P is $\bar{r} = \overline{OP} = \dfrac{\bar{a}+\bar{b}}{2}.$

Proof: In Theorem 4.5.4, take $m=n=1$.

4.5.7 Theorem:

Let $\bar{a}$ and $\bar{b}$ be any two non-collinear vectors. If $\bar{r}$ is any vector in the plane Π determined by a pair of supports $\bar{a}$ and $\bar{b},$ then there exist unique scalars x and y such that $\bar{r} = x\bar{a}+y\bar{b}.$

Proof: Choose a point O in the plane Π as the origin and points A and B in $\Pi.$ $\bar{a}=\overline{OA}$ and $\bar{b}=\overline{OB}$ so that O, A and B are not collinear. Let P be a point in the plane Π such that $\overline{OP}=\bar{r}.$ If P lies either on the support of $\bar{a}$ $(i.e\ on\ the\ line\ \overline{OA})$ or on the support of $\bar{b}$ $(i.e\ on\ the\ line\ \overline{OB}),$ then $y=0$ or $x=0.$



Suppose P does not lie on the supports of $\bar{a}$ and $\bar{b}.$ Through P draw lines parallel to $\bar{b}$ meeting the support of $\bar{a}$ in L and parallel to $\bar{a}$ meeting the support of $\bar{b}$ in M.

Thus $\overline{OL}$ is collinear with $\bar{a}$ and $\overline{OM}$ is collinear with $\bar{b}$. Hence there exist scalars x and y such that $\overline{OL} = x\bar{a}$ and $\overline{OM} = y\bar{b}$.

Then $\bar{r} = \overline{OP} = \overline{OL} + \overline{LP}$

$$= \overline{OL} + \overline{OM} = x\bar{a} + y\bar{b}.$$

If $\bar{r} = x'\bar{a} + y'\bar{b}$, then $(x - x')\bar{a} = (y - y')\bar{b}$ so that $x = x'$, $y = y'$, otherwise $\bar{a}$ and $\bar{b}$ will be collinear vectors. Thus x and y are unique.

4.5.8 Corollary:

Let $\bar{a}$ and $\bar{b}$ be vectors and $x,\ y$ are scalars then $x\bar{a} + y\bar{b} = \bar{0}$ if and only if $x = y = 0.$

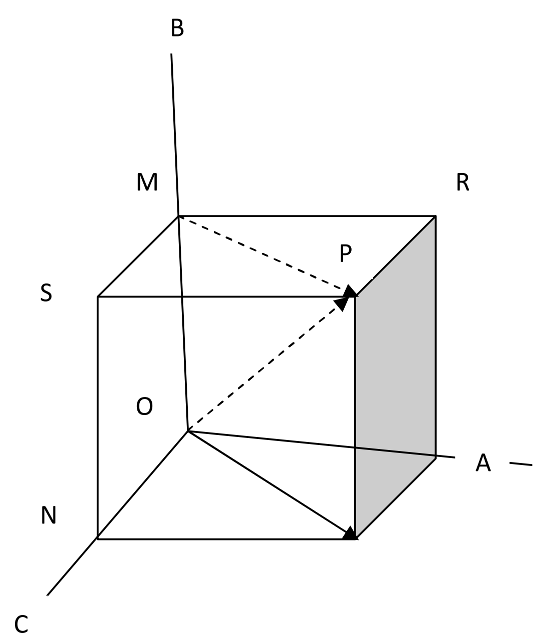
Proof: If $x = y = 0,$ then $x\bar{a} + y\bar{b} = \bar{0}.$

Since $\bar{0} = 0.\bar{a} + 0.\bar{b},$ by Theorem 4.5.7, $x = y = 0.$

It is known that non-coplanar vectors do exist in the space and in particular three non-zero vectors with the same initial point exist. Now, we have the following theorem which we called *representation theorem.*

4.5.9 Theorem:

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be any three non-coplanar vectors and $\bar{r}$ is any vector in the space Π. Then, there exist unique triad of scalars x, y, z such that $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}.$



Proof: Let O be the origin, $\bar{a} = \overline{OA}$, $\bar{b} = \overline{OB}$ and $\bar{c} = \overline{OC}$. Let P be a point in the space. If P lies on the supports of $\bar{a}$, that is $\bar{r}$ is collinear with $\bar{a}$ then we choose $y = z = 0.$ Similarly, if P lies on the supports of $\bar{b}$ or $\bar{c}$, then choose $z = x = 0$ or $x = y = 0$ respectively. Suppose P lies in the plane $AOB.$ Then by Theorem 4.5.7, $\bar{r} = x\bar{a} + y\bar{b}$ so that $z = 0.$ Similarly if P lies in the plane BOC, then $\bar{r} = y\bar{b} + z\bar{c}, x = 0$ and P lies in the plane $COA,$ then

$\overline{r} = x\overline{a} + z\overline{c}, y = 0.$

Now suppose P does not belong to any one of the planes AOB, BOC and COA. Through P draw planes parallel to the planes AOB, BOC and COA meeting the support of $\overline{c}, \overline{a}$ and $\overline{b}$ in N, L and M respectively.

Now $\overline{r} = \overline{OP} = \overline{OQ} + \overline{QP} = (\overline{OL} + \overline{LQ}) + \overline{OM} = (\overline{OL} + \overline{ON}) + \overline{OM} = \overline{OL} + \overline{OM} + \overline{ON}.$

Since $\overline{OL}, \overline{OM}$ and $\overline{ON}$ are collinear with $\overline{a}, \overline{b}$ and $\overline{c}$ respectively, then there exist scalars x, y and z such that $\overline{OL} = x\overline{a}, \overline{OM} = y\overline{b}$ and $\overline{ON} = z\overline{c}.$

Then $\overline{r} = x\overline{a} + y\overline{b} + z\overline{c}.$

If $\overline{r} = x'\overline{a} + y'\overline{b} + z'\overline{c}$, then $(x' - x)\overline{a} = (y - y')\overline{b} + (z - z')\overline{c}.$

If $x \neq x'$, then $\overline{a}$ is coplanar with $\overline{b}$ and $\overline{c}$ which is not true.

$\therefore x = x'$. Similarly $y = y'$ and $z = z'$.

4.5.10 Corollary:

Let $\overline{a}, \overline{b}$ and $\overline{c}$ are non-coplanar vectors, then $x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}$ if and only if $x = y = z = 0.$

Proof: If $x = y = z = 0,$ then clearly $x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}.$

Suppose $x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}.$ Since $\overline{0} = 0.\overline{a} + 0.\overline{b} + 0.\overline{c}$ by Theorem 4.5.9, $x = y = z = 0.$

4.5.11 Definition:

Let $\overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots \overline{a_n}$ be vectors and $x_1, x_2, x_3, \ldots x_n$ be scalars. Then the vector $x_1\overline{a_1} + x_2\overline{a_2} + x_3\overline{a_3} + \ldots + x_n\overline{a_n}$ is called the *linear combination* of the vectors $\overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots \overline{a_n}$.

4.5.12 Note:

(i) If $\overline{a}, \overline{b}$ are non-collinear vectors, then by Theorem 4.5.7, every vector in the plane determined by pair of supports of $\overline{a}$ and $\overline{b}$ can be expressed as linear combination of $\overline{a}$ and $\overline{b}$ in one and only one way.

(ii) If $\overline{a}, \overline{b}$ and $\overline{c}$ are non-coplanar vectors, then by Theorem 4.5.9, every vector in the space can be expressed as linear combination of $\overline{a}, \overline{b}$ and $\overline{c}$ in one and only one way.

(iii) Three vectors are coplanar vectors if and only if one of them is a scalar multiple of the other two.

4.5.13 Components of a vector in Three Dimensions:

In Theorem 4.5.9, we have proved that every vector can be expressed as a linear combination of three non-coplanar vectors. Here we introduce the concept of components of a vector with respect to non-coplanar vectors $\bar{a}, \bar{b}$ and $\bar{c}$.

4.5.14 Definition (Components):

Consider an ordered triad $(\bar{a}, \bar{b}, \bar{c})$ of non-coplanar vectors $\bar{a}, \bar{b}, \bar{c}$. If $\bar{r}$ is any vector then there exists a unique triad (x, y, z) of scalars such that $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$. These scalars x, y, z are called the components of $\bar{r}$ with respect to the triad $(\bar{a}, \bar{b}, \bar{c})$.

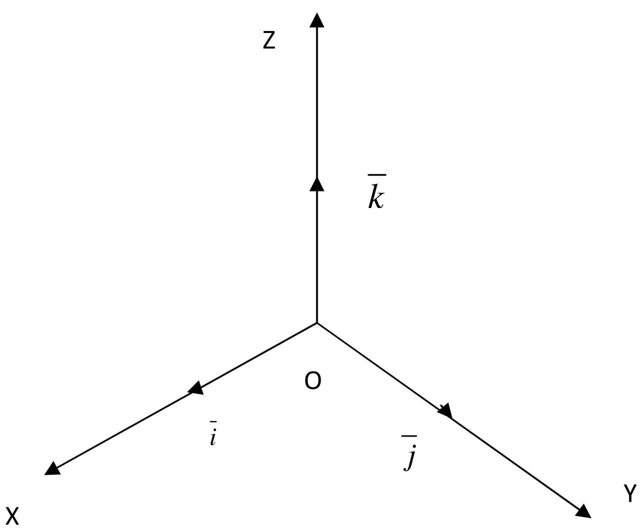
Any ordered triad of non-coplanar vectors is called a base for the space. The components of a vector depend on the choice of the base.

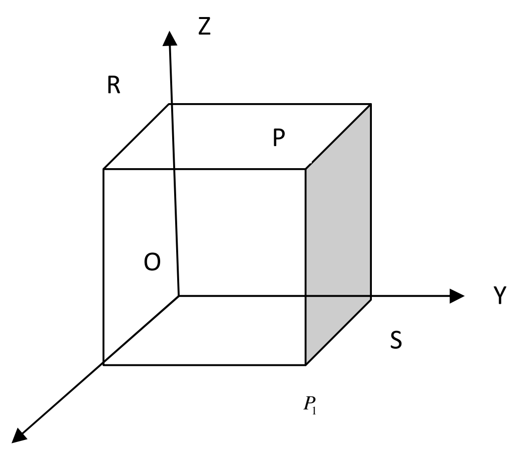
4.5.15 Representing a vector in component form:

We shall now express a given vector in component form. Let O be a point in space. Call it the origin. Take three mutually perpendicular X, Y and Z axes. Let us take the points $A(1,0,0), B(0,1,0)$ and $C(0,0,1)$ on the X^- axis, Y^- axis and Z^- axis respectively. Then clearly $|\overline{OA}| = 1, |\overline{OB}| = 1$ and $|\overline{OC}| = 1$.

The vectors $\overline{OA}, \overline{OB}$ and $\overline{OC}$, each having magnitude 1 are called unit vectors along $\overline{OX}$, $\overline{OY}$ and $\overline{OZ}$ respectively, and denoted by $\bar{i}$, $\bar{j}$ and $\bar{k}$ respectively.

Now consider the position vector $\overline{OP}$ of a point $P(x, y, z)$. Let P_1 be the foot of the perpendicular from P on the plane XOY. We thus see that P_1P is parallel to Z^- axis.





As $\bar{i}$, $\bar{j}$ and $\bar{k}$ are unit vectors along the X^- axis, Y^- axis and Z^- axis, respectively, and by the coordinates of P we have $\overline{P_1P} = \overline{OR} = z\bar{k}$.

Similarly $\overline{QP_1} = \overline{OS} = y\overline{j}$ and $\overline{OQ} = x\overline{i}$.

Therefore, it follows that $\overline{OP_1} = \overline{OQ} + \overline{QP_1} = x\overline{i} + y\overline{j}$ and $\overline{OP} = \overline{OP_1} + \overline{P_1P} = x\overline{i} + y\overline{j} + z\overline{k}$.

Hence the position vector of P with respect to O is given by $\overline{OP} = \overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$.

This form of any vector is called its component form. Here x, y, z are called the *scalar components* of $\overline{r}$ and $x\overline{i}, y\overline{j}, z\overline{k}$ are called the *vector components* of $\overline{r}$ along the respective axes. Sometimes x, y, z are also termed as the *rectangular components* of $\overline{r}$.

4.5.16 Length of a vector in terms of its components:

The length of any vector $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1

$$|OP_1| = \sqrt{|OQ|^2 + |QP_1|^2} = \sqrt{x^2 + y^2}$$

and in the right angle triangle OP_1P, we have

$$|OP| = \sqrt{|OP_1|^2 + |P_1P|^2} = \sqrt{x^2 + y^2 + z^2}$$

Hence the length of any vector $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$, is given by

$$|\overline{r}| = |x\overline{i} + y\overline{j} + z\overline{k}| = \sqrt{x^2 + y^2 + z^2}$$

4.5.17 Note:

If $\overline{a}$ and $\overline{b}$ are any two vectors given in the component form $a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and $b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$ respectively, then the following results of addition, subtraction and scalar multiplication to vectors hold in component form:

(i) The sum (or resultant) of the vectors $\overline{a}$ and $\overline{b}$ is given by
$\overline{a} + \overline{b} = (a_1 + b_1)\overline{i} + (a_2 + b_2)\overline{j} + (a_3 + b_3)\overline{k}$

(ii) The difference of the vectors $\overline{a}$ and $\overline{b}$ is given by
$\overline{a} - \overline{b} = (a_1 - b_1)\overline{i} + (a_2 - b_2)\overline{j} + (a_3 - b_3)\overline{k}$

(iii) The vectors $\overline{a}$ and $\overline{b}$ are equal if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

(iv) The multiplication of vectors $\overline{a}$ by any scalar is given by
$\lambda\overline{a} = \lambda a_1\overline{i} + \lambda a_2\overline{j} + \lambda a_3\overline{k}$

4.5.18 Vectors joining two points:

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overline{P_1P_2}$.

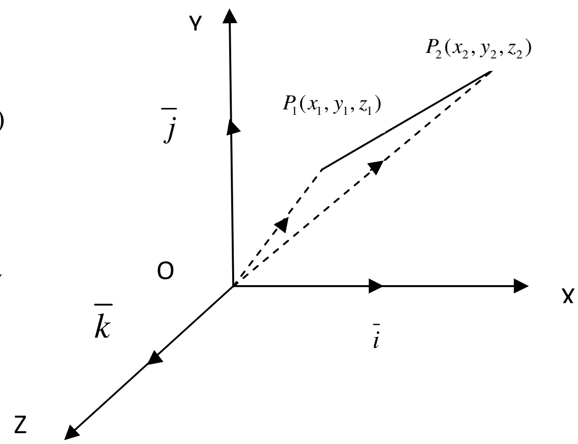
Joining the points P_1 and P_2 with the origin O and applying triangle law, to the triangle OP_1P_2, we have $\overline{OP_1} + \overline{P_1P_2} = \overline{OP_2}$

Using the properties of vector addition, the above equation becomes

$$\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1}$$

i.e, $\overline{P_1P_2} = (x_2\overline{i} + y_2\overline{j} + z_2\overline{k}) - (x_1\overline{i} + y_1\overline{j} + z_1\overline{k})$

$$= (x_2 - x_1)\overline{i} + (y_2 - y_1)\overline{j} + (z_2 - z_1)\overline{k}$$



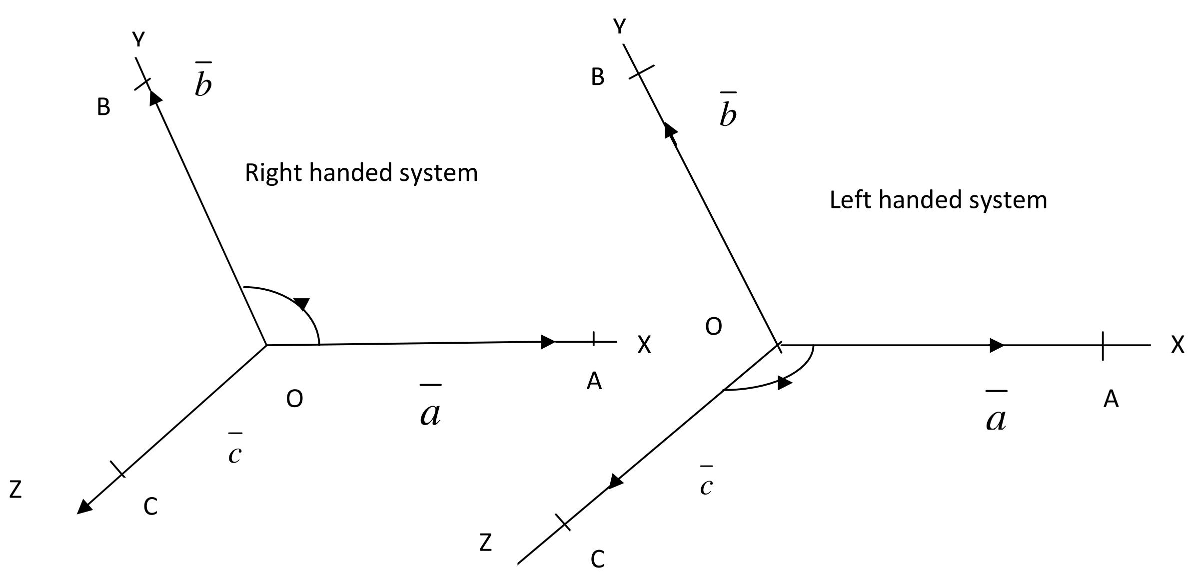
Therefore the magnitude of the vector $\overline{P_1P_2}$ is given by

$$\left|\overline{P_1P_2}\right| = \left|(x_2 - x_1)\overline{i} + (y_2 - y_1)\overline{j} + (z_2 - z_1)\overline{k}\right|$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4.5.19 Definition (Right handed and left handed triads):

Let $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}, \overline{OC} = \overline{c}$ be three non-coplanar vectors.

Viewing from the point C, if the rotation of $\overline{OA}$ to $\overline{OB}$ does not exceed angle 180^0 in anti-clock sense, then $\overline{a}, \overline{b}, \overline{c}$ are said to form a *right handed system of vectors* and we say simply that $(\overline{a}, \overline{b}, \overline{c})$ is a right handed system. If $(\overline{a}, \overline{b}, \overline{c})$ is not a right handed system ten it is called a *left handed system of vectors*.

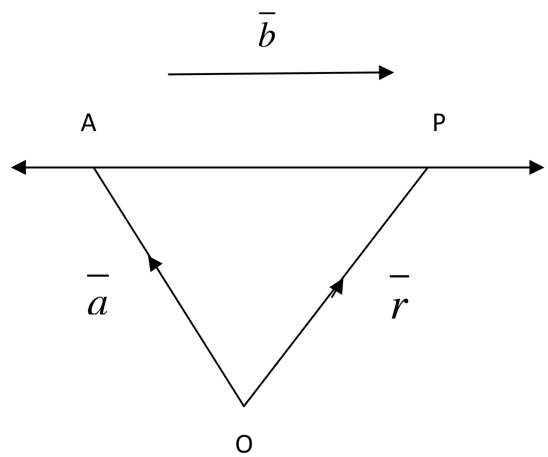


4.6 Vector Equations of Line and Plane:

In this section we discuss the parametric vector equations of a straight line and plane which are useful in solving certain geometric problems. Hereafter $P(\bar{r})$ means, P is a point with position vector $\bar{r}$.

4.6.1 Theorem:

The vector equation of the straight line passing through the point $A(\bar{a})$ and parallel to the vector $\bar{b}$ is $\bar{r} = \bar{a} + t\bar{b},\ t \in R.$



Proof: Let $P(\bar{r})$ be any point on the line.

Then $\overline{AP}$ and $\bar{b}$ are collinear vectors.

$\therefore \bar{r} - \bar{a} = t\bar{b}$ for some $t \in R.$

Conversely suppose $\bar{r} = \bar{a} + t\bar{b},\ t \in R.$

Then $\bar{r} - \bar{a} = t\bar{b}$

$\therefore \overline{AP} = t\bar{b}$

$\therefore \overline{AP}$ and $\bar{b}$ are collinear vectors.

$\therefore P(\bar{r})$ lies on the line.

4.6.2 Corollary:

The equation of the line passing through origin O and parallel to the vector $\bar{b}$ is $\bar{r} = t\bar{b},\ t \in R.$

4.6.3 Cartesian form:

Cartesian equation for the line passing through $A(x_1, y_1, z_1)$ and parallel to the vector $\bar{b} = l\bar{i} + m\bar{j} + n\bar{k}$ is $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}.$

Fix the origin at O so that $\overline{OA} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}.$

If $P(\bar{r}) = (x, y, z)$ so that $\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$ then P lies on the above line

$\Leftrightarrow \bar{r} = \bar{a} + t\bar{b}$ for some $t \in R.$

Now $\bar{r} = \bar{a} + t\bar{b} \Leftrightarrow x\bar{i} + y\bar{j} + z\bar{k} = (x_1\bar{i} + y_1\bar{j} + z_1\bar{k}) + t(l\bar{i} + m\bar{j} + n\bar{k})$

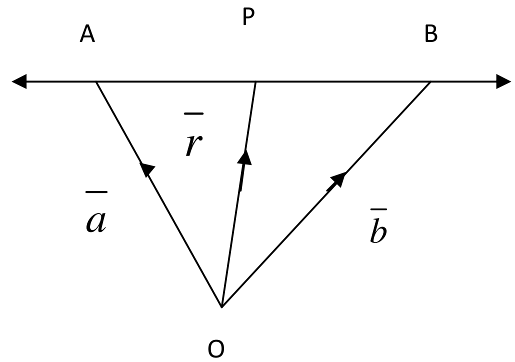
$\Leftrightarrow x = x_1 + lt,\ y = y_1 + mt,\ z = z_1 + nt \Leftrightarrow \dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} = t$

We represent these equations by $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$.

4.6.4 Theorem:

The vector equation of the plane passing through the points $A(\bar{a})$ and $B(\bar{b})$ is $\bar{r} = (1 - t)\bar{a} + t\bar{b},\ t \in R.$

Proof: Let O be the origin so that $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$



Let $P(\bar{r})$ be any point on the line.

$\Leftrightarrow \overline{AP}$ and $\overline{AB}$ are collinear vectors.

$\Leftrightarrow \overline{AP} = t\ \overline{AB}, t \in R$

$\Leftrightarrow \bar{r} - \bar{a} = t(\bar{b} - \bar{a}), t \in R$

$\Leftrightarrow \bar{r} = (1 - t)\bar{a} + t\bar{b},\ t \in R.$

4.6.5 Cartesian form:

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, $P(\bar{r})$ be any point on the line.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Then P lies on the above line $\overline{AB}$.

$\Leftrightarrow \bar{r} = (1 - t)\bar{a} + t\bar{b}$ for some $t \in R.$

$\Leftrightarrow x\bar{i} + y\bar{j} + z\bar{k} = (1 - t)(x_1\bar{i} + y_1\bar{j} + z_1\bar{k}) + t(x_2\bar{i} + y_2\bar{j} + z_2\bar{k})$

$\Leftrightarrow (x - x_1)\bar{i} + (y - y_1)\bar{j} + (z - z_1)\bar{k} = t\left[(x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}\right]$

$\Leftrightarrow (x - x_1) = t(x_2 - x_1),\ (y - y_1) = t(y_2 - y_1),\ (z - z_1) = t(z_2 - z_1)$

$\Leftrightarrow \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1} = t$

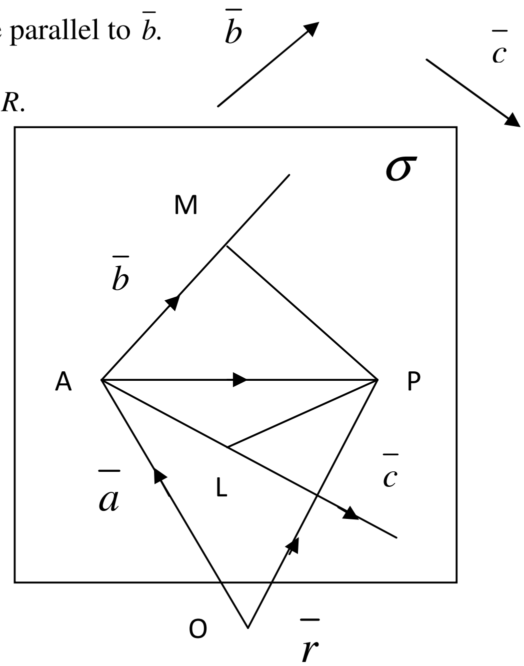
We represent these equations by $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

4.6.6 Theorem:

The vector equation of the plane passing through the points $A(\overline{a})$ and parallel to the vectors $\overline{b}$ and $\overline{c}$ is $\overline{r} = \overline{a} + t\overline{b} + s\overline{c};\ t, s \in R.$

Proof: Let σ be the plane passing through the point $A(\overline{a})$ and parallel to the vectors $\overline{b}$ and $\overline{c}$ and $P(\overline{r})$ be any point in σ.

In the plane σ, through the point A, draw lines parallel to the vectors $\overline{b}$ and $\overline{c}$. With the line segment $\overline{AP}$ as diagonal, complete the parallelogram $ALPM$ in σ with the point L on the line parallel to $\overline{c}$ and M on the line parallel to $\overline{b}$.



$\therefore \overline{AL} = s\overline{c}$, for some $s \in R$ and $\overline{AM} = t\overline{b}$, for some $t \in R$.

Now $\overline{r} - \overline{a} = \overline{AP} = \overline{AL} + \overline{AM} = s\overline{c} + t\overline{b}$

$\therefore \overline{r} = \overline{a} + t\overline{b} + s\overline{c};\ t, s \in R.$

Conversely suppose $\overline{r} = \overline{a} + t\overline{b} + s\overline{c};\ t, s \in R.$

Then $\overline{r} - \overline{a} = t\overline{b} + s\overline{c}$

$\therefore \overline{AP} = t\overline{b} + s\overline{c}$

$\therefore P(\overline{r})$ lies in the plane σ.

4.6.7 Corollary:

The vector equation of the plane passing through the points $A(\overline{a}), B(\overline{b})$ and parallel to the vector $\overline{c}$ is $\overline{r} = (1-t)\overline{a} + t\overline{b} + s\overline{c};\ t, s \in R.$

Proof: In Theorem 4.6.6, replace the vector $\overline{b}$ with $\overline{AB}$.

Then the equation of the plane is $\overline{r} = \overline{a} + t\overline{AB} + s\overline{c}$

$$i.e\quad \overline{r} = \overline{a} + t(\overline{b} - \overline{a}) + s\overline{c}$$

$$i.e\quad \overline{r} = (1-t)\overline{a} + t\overline{b} + s\overline{c} \text{ for some } t, s \in R.$$

4.6.8 Corollary:

The vector equation of the plane passing through the points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is $\bar{r} = (1-t-s)\bar{a} + t\bar{b} + s\bar{c};\ t, s \in R.$

Proof: In Theorem 4.6.6, replace the vector $\bar{b}$ with $\overline{AB}$ and $\bar{c}$ with $\overline{AC}$.

Then the equation of the plane is $\bar{r} = \bar{a} + t\overline{AB} + s\overline{AC}$

$$i.e\quad \bar{r} = \bar{a} + t(\bar{b} - \bar{a}) + s(\bar{c} - \bar{a})$$

$$i.e\quad \bar{r} = (1-t-s)\bar{a} + t\bar{b} + s\bar{c} \text{ for some } t, s \in R.$$

4.6.9 Theorem:

Three points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ are collinear if and only if there exist scalars x, y, z (not all zero) such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0.$

Proof: Suppose $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ are collinear.

Then $\overline{AB} = \lambda\overline{BC}$ for some $\lambda \in R.$

$\Rightarrow \bar{b} - \bar{a} = \lambda(\bar{c} - \bar{a})$

$\Rightarrow \bar{a} + (-1-\lambda)\bar{b} + \lambda\bar{c} = \bar{0}$

Take $x = 1, y = -1-\lambda$ and $z = \lambda$ so that $x + y + z = 0$ and $x \neq 0.$

Conversely, let x, y, z be scalars such that at least one of them is not zero, $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0.$

Suppose $z \neq 0.$ Since $z = -x - y$ and $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$

$\Rightarrow x\bar{a} + y\bar{b} - (x+y)\bar{c} = \bar{0}$

$\Rightarrow x(\bar{a} - \bar{c}) + y(\bar{b} - \bar{c}) = \bar{0}$

$\therefore x\ \overline{CA} + y\ \overline{CB} = \bar{0}$ and $x + y \neq 0.$

$\therefore \overline{CA}$ and $\overline{CB}$ are collinear vectors and hence $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ are collinear points.

4.6.10 Theorem:

Four points $A(\overline{a}),\ B(\overline{b}), C(\overline{c})$ and $D(\overline{d})$ are coplanar if and only if there exist scalars x, y, z and u (not all zero) such that $x\overline{a} + y\overline{b} + z\overline{c} + u\overline{d} = \overline{0}$ and $x + y + z + u = 0.$

Proof: Suppose $A(\overline{a}),\ B(\overline{b}), C(\overline{c})$ and $D(\overline{d})$ are coplanar.

Then the vectors $\overline{AB},\ \overline{AC}$ and $\overline{AD}$ are coplanar.

Therefore there exist scalars λ and μ such that $\overline{AD} = \lambda\overline{AB} + \mu\,\overline{AC}$

i.e $\overline{d} - \overline{a} = \lambda(\overline{b} - \overline{a}) + \mu(\overline{c} - \overline{a})$

Take $x = 1 - \lambda - \mu, y = \lambda, z = \mu$ and $u = -1$.

Then $x\overline{a} + y\overline{b} + z\overline{c} + u\overline{d} = \overline{0}$ and $x + y + z + u = 0.$

Conversely, let x, y, z and u be scalars such that at least one of them is not zero, $x\overline{a} + y\overline{b} + z\overline{c} + u\overline{d} = \overline{0}$ and $x + y + z + u = 0.$

Suppose $u \neq 0.$ Since $u = -x - y - z$ and $x\overline{a} + y\overline{b} + z\overline{c} + u\overline{d} = \overline{0}$

$$\Rightarrow x\overline{a} + y\overline{b} + z\overline{c} - (x + y + z)\overline{d} = \overline{0}$$

$$\Rightarrow x(\overline{a} - \overline{d}) + y(\overline{b} - \overline{d}) + z(\overline{c} - \overline{d}) = \overline{0}$$

$\therefore x\ \overline{DA} + y\ \overline{DB} + z\ \overline{DC} = \overline{0}$ and one of x, y, z is non zero. ($\because x + y + z \neq 0$)

$\therefore \overline{DA},\ \overline{DB}$ and $\overline{DC}$ are coplanar vectors

$\therefore A(\overline{a}),\ B(\overline{b}), C(\overline{c})$ and $D(\overline{d})$ are coplanar.

4.6.11 Solved Problems:

1. **Problem:** If $\overline{a} = \overline{i} + 2\overline{j} + 3\overline{k}$ and $\overline{b} = 3\overline{i} + \overline{j}$ then find the unit vector in the direction of $\overline{a} + \overline{b}.$

Solution: Given $\overline{a} = \overline{i} + 2\overline{j} + 3\overline{k}$ and $\overline{b} = 3\overline{i} + \overline{j}$

We have $\overline{a} + \overline{b} = \overline{i} + 2\overline{j} + 3\overline{k} + 3\overline{i} + \overline{j} = 4\overline{i} + 3\overline{j} + 3\overline{k}$

$$\left|\overline{a} + \overline{b}\right| = \left|4\overline{i} + 3\overline{j} + 3\overline{k}\right| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{16 + 9 + 9} = \sqrt{34}$$

Unit vector in the direction of $\bar{a}+\bar{b}$ is $\dfrac{\bar{a}+\bar{b}}{|\bar{a}+\bar{b}|}=\dfrac{4\bar{i}+3\bar{j}+3\bar{k}}{\sqrt{34}}$

2. Problem: If $-3\bar{i}+4\bar{j}+\lambda\bar{k}$ and $\mu\bar{i}+8\bar{j}+6\bar{k}$ are collinear vectors then find λ and μ

Solution: Let $\bar{a}=-3\bar{i}+4\bar{j}+\lambda\bar{k}$ and $\bar{b}=\mu\bar{i}+8\bar{j}+6\bar{k}$

Two vectors $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ and $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ are collinear then

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}\Rightarrow\frac{-3}{\mu}=\frac{4}{8}=\frac{\lambda}{6}\Rightarrow\frac{-3}{\mu}=\frac{4}{8},\frac{\lambda}{6}=\frac{4}{8}\Rightarrow\frac{-3}{\mu}=\frac{1}{2},\frac{\lambda}{6}=\frac{1}{2}$$

$$\Rightarrow\mu=-6,\lambda=3$$

3. Problem: If the points whose position vectors are $3\bar{i}-2\bar{j}-\bar{k},\ 2\bar{i}+3\bar{j}-4\bar{k},$

$-\bar{i}+\bar{j}+2\bar{k}$ and $4\bar{i}+5\bar{j}+\lambda\bar{k}$ are coplanar then show that $\lambda=\dfrac{-146}{17}$.

Solution: Let A,B,C,D be the given points respectively

$$\therefore\overline{OA}=3\bar{i}-2\bar{j}-\bar{k},\ \overline{OB}=2\bar{i}+3\bar{j}-4\bar{k},\ \overline{OC}=-\bar{i}+\bar{j}+2\bar{k},\ \overline{OD}=4\bar{i}+5\bar{j}+\lambda\bar{k}$$

We have $\overline{AB}=\overline{OB}-\overline{OA}=(2\bar{i}+3\bar{j}-4\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-\bar{i}+5\bar{j}-3\bar{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=(-\bar{i}+\bar{j}+2\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=-4\bar{i}+3\bar{j}+3\bar{k}$

$\overline{AD}=\overline{OD}-\overline{OA}=(4\bar{i}+5\bar{j}+\lambda\bar{k})-(3\bar{i}-2\bar{j}-\bar{k})=\bar{i}+7\bar{j}+(\lambda+1)\bar{k}$

Since A,B,C,D are coplanar $\begin{vmatrix}-1 & 5 & -3\\ -4 & 3 & 3\\ 1 & 7 & \lambda+1\end{vmatrix}=0$

$$\Rightarrow -1\begin{vmatrix}3 & 3\\ 7 & \lambda+1\end{vmatrix}-5\begin{vmatrix}-4 & 3\\ 1 & \lambda+1\end{vmatrix}-3\begin{vmatrix}-4 & 3\\ 1 & 7\end{vmatrix}=0$$

$$\Rightarrow -1\big(3(\lambda+1)-21\big)-5\big(-4(\lambda+1)-3\big)-3\big(-28-3\big)=0$$

$$\Rightarrow -1\big(3\lambda-18\big)-5\big(-4\lambda-7\big)-3\big(-31\big)=0\Rightarrow -3\lambda+18+20\lambda+35+93=0$$

$$\Rightarrow 17\lambda+146=0\Rightarrow\lambda=\frac{-146}{17}$$

4. Problem: If $\overline{OA}=\bar{i}+\bar{j}+\bar{k},\ \overline{AB}=3\bar{i}-2\bar{j}+\bar{k},\ \overline{BC}=\bar{i}+2\bar{j}-2\bar{k}$ and $\overline{CD}=2\bar{i}+\bar{j}+3\bar{k}$

then find the vector $\overline{OD}$.

Solution: Given $\overline{OA} = \bar{i} + \bar{j} + \bar{k}$, $\overline{AB} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\overline{BC} = \bar{i} + 2\bar{j} - 2\bar{k}$ and $\overline{CD} = 2\bar{i} + \bar{j} + 3\bar{k}$

We have $\overline{AB} = \overline{OB} - \overline{OA} \Rightarrow \overline{OB} = \overline{AB} + \overline{OA}$

$\Rightarrow \overline{OB} = 3\bar{i} - 2\bar{j} + \bar{k} + \bar{i} + \bar{j} + \bar{k} \Rightarrow \overline{OB} = 4\bar{i} - \bar{j} + 2\bar{k}$

We have $\overline{BC} = \overline{OC} - \overline{OB} \Rightarrow \overline{OC} = \overline{BC} + \overline{OB}$

$\Rightarrow \overline{OC} = \bar{i} + 2\bar{j} - 2\bar{k} + 4\bar{i} - \bar{j} + 2\bar{k} \Rightarrow \overline{OC} = 5\bar{i} + \bar{j}$

We have $\overline{CD} = \overline{OD} - \overline{OC} \Rightarrow \overline{OD} = \overline{CD} + \overline{OC}$

$\Rightarrow \overline{OD} = 2\bar{i} + \bar{j} + 3\bar{k} + 5\bar{i} + \bar{j} \Rightarrow \overline{OD} = 7\bar{i} + 2\bar{j} + 3\bar{k}$

5. Problem: If $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{j} + 2\bar{k}$ then find the unit vector in the opposite direction of $\bar{a} + \bar{b} + \bar{c}$.

Solution: Given $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{j} + 2\bar{k}$

We have $\bar{a} + \bar{b} + \bar{c} = 2\bar{i} + 4\bar{j} - 5\bar{k} + \bar{i} + \bar{j} + \bar{k} + \bar{j} + 2\bar{k} = 3\bar{i} + 6\bar{j} - 2\bar{k}$

$\left|\bar{a} + \bar{b} + \bar{c}\right| = \left|3\bar{i} + 6\bar{j} - 2\bar{k}\right| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$

Unit vector in the direction opposite to $\bar{a} + \bar{b} + \bar{c}$ is $-\left(\dfrac{\bar{a} + \bar{b} + \bar{c}}{\left|\bar{a} + \bar{b} + \bar{c}\right|}\right) = -\left(\dfrac{3\bar{i} + 6\bar{j} - 2\bar{k}}{7}\right)$

$$= \frac{-3\bar{i} - 6\bar{j} + 2\bar{k}}{7}$$

6. Problem: Find the vector equation of the plane passing through the points $\bar{i} - 2\bar{j} + 5\bar{k}$, $-5\bar{j} - \bar{k}$ and $-3\bar{i} + 5\bar{j}$

Solution: Let $\bar{a} = \bar{i} - 2\bar{j} + 5\bar{k}$, $\bar{b} = -5\bar{j} - \bar{k}$ and $\bar{c} = -3\bar{i} + 5\bar{j}$

The vector equation of the plane passing through the points $\bar{a}, \bar{b}, \bar{c}$ is

$\bar{r} = (1 - s - t)\bar{a} + s\bar{b} + t\bar{c}$ where $s, t \in R$

$$\Rightarrow \bar{r} = (1-s-t)(\bar{i}-2\bar{j}+5\bar{k}) + s(-5\bar{j}-\bar{k}) + t(-3\bar{i}+5\bar{j})$$

$$\Rightarrow \bar{r} = (1-s-t-3t)\bar{i} + (-2(1-s-t)-5s+5t)\bar{j} + (5(1-s-t)-s)\bar{k}$$

$$\Rightarrow \bar{r} = (1-s-4t)\bar{i} + (-2-3s+7t)\bar{j} + (5-6s-5t)\bar{k}$$

7. Problem: If $\bar{a} = 2\bar{i} - \bar{j} + \bar{k}$ and $\bar{b} = \bar{i} - 3\bar{j} - 5\bar{k}$ then find the vector $\bar{c}$ such that $\bar{a}$, $\bar{b}$ and $\bar{c}$ forms the sides of a triangle.

Solution: Let $\bar{a} = 2\bar{i} - \bar{j} + \bar{k}$ and $\bar{b} = \bar{i} - 3\bar{j} - 5\bar{k}$

Given that $\bar{a}$, $\bar{b}$ and $\bar{c}$ forms the sides of a triangle $\Rightarrow \bar{a} + \bar{b} + \bar{c} = \bar{0}$

$$\Rightarrow \bar{c} = -(\bar{a}+\bar{b}) \Rightarrow \bar{c} = -(2\bar{i}-\bar{j}+\bar{k}+\bar{i}-3\bar{j}-5\bar{k}) \Rightarrow \bar{c} = -(3\bar{i}-4\bar{j}-4\bar{k})$$

$$\Rightarrow \bar{c} = -3\bar{i} + 4\bar{j} + 4\bar{k}$$

8. Problem: Find the point of intersection of the line $\bar{r} = 2\bar{a} + \bar{b} + t(\bar{b} - \bar{c})$ and the plane $\bar{r} = \bar{a} + x(\bar{b} + \bar{c}) + y(\bar{a} + 2\bar{b} - \bar{c})$ where $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar vectors.

Solution: At the point of intersection of the line and plane, we have

$$2\bar{a} + \bar{b} + t(\bar{b} - \bar{c}) = \bar{a} + x(\bar{b} + \bar{c}) + y(\bar{a} + 2\bar{b} - \bar{c})$$

$$\Rightarrow 2\bar{a} + (1+t)\bar{b} - t\bar{c} = (1+y)\bar{a} + (x+2y)\bar{b} + (x-y)\bar{c}$$

On comparing the corresponding coefficients,

$$\Rightarrow 2 = 1+y, 1+t = x+2y, -t = x-y$$

$$\Rightarrow 1 = y, t = x+2y-1, -t = x-y$$

$$\Rightarrow y = 1, x = 0, t = 1$$

The point of intersection is $2\bar{a} + 2\bar{b} - \bar{c}$

Exercise 4

1. Find unit vector in the direction of $\bar{a} = 2\bar{i} + 3\bar{j} + \bar{k}$.
2. Find a vector in the direction of vector $\bar{a} = \bar{i} - 2\bar{j}$ that has magnitude 7 units.
3. Find unit vector in the direction of sum of the vectors $2\bar{i} + 2\bar{j} - 5\bar{k}$ and $2\bar{i} + \bar{j} + 3\bar{k}$.

4. Find the direction ratios and direction cosines of the vector $\bar{i}+\bar{j}-2\bar{k}.$

5. Consider two points P and Q with position vectors $\overline{OP}=3\bar{a}-2\bar{b}$ and $\overline{OQ}=\bar{a}+\bar{b}.$ Find the position vector of a point R which divides the line joining P and Q in the ratio $2:1$, (i) internally and (ii) externally.

6. Show that the points $A(2\bar{i}-\bar{j}+\bar{k}), B(\bar{i}-3\bar{j}-5\bar{k})$ and $C(3\bar{i}-4\bar{j}-4\bar{k})$ are the vertices of a right angled triangle.

7. If $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors. Prove that $-\bar{a}+4\bar{b}-3\bar{c}, 3\bar{a}+2\bar{b}-5\bar{c}, -3\bar{a}+8\bar{b}-5\bar{c}, -3\bar{a}+2\bar{b}+\bar{c}$ are coplanar.

8. $OABC$ is a parallelogram. If $\overline{OA}=\bar{a}$ and $\overline{OC}=\bar{c}$ then find the vector equation of the side $\overline{BC}$.

9. Find the vector equation of the line passing through the point $2\bar{i}+3\bar{j}+\bar{k}$ and parallel to the vector $4\bar{i}-2\bar{j}+3\bar{k}.$

10. Find the vector equation of the line joining the points $2\bar{i}+\bar{j}+3\bar{k}$ and $-4\bar{i}+3\bar{j}-\bar{k}.$

11. Find the vector equation of the plane passing through the points $\bar{i}-2\bar{j}+5\bar{k}, -5\bar{j}-\bar{k}$ and $-3\bar{i}+5\bar{j}.$

Key Concepts

1. The set of all ordered triads (a,b,c) of real numbers is denoted by R^3.

2. The position vector of the point with respect to O is denoted by $\bar{r}$. The magnitude of $\overline{OP}$, is given by $|\overline{OP}|=|\bar{r}|=\sqrt{x^2+y^2+z^2}$.

3. Let α,β,γ be the angles made by the vector $\bar{r}$ with the positive direction (counter clockwise direction) of X,Y,Z axes respectively. Then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the direction cosines of the vector $\bar{r}$. These direction cosines are usually denoted by l,m,n respectively. $l^2+m^2+n^2=1$

4. A vector whose magnitude is unity (*i.e* 1 unit) is called a *unit vector*. It is denoted by $\bar{e}$. The unit vector in the direction of a given vector $\bar{a}$ is usually denoted by $\hat{a}$.

5. Two vectors $\bar{a}$ and $\bar{b}$ are said to be *equal vectors* and written as $\bar{a}=\bar{b}$, if they have the same magnitude and direction, regardless of the positions of their initial points.

6. Two or more vectors are said to be *collinear vectors* if they are parallel to the same line, irrespective of their magnitudes and direction.

7. Two or more vectors having same initial point are called *co-initial vectors* and same final point are called *co-terminal vectors.*

8. Vectors whose supports are in the same plane or parallel to the same plane are called *coplanar vectors*. Vectors which are not coplanar are called *non-coplanar vectors.*

9. Triangle law of vector addition $\overline{AC} = \overline{AB} + \overline{BC}$

10. Parallelogram law of vector addition $\overline{OA} + \overline{AC} = \overline{OC}$ or $\overline{OA} + \overline{OB} = \overline{OC} (\because \overline{OB} = \overline{AC})$

11. Properties of vector addition:

(i) *Commutative property*: For any two vectors $\overline{a}$ and $\overline{b}$, $\overline{a} + \overline{b} = \overline{b} + \overline{a}$.

(ii) *Associative property*: For any three vectors $\overline{a}, \overline{b}$ and $\overline{c}$, $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$.

(iii) *Identity property*: For any vector $\overline{a}$, $\overline{a} + \overline{0} = \overline{0} + \overline{a} = \overline{a}$. Here the $\overline{0}$ is the additive identity vector.

(iv) *Inverse property*: For any vector $\overline{a}$, $\overline{a} + \overline{b} = \overline{b} + \overline{a} = \overline{0}$. Here the vector $\overline{b}$ is the additive inverse of the vector $\overline{a}$.

12. Let $\overline{a}$ be a given non-zero vector and λ a scalar. Then the product of the vector $\overline{a}$ by a scalar λ, denoted as $\lambda\overline{a}$, is defined as a vector $\lambda\overline{a}$ collinear with $\overline{a}$.

(i) If $\lambda = -1$ then $\lambda\overline{a} = -\overline{a}$, (ii) $\overline{a} + (-\overline{a}) = (-\overline{a}) + \overline{a} = \overline{0}$.

(iii) If $\lambda = \frac{1}{|\overline{a}|}$, provided $\overline{a} \neq \overline{0}$ then $|\lambda\overline{a}| = |\lambda||\overline{a}| = \frac{1}{|\overline{a}|}|\overline{a}| = 1$.

13. The unit vector $\hat{a}$ in the direction of $\overline{a}$ is $\hat{a} = \frac{1}{|\overline{a}|}\overline{a}$.

14. (i) If $\overline{a}$ is a vector and λ is a scalar, then $(-\lambda)\overline{a} = \lambda(-\overline{a}) = -(\lambda\overline{a})$.

(ii) If $\overline{a}$ is a vector and m, n are scalars, then $m(n\overline{a}) = (mn)\overline{a} = (nm)\overline{a} = n(m\overline{a})$.

(iii) If $\overline{a}$ is a vector and m, n are scalars, then $(m+n)\overline{a} = m\overline{a} + n\overline{a}$.

(iv) If m is a scalar and $\overline{a}, \overline{b}$ are any two vectors, then $m(\overline{a} + \overline{b}) = m\overline{a} + m\overline{b}$.

15. Two vectors are collinear (parallel) iff one is a scalar multiple of the other.

16. Let $\overline{a}$ and $\overline{b}$ be two non-zero vectors. Let $\overline{OA} = \overline{a}$ and $\overline{OB} = \overline{b}$. Then the measure of $\angle AOB$ which lies between 0^0 and 180^0 is called the angle between $\overline{a}$ and $\overline{b}$ and is denoted by $(\overline{a}, \overline{b})$.

17. Let $\bar{a}$ and $\bar{b}$ be non-zero vectors. Then

$(i)\,(\bar{a},\bar{b}) = 0^0 \Leftrightarrow \bar{a}$ and $\bar{b}$ are like vectors.

$(ii)\,(\bar{a},\bar{b}) = 180^0 \Leftrightarrow \bar{a}$ and $\bar{b}$ are unlike vectors.

$(iii)\,(\bar{a},\bar{b}) = 0^0$ *or* $180^0 \Leftrightarrow \bar{a}$ and $\bar{b}$ are collinear vectors.

(iv) If $(\bar{a},\bar{b}) = 90^0$ then $\bar{a}$ and $\bar{b}$ are perpendicular vectors.

18. Let $\bar{a}$ and $\bar{b}$ be non-zero vectors and m,n be positive scalars. Then

$(i)\,(\bar{a},\bar{b}) = (\bar{b},\bar{a})$, $(ii)\,(\bar{a},\bar{b}) = (-\bar{a},-\bar{b})$, $(iii)\,(\bar{a},-\bar{b}) = (-\bar{a},\bar{b}) = 180^0 - (\bar{a},\bar{b})$,

$(iv)\,(\bar{a},\bar{b}) = (m\bar{a},n\bar{b})$, $(v)\,(m\bar{a},-n\bar{b}) = (-m\bar{a},n\bar{b}) = 180^0 - (m\bar{a},n\bar{b})$.

19. Let $\bar{a}$ and $\bar{b}$ be position vectors of the points A and B with respect to the origin O.

(i) If P divides the line segment AB in the ratio $m:n\,(m+n \neq 0)$ internally then the position vector of P is $\dfrac{m\bar{b}+n\bar{a}}{m+n}$.

(ii) If P divides the line segment AB in the ratio $m:n\,(m-n \neq 0)$ externally then the position vector of P is $\dfrac{m\bar{b}-n\bar{a}}{m-n}$.

(iii) If P is the mid point of AB then $m = n$, and hence the position vector of P is $\bar{r} = \overline{OP} = \dfrac{\bar{a}+\bar{b}}{2}$.

20. Let $\bar{a}$ and $\bar{b}$ be vectors and $x,\ y$ are scalars then $x\bar{a} + y\bar{b} = \bar{0} \Leftrightarrow x = y = 0$.

21. Let $\bar{a},\bar{b}$ and $\bar{c}$ are non-coplanar vectors, then $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0} \Leftrightarrow x = y = z = 0$.

22. Let $\overline{a_1},\overline{a_2},\overline{a_3},...\overline{a_n}$ be vectors and $x_1, x_2, x_3,...x_n$ be scalars. Then the vector $x_1\overline{a_1} + x_2\overline{a_2} + x_3\overline{a_3} + ... + x_n\overline{a_n}$ is called the *linear combination* of the vectors $\overline{a_1},\overline{a_2},\overline{a_3},...\overline{a_n}$.

23. (i) If $\bar{a},\bar{b}$ are non-collinear vectors, then every vector in the plane can be expressed as linear combination of $\bar{a}$ and $\bar{b}$ in one and only one way.

(ii) If $\bar{a},\bar{b}$ and $\bar{c}$ are non-coplanar vectors, then every vector in the space can be expressed as linear combination of $\bar{a},\bar{b}$ and $\bar{c}$ in one and only one way.

(iii) Three vectors are coplanar vectors if and only if one of them is a scalar multiple of the other two.

24. The ordered triad of non-coplanar vectors $\bar{a},\bar{b},\bar{c}$ is $(\bar{a},\bar{b},\bar{c})$. If $\bar{r}$ is the position vector scalars such that $\bar{r}=x\bar{a}+y\bar{b}+z\bar{c}$. These scalars x,y,z are called the components of $\bar{r}$ with respect to the triad $(\bar{a},\bar{b},\bar{c})$.

25. The length of any vector $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$, is given by $|\bar{r}|=|x\bar{i}+y\bar{j}+z\bar{k}|=\sqrt{x^2+y^2+z^2}$

26. If $\bar{a}$ and $\bar{b}$ are any two vectors given in the component form $a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ and $b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ respectively.

(i) The sum of the vectors $\bar{a}$ and $\bar{b}$ is $\bar{a}+\bar{b}=(a_1+b_1)\bar{i}+(a_2+b_2)\bar{j}+(a_3+b_3)\bar{k}$

(ii) The difference of the vectors $\bar{a}$ and $\bar{b}$ is $\bar{a}-\bar{b}=(a_1-b_1)\bar{i}+(a_2-b_2)\bar{j}+(a_3-b_3)\bar{k}$

(iii) The vectors $\bar{a}$ and $\bar{b}$ are equal if and only if $a_1=b_1, a_2=b_2, a_3=b_3$.

(iv) The multiplication of vectors $\bar{a}$ by any scalar is $\lambda\bar{a}=\lambda a_1\bar{i}+\lambda a_2\bar{j}+\lambda a_3\bar{k}$

27. If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overline{P_1P_2}$. *i.e*, $\overline{P_1P_2}=(x_2-x_1)\bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}$ and the magnitude is

$$|\overline{P_1P_2}|=|(x_2-x_1)\bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

28. Let $\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{OC}=\bar{c}$ be three non-coplanar vectors. From the point C, if the rotation of $\overline{OA}$ to $\overline{OB}$ does not exceed angle 180^0 in anti-clock sense, then $\bar{a},\bar{b},\bar{c}$ are said to form a *right handed system of vectors* and we say simply that $(\bar{a},\bar{b},\bar{c})$ is a right handed system. If $(\bar{a},\bar{b},\bar{c})$ is not a right handed system ten it is called a *left handed system of vectors*.

29. The vector equation of the straight line passing through the point $A(\bar{a})$ and parallel to the vector $\bar{b}$ is $\bar{r}=\bar{a}+t\bar{b},\ t\in R$.

30. The equation of the line passing through origin O and parallel to the vector $\bar{b}$ is $\bar{r}=t\bar{b},\ t\in R$.

31. Cartesian equation for the line passing through $A(x_1,y_1,z_1)$ and parallel to the vector $\bar{b}=l\bar{i}+m\bar{j}+n\bar{k}$ is $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$.

32. The vector equation of the plane passing through the points $A(\bar{a})$ and $B(\bar{b})$ is $\bar{r} = (1-t)\bar{a} + t\bar{b},\ t \in R.$

33. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, $P(\bar{r})$ be any point on the line. Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}.$ Then P lies on the above line $\overline{AB}$. The equation of $\overline{AB}$ is $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}.$

34. The vector equation of the plane passing through the points $A(\bar{a})$ and parallel to the vectors $\bar{b}$ and $\bar{c}$ is $\bar{r} = \bar{a} + t\bar{b} + s\bar{c};\ t, s \in R.$

35. The vector equation of the plane passing through the points $A(\bar{a}), B(\bar{b})$ and parallel to the vector $\bar{c}$ is $\bar{r} = (1-t)\bar{a} + t\bar{b} + s\bar{c};\ t, s \in R.$

36. The vector equation of the plane passing through the points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is $\bar{r} = (1-t-s)\bar{a} + t\bar{b} + s\bar{c};\ t, s \in R.$

37. Three points $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ are collinear if and only if there exist scalars x, y, z (not all zero) such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0.$

38. Four points $A(\bar{a}), B(\bar{b}), C(\bar{c})$ and $D(\bar{d})$ are coplanar if and only if there exist scalars x, y, z and u (not all zero) such that $x\bar{a} + y\bar{b} + z\bar{c} + u\bar{d} = \bar{0}$ and $x + y + z + u = 0.$

Answers

Exercise 4

(1) $\hat{a} = \dfrac{2}{\sqrt{14}}\bar{i} + \dfrac{3}{\sqrt{14}}\bar{j} + \dfrac{1}{\sqrt{14}}\bar{k}$ (2) $7\bar{a} = \dfrac{7}{\sqrt{5}}\bar{i} - \dfrac{14}{\sqrt{5}}\bar{j}$ (3) $\dfrac{4}{\sqrt{29}}\bar{i} + \dfrac{3}{\sqrt{29}}\bar{j} - \dfrac{2}{\sqrt{29}}\bar{k}$

(4) $\sqrt{6}, \left(\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, -\dfrac{2}{\sqrt{6}}\right)$ (5) $(i)\ \dfrac{5}{3}\bar{a},\ (ii)\ 4\bar{b} - \bar{a}$ (8) $\bar{r} = \bar{c} + t\bar{a}, t \in \mathrm{R}$

(9) $\bar{r} = (2+4t)\bar{i} + (3-2t)\bar{j} + \bar{k}(1+3t), t \in \mathrm{R}$

(10) $\bar{r} = (2-6t)\bar{i} + (1+2t)\bar{j} + \bar{k}(3-4t), t \in \mathrm{R}$

(11) $\bar{r} = (1-s-t)(\bar{i} - 2\bar{j} + 5\bar{k}) + s(-5\bar{j} - \bar{k}) + t(-3\bar{i} + 5\bar{j}), s, t \in \mathrm{R}$

5. PRODUCT OF VECTORS

Introduction:

In Chapter 4, we studied about the addition and subtraction of vectors. We also introduced the concept of multiplication of a vector with a scalar and derived the parametric vectorial equations of straight line and plane. In this unit, we intend to introduce another algebraic operation, called the product of vectors.

Recall that product of two real numbers is a real number and product of two matrices that are compatible for multiplication, is again a matrix. But in case of functions, we may operate them in many ways, Two such operations are multiplication of functions point and composition of two functions. Similarly we define two different types of products, namely, scalar(or dot) product where the resultant is a scalar and vector (or cross) product where the resultant is a vector. In the case of vectors, both the types of products have several applications in Geometry, Mechanics, Physics and Engineering.

We shall conclude this chapter by introducing the concept of scalar triple product of three vectors, explain its geometrical interpretation, indicate its use in obtaining the shortest distance between two skew lines and also discuss the vector triple product of three vectors.

5.1 Scalar or Dot product of two vectors- Geometrical Interpretation-Orthogonal Projections:

5.1.1 Definition (Scalar or Dot product):

Let $\bar{a}$ and $\bar{b}$ be two vectors. The *scalar* (*or dot*) *product* of $\bar{a}$ and $\bar{b}$ written as $\bar{a}.\bar{b}$, is defined by

$$\bar{a}.\bar{b} = \begin{cases} 0 \ \textit{if one of } \bar{a}, \bar{b} \textit{ is } \bar{0} \\ |\bar{a}||\bar{b}|\cos\theta, \textit{if } \bar{a} \neq \bar{0} \neq \bar{b} \textit{ and } \theta \textit{ is the angle between } \bar{a} \textit{ and } \bar{b}. \end{cases}$$

5.1.2 Note:

1. For any two vectors $\bar{a}$ and $\bar{b}$, $\bar{a}.\bar{b}$ is a scalar.

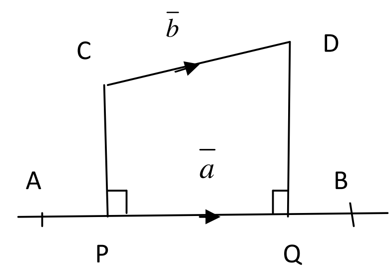
2. If $\bar{a}, \bar{b}$ are non-zero vectors, then $\bar{a}.\bar{b}$ is positive or zero or negative according as the angle θ, is acute or right or obtuse angle.

3. If $\theta = 0^0$ then $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|$. In particular $\bar{a}.\bar{a} = |\bar{a}||\bar{a}|\cos 0^0 = |\bar{a}|^2$ and $\bar{a}.\bar{a}$ is generally denoted by $|\bar{a}|^2$ $or\,(\bar{a})^2$.

4. If $\theta = 180^0$ then $\bar{a}.\bar{b} = -|\bar{a}||\bar{b}|$. In particular $\bar{a}.\bar{a} = |\bar{a}||\bar{a}|\cos 180^0 = -|\bar{a}|^2$.

5.1.3 Orthogonal Projection:

We introduce the concept of orthogonal projection of a vector $\bar{b}$ on a vector $\bar{a}$ and derive formulae for orthogonal projection of $\bar{b}$ on $\bar{a}$ and its magnitude, we notice that the orthogonal projection of $\bar{b}$ on $\bar{a}$ is same as the orthogonal projection of $\bar{b}$ on any vector collinear with $\bar{a}$.



5.1.4 Definition:

Let $\overline{AB} = \bar{a}$ and $\overline{CD} = \bar{b}$ be two non-zero vectors. Let P and Q be the feet of the perpendiculars from C and D respectively onto the line $\overline{AB}$. Then $\overline{PQ}$ is called the *orthogonal projection of* $\bar{b}$ *on* $\bar{a}$ and the magnitude $|PQ|$ is called the *magnitude projection of* $\bar{b}$ *on* $\bar{a}$. If $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$ then the projection vector of $\bar{b}$ on $\bar{a}$ is defined as the zero vector.

5.1.5 Note:

(i). Some people use the word ' projection' for the projection of a vector as well as the magnitude of the projected vector. It should be understood according to the context.

(ii). The projection remains unchanged even if the supports of the vector are replaced by parallel lines. Hence we may choose $\bar{a}, \bar{b}$ are coinitial vectors.

5.1.6 Theorem:

The projection vector of $\bar{b}$ on $\bar{a}$ is $\left(\dfrac{\bar{a}.\bar{b}}{|\bar{a}|^2}\right)\bar{a}$ and its magnitude is $\dfrac{|\bar{a}.\bar{b}|}{|\bar{a}|}$

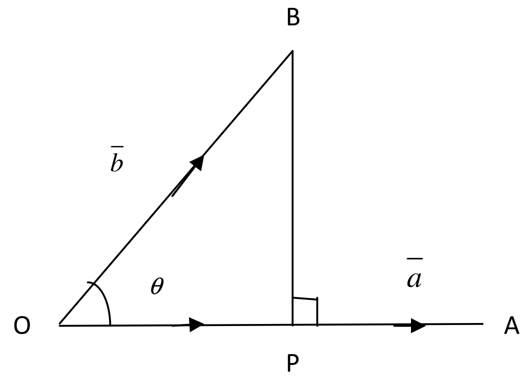
Proof: Let $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$; P be the foot of the perpendicular from B on $\overline{OA}$ and $\theta = \angle AOB$.

Case 1: θ is acute. Then by definition, the projection of $\bar{b}$ on $\bar{a} = \overline{OP}$

$$= \left|\overline{OP}\right| \left(\frac{\overline{a}}{\left|\overline{a}\right|} \right)$$

$$= \left|\overline{OB}\right| \cos\theta \left(\frac{\overline{a}}{\left|\overline{a}\right|} \right)$$

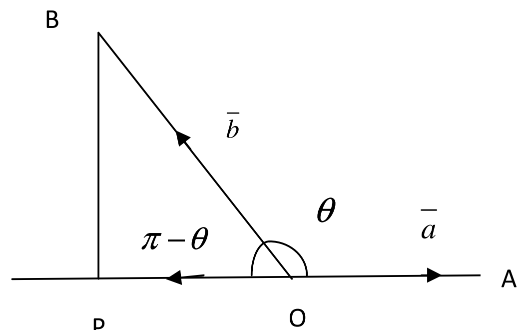
$$= \left|\overline{b}\right| \cos\theta \left(\frac{\overline{a}}{\left|\overline{a}\right|} \right)$$



$$= \left(\left|\overline{a}\right| \left|\overline{b}\right| \cos\theta \right) \left(\frac{\overline{a}}{\left|\overline{a}\right|^2} \right) = \left(\overline{a}.\overline{b} \right) \left(\frac{\overline{a}}{\left|\overline{a}\right|^2} \right) = \left(\frac{\overline{a}.\overline{b}}{\left|\overline{a}\right|^2} \right) \overline{a}$$

Case 2: θ is obtuse. In this case, $\overline{OP}$ is in the opposite direction of $\overline{a}$ and hence the angle $(\overline{b}, \overline{OP}) = \pi - \theta.$

$\therefore$ the projection of $\overline{b}$ on $\overline{a} = \overline{OP} = \left|\overline{OP}\right| \left(\frac{-\overline{a}}{\left|\overline{a}\right|} \right)$



$$= \left|\overline{OB}\right| \cos(\pi - \theta) \left(\frac{-\overline{a}}{\left|\overline{a}\right|} \right)$$

$$= -\left|\overline{OB}\right| \cos\theta \left(\frac{-\overline{a}}{\left|\overline{a}\right|} \right) = \left|\overline{b}\right| \cos\theta \left(\frac{\overline{a}}{\left|\overline{a}\right|} \right)$$

$$= \left(\left|\overline{a}\right| \left|\overline{b}\right| \cos\theta \right) \left(\frac{\overline{a}}{\left|\overline{a}\right|^2} \right) = \left(\overline{a}.\overline{b} \right) \left(\frac{\overline{a}}{\left|\overline{a}\right|^2} \right) = \left(\frac{\overline{a}.\overline{b}}{\left|\overline{a}\right|^2} \right) \overline{a}$$

Case 3: When θ is a right angle, P coincides with O so that $\overline{OP} = \overline{0}$ and also $\overline{a}.\overline{b} = 0.$

Hence $\overline{OP} = \left(\frac{\overline{a}.\overline{b}}{\left|\overline{a}\right|^2} \right) \overline{a}$

Thus the the projection of $\overline{b}$ on $\overline{a} = \left(\frac{\overline{a}.\overline{b}}{\left|\overline{a}\right|^2} \right) \overline{a}$ and magnitude is $\frac{\left|\overline{a}.\overline{b}\right|}{\left|\overline{a}\right|}$

5.1.7 Definition:

Let $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ be two non-zero vectors. Let P be the feet of the perpendicular from B on the line $\overline{OA}$. Then $\overline{OP}$ is called the *component of* $\bar{b}$ *parallel to* $\bar{a}$ and $\overline{PB}$ is called *component of* $\bar{b}$ *perpendicular to* $\bar{a}$

$$\overline{PB} = \bar{b} - \frac{(\bar{a}.\bar{b})}{|\bar{a}|^2}\bar{a}.$$

5.1.8 Geometrical interpretation of the scalar product:

Let $\bar{a}$ and $\bar{b}$ be two non-zero vectors and θ is the angle between $\bar{a}$ and $\bar{b}$

Let $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$. Let P be the feet of the perpendicular from B on the line $\overline{OA}$.

Then $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

$\therefore |\bar{a}.\bar{b}| = |\bar{a}||\bar{b}||\cos\theta| = |\bar{a}||\overline{OP}| =$ Area of the rectangle whose sides are $|\bar{a}|$ and $|\overline{OP}|$.

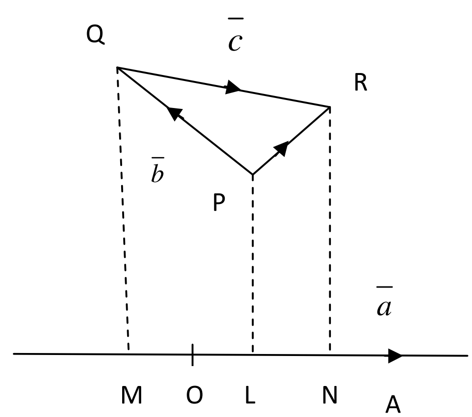
projection vector of $\bar{b}$ on $\bar{a}$ is defined as the zero vector.

5.1.9 Theorem:

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three non-zero vectors. Then the projection $\bar{b}+\bar{c}$ of on $\bar{a}$ is equal to the sum of the projections of $\bar{b}$ and $\bar{c}$ on $\bar{a}$ and hence

$$\frac{\bar{a}.(\bar{b}+\bar{c})}{|\bar{a}|^2}\bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{a}|^2}\bar{a} + \frac{\bar{a}.\bar{c}}{|\bar{a}|^2}\bar{a}$$

Proof: Let $\overline{OA} = \bar{a}, \overline{PQ} = \bar{b}$ and $\overline{QR} = \bar{c}$; so that $\overline{PR} = \bar{b}+\bar{c}$.



We may assume that $\bar{b}+\bar{c} \neq \bar{0}$. Let L, M and N be the foot of the perpendiculars from P, Q and R respectively on the line $\overline{OA}$.

$\frac{\bar{a}.(\bar{b}+\bar{c})}{|\bar{a}|^2}\bar{a} =$ projection of $\bar{b}+\bar{c}$ on $\bar{a} = \overline{LN} = \overline{LM} + \overline{MN}$

$=$ projection of $\bar{b}$ on $\bar{a}$ + projection of $\bar{c}$ on $\bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{a}|^2}\bar{a} + \frac{\bar{a}.\bar{c}}{|\bar{a}|^2}\bar{a}$

5.1.10 Corollary:

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three vectors then $\bar{a}.(\bar{b}+\bar{c}) = \bar{a}.\bar{b} + \bar{a}.\bar{c}$

Proof: We may assume that $\bar{a}, \bar{b}, \bar{c}$ and $\bar{b}+\bar{c}$ are non-zero vectors.

From Theorem 5.1.9,

the projection of $\bar{b}+\bar{c}$ on $\bar{a}$ = projection of $\bar{b}$ on $\bar{a}$ + projection of $\bar{c}$ on $\bar{a}$

$$\therefore \frac{\bar{a}.(\bar{b}+\bar{c})}{|\bar{a}|^2}\bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{a}|^2}\bar{a} + \frac{\bar{a}.\bar{c}}{|\bar{a}|^2}\bar{a}$$

$$= \frac{\bar{a}.\bar{b} + \bar{a}.\bar{c}}{|\bar{a}|^2}\bar{a}$$

$$\therefore \bar{a}.(\bar{b}+\bar{c}) = \bar{a}.\bar{b} + \bar{a}.\bar{c}$$

5.2 Properties of dot product:

In this section, we discuss some of the basic laws of dot product of two vectors.

5.2.1 Theorem:

Let $\bar{a}, \bar{b}$ be two vectors. Then

$(i)\ \bar{a}.\bar{b} = \bar{b}.\bar{a}\ (commutative\ law)$

$(ii)\ (l\bar{a}).\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}),\ l \in R$

$(iii)\ (l\bar{a}).(m\bar{b}) = lm(\bar{a}.\bar{b}),\ l\ and\ m \in R$

$(iv)\ (-\bar{a}).\bar{b} = \bar{a}.(-\bar{b}) = -(\bar{a}.\bar{b})$

$(v)\ (-\bar{a}).(-\bar{b}) = \bar{a}.\bar{b}$

Proof: If one of $\bar{a}, \bar{b}$ is a zero vectors, then by the definition of dot product (i) to (v) hold.

Suppose $\bar{a} \neq \bar{0}$ and $\bar{b} \neq \bar{0}$. Let $(\bar{a}, \bar{b}) = \theta$. Then

$(i)\ (\bar{a}, \bar{b}) = \theta = (\bar{b}, \bar{a})$.

$\therefore \bar{b}.\bar{a} = |\bar{b}||\bar{a}|\cos\theta = |\bar{a}||\bar{b}|\cos\theta = \bar{a}.\bar{b}.$

(ii) **Case 1:** $l > 0$. Then $(l\bar{a}, \bar{b}) = (\bar{a}, l\bar{b}) = (\bar{a}, \bar{b}) = \theta$

$$\therefore (l\bar{a}).\bar{b} = |l\bar{a}||\bar{b}|\cos\theta = l|\bar{a}||\bar{b}|\cos\theta \quad [\because l > 0]$$

$$= |\bar{a}|(l|\bar{b}|\cos\theta) = \bar{a}.(l\bar{b})$$

$$\therefore (l\bar{a}).\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}),\ l > 0$$

Case 2: $l < 0$. Then $(l\bar{a}, \bar{b}) = (\bar{a}, l\bar{b}) = \pi - \theta.$

$$\text{Now } (l\bar{a}).\bar{b} = |l\bar{a}||\bar{b}|\cos(\pi - \theta)$$

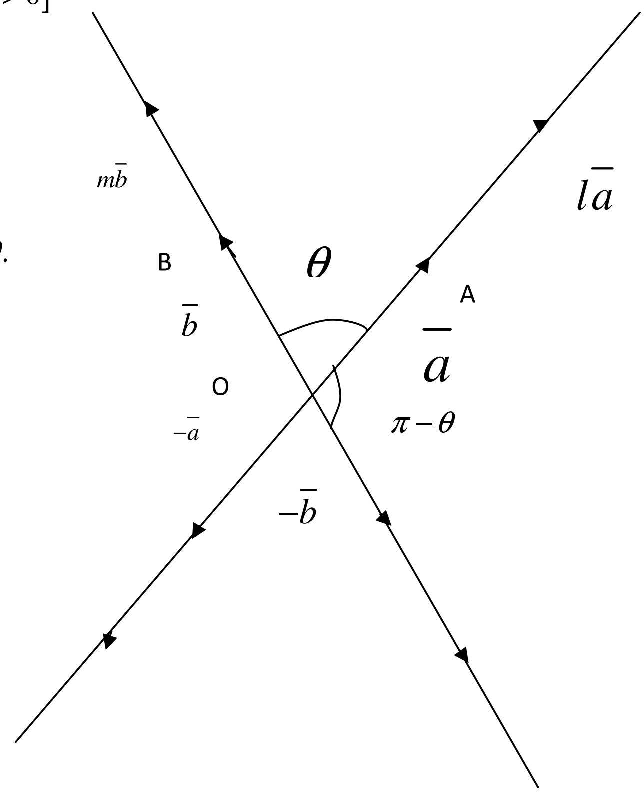
$$= -l|\bar{a}||\bar{b}|(-\cos\theta) \quad [\because l < 0]$$

$$= l(\bar{a}.\bar{b})$$

Similarly $\bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}),\ l < 0$ magnitu

$$\therefore (l\bar{a}).\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}),\ l < 0$$

$$\therefore (l\bar{a}).\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}) \forall l \in R$$



(iii) In (ii) if we replace $\bar{b}$ with $m\bar{b} (m \in R)$, then

$$(l\bar{a}).(m\bar{b}) = l(\bar{a}.(m\bar{b})) = l((m\bar{b}).\bar{a}) = lm(\bar{a}.\bar{b}).$$

$$\therefore (l\bar{a}).(m\bar{b}) = lm(\bar{a}.\bar{b}),\ l \text{ and } m \in R$$

(iv) In (ii) if we replace l with -1, then we get

$$(-\bar{a}).\bar{b} = \bar{a}.(-\bar{b}) = -(\bar{a}.\bar{b})$$

(v) In (iii) if we replace l with -1, m with -1, we get

$$(-\bar{a}).(-\bar{b}) = \bar{a}.\bar{b}$$

5.2.2 Note:

(i) $(\bar{b} + \bar{c}).\bar{a} = \bar{b}.\bar{a} + \bar{c}.\bar{a}$ $\quad$ (ii) $(\bar{a} + \bar{b})^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}.\bar{b}$

5.3 Expression for scalar(dot)product, Angle between two vectors:

In this section, we derive formula for the dot product $\bar{a}.\bar{b}$ when $\bar{a}$ and $\bar{b}$ are expressed in terms of a right handed system $(\bar{i},\bar{j},\bar{k})$. We observe that, if $\bar{i},\bar{j},\bar{k}$ are mutually perpendicular unit vectors, then $\bar{i}.\bar{i}=\bar{j}.\bar{j}=\bar{k}.\bar{k}=1$ and $\bar{i}.\bar{j}=\bar{j}.\bar{k}=\bar{k}.\bar{i}=0.$

5.3.1 Theorem:

Let $(\bar{i},\bar{j},\bar{k})$ be the orthogonal unit triad. Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ and $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ be the vectors where a_j, b_j are scalars for $j=1,2,3.$ Then $\bar{a}.\bar{b}=a_1b_1+a_2b_2+a_3b_3$

Proof: By Corollary 5.1.10, and Theorem 5.2.1 we have

$$a_1\bar{i}.(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})=a_1b_1(\bar{i}.\bar{i})+a_1b_2(\bar{i}.\bar{j})+a_1b_3(\bar{i}.\bar{k})$$

$$=a_1b_1(1)+a_1b_2(0)+a_1b_3(0)=a_1b_1.$$

$$i.e\ \ a_1\bar{i}.(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})=a_1b_1$$

Similarly $a_2\bar{j}.(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})=a_2b_2$ and $a_3\bar{k}.(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})=a_3b_3$

Again by Corollary 5.1.10, we have $\bar{a}.\bar{b}=a_1b_1+a_2b_2+a_3b_3$

5.3.2 Note:

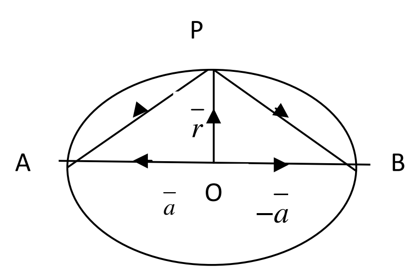
(i) If θ is the angle between two non-zero $\bar{a}$ and $\bar{b}$, from the definition of $\bar{a}.\bar{b}$, we have $\theta=\cos^{-1}\left(\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$ and in particular if $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ and

$\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ then $\theta=\cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$

(ii) If $\bar{a},\bar{b}$ are perpendicular to each other if and only if $a_1b_1+a_2b_2+a_3b_3=0.$

5.3.3 Theorem:

Angle in a semi circle is a right angle.



Proof: Let $\overline{AB}$ be a diameter of a circle with centre O.

Let $\overline{OA}=\bar{a}$ so that $\overline{OB}=-\bar{a}.$

Let P be a point on the circle and $\overline{OP} = \overline{r}$

Then $\overline{PA}.\overline{PB} = (\overline{a} - \overline{r}).(-\overline{a} - \overline{r}) \; = -(\left|\overline{a}\right|^2 - \left|\overline{r}\right|^2) = 0 \left[\because \left|\overline{a}\right| = \left|\overline{r}\right| = radius \right]$

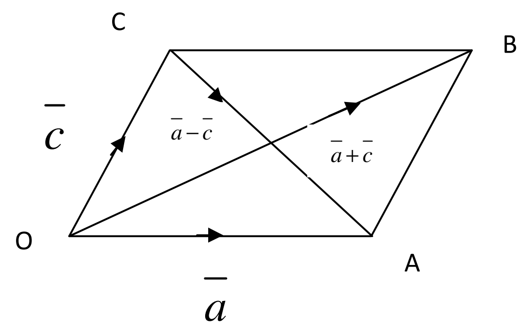
$\angle APB = 90^0$.

5.3.4 Theorem (Parallelogram law):

In a parallelogram, the sum of squares of lengths of the diagonals is equal to sum of squares of lengths of its ides.

Proof: Let $OABC$ be a parallelogram in which $\overline{OB}$ and $\overline{CA}$ are diagonals.

Let $\overline{OA} = \overline{a}$ and $\overline{OC} = \overline{c}$.



Then $\overline{OB} = \overline{a} + \overline{c}$ and $\overline{CA} = \overline{a} - \overline{c}$

$$\therefore \left|\overline{OB}\right|^2 + \left|\overline{CA}\right|^2 = \left|\overline{a} + \overline{c}\right|^2 + \left|\overline{a} - \overline{c}\right|^2$$

$$= \left|\overline{a}\right|^2 + \left|\overline{c}\right|^2 + 2\overline{a}.\overline{c} + \left|\overline{a}\right|^2 + \left|\overline{c}\right|^2 - 2\overline{a}.\overline{c}$$

$$= 2\left|\overline{a}\right|^2 + 2\left|\overline{c}\right|^2$$

$$= \left|\overline{OA}\right|^2 + \left|\overline{AB}\right|^2 + \left|\overline{CB}\right|^2 + \left|\overline{OC}\right|^2 \left[\because \left|\overline{OA}\right| = \left|\overline{BC}\right|, \left|\overline{AB}\right| = \left|\overline{OC}\right| \right]$$

5.4 Vector (cross) product of two vectors and properties:

In this section, we recall 'Right and Left handed system' of a vector triad introduced in Chapter4.We shall define the vector(cross) product of two vectors and some of the properties of cross product of vectors.

5.4.1 Right handed and Left handed system:

Let O, A, B and C be points in the space such that no three of them are collinear. Let $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}$ and $\overline{OC} = \overline{c}$. Observing from the point C, if the angle of rotation (in the counter clock wise sense) of $\overline{OA}$ to $\overline{OB}$ does not exceed 180^0, then the vector triad $(\overline{a}, \overline{b}, \overline{c})$ is said to be a Right handed triad or Right handed system.

If $(\overline{a}, \overline{b}, \overline{c})$ is not a Right handed triad, then it is said to be a Left handed triad.

5.4.2 Note:

(i) If $(\bar{a},\bar{b},\bar{c})$ is a Right (Left) handed system, then the triads $(\bar{b},\bar{c},\bar{a})$ and $(\bar{c},\bar{a},\bar{b})$ also form Right (Left) handed systems.

(ii) If $(\bar{a},\bar{b},\bar{c})$ is a Right handed system and $\bar{a},\bar{b},\bar{c}$ are mutually perpendicular to each other, then $(\bar{a},\bar{b},\bar{c})$ is called an orthogonal triad. Thus the vector triad $(\bar{i},\bar{j},\bar{k})$ is an orthogonal triad.

(iii) If any two vectors in a triad are interchanged, then the system will change. For example $(\bar{a},\bar{b},\bar{c})$ and $(\bar{b},\bar{a},\bar{c})$ form opposite systems.

5.4.3 Definition:

. Let $\bar{a}$ and $\bar{b}$ be two non zero non collinear vectors. The cross (or vector) product of $\bar{a}$ and $\bar{b}$, is written as $\bar{a}\times\bar{b}$ (read as $\bar{a}$ cross $\bar{b}$) is defined to be the vector $|\bar{a}||\bar{b}|\sin\theta\,\hat{n}$ where θ is the angle between the vectors $\bar{a}$ and $\bar{b}$ and $\hat{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ such that $(\bar{a},\bar{b},\hat{n})$ is a right handed system.

If one of the vectors $\bar{a},\bar{b}$ is the null vector or $\bar{a},\bar{b}$ are collinear vectors then the cross product $\bar{a}\times\bar{b}$ is defined as the null vector.

5.4.4 Note:

If $\bar{a},\bar{b}$ are non-zero and non-collinear vectors, then $\bar{a}\times\bar{b}$ is a vector, perpendicular to the plane determined by $\bar{a}$ and $\bar{b}$, whose magnitude is $|\bar{a}||\bar{b}|\sin\theta$ defined as the null vector.

In the following theorem we prove that, the cross product of two non-zero and non-collinear vectors does not obey the commutative law.

5.4.5 Theorem:

Let $\bar{a},\bar{b}$ be two vectors. Then $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$

Proof: If one of $\bar{a},\bar{b}$ is a zero vector or $\bar{a},\bar{b}$ are collinear vectors then $\bar{a}\times\bar{b}=\bar{0}$ and $\bar{b}\times\bar{a}=\bar{0}$ and hence $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$

Suppose $\bar{a},\bar{b}$ are non-zero and non-collinear vectors.

Let θ is the angle between the vectors $\bar{a}$ and $\bar{b}$ and $\hat{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ such that $(\bar{a},\bar{b},\hat{n})$ is a right handed system. Hence by definition $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$. In this case θ is traversed from $\bar{a}$ to $\bar{b}$. If $(\bar{b},\bar{a},-\hat{n})$ is a right handed triad, here θ is traversed from $\bar{b}$ to $\bar{a}$.

$$\therefore \bar{b}\times\bar{a} = |\bar{a}||\bar{b}|\sin\theta(-\hat{n}) = -|\bar{a}||\bar{b}|\sin\theta\,\hat{n} = -(\bar{a}\times\bar{b}).$$

Thus $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$

5.4.6 Note:

$$|\bar{a}\times\bar{b}| = |\bar{b}\times\bar{a}| = |\bar{a}||\bar{b}|\sin\theta.$$

5.4.7 Theorem:

Let $\bar{a},\bar{b}$ be two vectors and l,m be scalars. Then

(i) $(-\bar{a})\times\bar{b} = \bar{a}\times(-\bar{b}) = -(\bar{a}\times\bar{b}) = \bar{b}\times\bar{a}$

(ii) $(-\bar{a})\times(-\bar{b}) = \bar{a}\times\bar{b}$

(iii) $(l\bar{a})\times\bar{b} = \bar{a}\times(l\bar{b}) = l(\bar{a}\times\bar{b})$

(iv) $(l\bar{a})\times(m\bar{b}) = lm(\bar{a}\times\bar{b})$

Proof: If one of $\bar{a},\bar{b}$ is a zero vector or $\bar{a},\bar{b}$ are collinear vectors or one of l,m is a zero scalar then all the above equalities hold good. Hence we assume that $\bar{a},\bar{b}$ are non-zero and non-collinear vectors and l,m are non-zero scalars. Let θ is the angle between the vectors $\bar{a}$ and $\bar{b}$ and $\hat{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ such that $(\bar{a},\bar{b},\hat{n})$ is a right handed system. The angle between the vectors $\bar{a}$ and $\bar{b}$ is $\pi-\theta$. The triad $(-\bar{a},\bar{b},\hat{n})$ is a left handed triad and $(-\bar{a},\bar{b},-\hat{n})$ is a right handed triad.

(i) $(-\bar{a})\times\bar{b} = |-\bar{a}||\bar{b}|\sin(\pi-\theta)(-\hat{n}) = -\left(|\bar{a}||\bar{b}|\sin\theta\,\hat{n}\right) = -(\bar{a}\times\bar{b}).$

Also $\bar{a}\times(-\bar{b}) = -\left((-\bar{b})\times\bar{a}\right)$ $[\because$ by Theorem 5.4.5$]$

$$= -\left(-(\bar{b}\times\bar{a})\right) = \bar{b}\times\bar{a} = -(\bar{a}\times\bar{b})$$

Thus $(-\bar{a})\times\bar{b} = \bar{a}\times(-\bar{b}) = -(\bar{a}\times\bar{b}) = \bar{b}\times\bar{a}$

$(ii)\quad (-\bar{a})\times(-\bar{b}) = -\left[\bar{a}\times(-\bar{b})\right]\ (\because\ by\,(i))$

$$= -\left[-(\bar{a}\times\bar{b})\right]\ (\because\ by\,(i))$$

$$= \bar{a}\times\bar{b}$$ (iii) Let $l > 0.$

The angle between the vectors $l\bar{a}$

and $\bar{b}$ is θ and $\left|l\bar{a}\right| = l\left|\bar{a}\right|$. Further, the vector triad

$(l\bar{a}, \bar{b}, \hat{n})$ is a right handed triad.

$$l\,\bar{a}\times\bar{b} = \left|l\bar{a}\right|\left|\bar{b}\right|\sin\theta\,\hat{n} = l\left(\left|\bar{a}\right|\left|\bar{b}\right|\sin\theta\,\hat{n}\right) = l(\bar{a}\times\bar{b})$$

Similarly, we can get $\bar{a}\times l\bar{b} = l(\bar{a}\times\bar{b})$

Thus $(l\bar{a})\times\bar{b} = \bar{a}\times(l\bar{b}) = l(\bar{a}\times\bar{b})$

(iv) $(l\bar{a})\times(m\bar{b}) = lm(\bar{a}\times\bar{b})$ follows from $(i),(ii),(iii)$.

The proof of the following Theorem 5.4.8, is beyond the scope of this book and hence we assume its validity without proof.

5.4.8 Theorem (Distributive law):

Let $\bar{a},\bar{b}$ and $\bar{c}$ are vectors. Then

$$(i)\quad \bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b}+\bar{a}\times\bar{c}$$

$$(ii)\quad (\bar{a}+\bar{b})\times\bar{c} = \bar{a}\times\bar{c}+\bar{b}\times\bar{c}.$$

5.4.9 Note:

If $(\bar{i},\bar{j},\bar{k})$ is an orthogonal triad, then from the definition of the cross product of vectors, it is easy to see that (i) $\bar{i}\times\bar{i} = \bar{j}\times\bar{j} = \bar{k}\times\bar{k} = \bar{0}$ and
(ii) $\bar{i}\times\bar{j} = \bar{k}, \bar{j}\times\bar{k} = \bar{i}, \bar{k}\times\bar{i} = \bar{j}.$

5.5 Vector (cross) product in $(\bar{i},\bar{j},\bar{k})$ system:

. In this section, we derive formula for $\bar{a}\times\bar{b}$ when $\bar{a}$ and $\bar{b}$ are given in $(\bar{i},\bar{j},\bar{k})$ system and deduce the formula for $\left|\bar{a}\times\bar{b}\right|$.

5.5.1 Theorem:

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$. Then

$$\bar{a}\times\bar{b} = (a_2b_3 - a_3b_2)\bar{i} - (a_1b_3 - a_3b_1)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}.$$

Proof: For proving the formula, we use Theorem 5.4.8, and property of cross product among $\bar{i}, \bar{j}$ and $\bar{k}$ as mentioned at the end of the Theorem 5.4.8.

$$\text{Now } \bar{a}\times\bar{b} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})\times(b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$$

$$\begin{aligned}
&= \left[a_1b_1(\bar{i}\times\bar{i}) + a_1b_2(\bar{i}\times\bar{j}) + a_1b_3(\bar{i}\times\bar{k})\right] \\
&\quad + \left[a_2b_1(\bar{j}\times\bar{i}) + a_2b_2(\bar{j}\times\bar{j}) + a_2b_3(\bar{j}\times\bar{k})\right] \\
&\quad + \left[a_3b_1(\bar{k}\times\bar{i}) + a_3b_2(\bar{k}\times\bar{j}) + a_3b_3(\bar{k}\times\bar{k})\right]
\end{aligned}$$

$$\begin{aligned}
&= \left[a_1b_1(\bar{0}) + a_1b_2(\bar{k}) + a_1b_3(-\bar{j})\right] \\
&\quad + \left[a_2b_1(-\bar{k}) + a_2b_2(\bar{0}) + a_2b_3(\bar{i})\right] \\
&\quad + \left[a_3b_1(\bar{j}) + a_3b_2(-\bar{i}) + a_3b_3(\bar{0})\right]
\end{aligned}$$

$$= \bar{i}(a_2b_3 - a_3b_2) - \bar{j}(a_1b_3 - a_3b_1) + \bar{k}(a_1b_2 - a_2b_1)$$

5.5.2 Notation:

Adopting the expansion of a 3×3 determinant of real matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The above formula for $\bar{a}\times\bar{b}$ can now expressed as

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \bar{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \bar{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \bar{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \bar{i}(a_2b_3 - a_3b_2) - \bar{j}(a_1b_3 - a_3b_1) + \bar{k}(a_1b_2 - a_2b_1)$$

5.5.3 Corollary:

Let $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}, \overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$ and θ is the angle between $\overline{a}$ and $\overline{b}$,

then $\sin\theta = \dfrac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}}$.

Proof: By Theorem 5.5.1, we have

$$\overline{a}\times\overline{b} = (a_2b_3 - a_3b_2)\overline{i} - (a_1b_3 - a_3b_1)\overline{j} + (a_1b_2 - a_2b_1)\overline{k}.$$

$$\therefore \left|\overline{a}\times\overline{b}\right| = \left|(a_2b_3 - a_3b_2)\overline{i} - (a_1b_3 - a_3b_1)\overline{j} + (a_1b_2 - a_2b_1)\overline{k}\right|$$
$$= \left|\sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}\right|$$

and $\left|\overline{a}\right| = \left|a_1\overline{i} + a_2\overline{j} + a_3\overline{k}\right| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$

and $\left|\overline{b}\right| = \left|b\overline{i} + b_2\overline{j} + b_3\overline{k}\right| = \sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}$

We have $\left|\overline{a}\times\overline{b}\right| = \left|\overline{a}\right|\left|\overline{b}\right|\sin\theta,$ so that $\sin\theta = \dfrac{\left|\overline{a}\times\overline{b}\right|}{\left|\overline{a}\right|\left|\overline{b}\right|}$

$$\Rightarrow \sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}}.$$

5.5.4 Notation:

To determine the angle between two vectors, we use the dot product of vectors rather than the cross product, as the cross product gives value of $\sin\theta$ which is positive for $\theta \in (0, \pi)$.

5.5.5 Theorem:

For any two vectors $\overline{a}$ and $\overline{b}$, $\left|\overline{a}\times\overline{b}\right|^2 = \left(\overline{a}.\overline{a}\right)\left(\overline{b}.\overline{b}\right) - \left(\overline{a}.\overline{b}\right)^2 = \left|\overline{a}\right|^2\left|\overline{b}\right|^2 - \left(\overline{a}.\overline{b}\right)^2$.

Proof: We have $\left|\overline{a}\times\overline{b}\right| = \left|\overline{a}\right|\left|\overline{b}\right|\sin\theta,$ where θ is the angle between vectors $\overline{a}$ and $\overline{b}$.

Now $\left|\overline{a}\times\overline{b}\right|^2 = \left|\overline{a}\right|^2\left|\overline{b}\right|^2\sin^2\theta = \left|\overline{a}\right|^2\left|\overline{b}\right|^2(1 - \cos^2\theta) = \left|\overline{a}\right|^2\left|\overline{b}\right|^2 - \left|\overline{a}\right|^2\left|\overline{b}\right|^2\cos^2\theta$

$$=\left|\overline{a}\right|^2\left|\overline{b}\right|^2-\left(\overline{a}.\overline{b}\right)^2=\left(\overline{a}.\overline{a}\right)\left(\overline{b}.\overline{b}\right)-\left(\overline{a}.\overline{b}\right)^2.$$

5.5.6 Note:

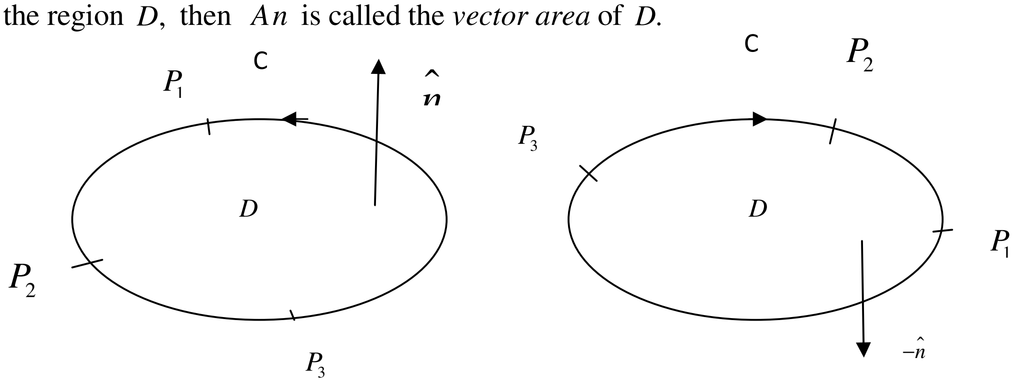
If $\overline{a}$ and $\overline{b}$ are non-collinear, then, unit vectors perpendicular to both $\overline{a}$ and $\overline{b}$ are $\pm\dfrac{\overline{a}\times\overline{b}}{\left|\overline{a}\times\overline{b}\right|}$.

5.6 Vector Areas:

. In the following, we introduce the concept of vector area of a plane region bounded by a closed plane curve (a curve in which initial point and terminal point are the same) and find vector area of a triangle and parallelogram.

5.6.1 Definition (Vector area):

. Let D be a plane region bounded by closed curve C. Let P_1, P_2, P_3 be three points on C (taken in this order). Let $\hat{n}$ be the unit vector perpendicular to the region D such that, from the side of $\hat{n}$, the points P_1, P_2 and P_3 are in anti clock sense. If A is the area of the region D, then $A\hat{n}$ is called the *vector area* of D.



In the following theorems, we derive the vector area of a triangle and parallelogram as applications of cross (or vector) product of vectors.

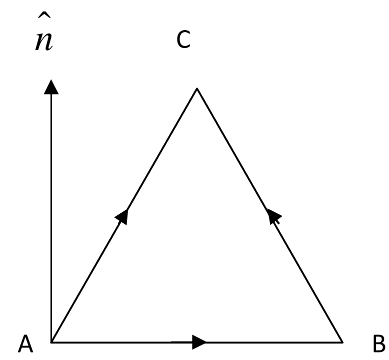
5.6.2 Theorem:

The vectors area of ΔABC is $\frac{1}{2}\left(\overline{AB}\times\overline{AC}\right)=\frac{1}{2}\left(\overline{BC}\times\overline{BA}\right)=\frac{1}{2}\left(\overline{CA}\times\overline{CB}\right)$.

Proof: Let the vertices A, B and C of the triangle be described in anti clock wise sense so that the closed boundary of the plane region ΔABC is $\overline{BC}\cup\overline{CA}\cup\overline{AB}$.

Let Δ be the area of ΔABC.

Let $\hat{n}$ be the unit vector in the direction of $\overline{AB}\times\overline{AC}$.



$\Delta = \frac{1}{2}\left|\overline{AB}\right|\left|\overline{AC}\right|\sin A \Rightarrow \Delta\hat{n} = \frac{1}{2}\left|\overline{AB}\right|\left|\overline{AC}\right|\sin A\hat{n}$

$\Rightarrow \Delta\hat{n} = \frac{1}{2}\left(\left|\overline{AB}\right|\left|\overline{AC}\right|\sin A\hat{n}\right) = \frac{1}{2}\left(\overline{AB}\times\overline{AC}\right).$

Similarly we can get $\Delta\hat{n} = \frac{1}{2}\left(\overline{BC}\times\overline{BA}\right)$ and $\Delta\hat{n} = \frac{1}{2}\left(\overline{CA}\times\overline{CB}\right)$.

Hence the vectors area of ΔABC is $\frac{1}{2}\left(\overline{AB}\times\overline{AC}\right) = \frac{1}{2}\left(\overline{BC}\times\overline{BA}\right) = \frac{1}{2}\left(\overline{CA}\times\overline{CB}\right)$.

5.6.3 Corollary:

If $\overline{a},\overline{b},\overline{c}$ are the position vectors of the vertices A, B and C (described in counter clock wise sense) of ΔABC, then the vectors area of ΔABC is $\frac{1}{2}\left(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right)$ and its area is $\frac{1}{2}\left|\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right|$.

Proof: By Theorem 5.6.2, the vector area of ΔABC is

$$\Delta = \frac{1}{2}\left(\overline{AB}\times\overline{AC}\right)$$

$$\Delta = \frac{1}{2}\left((\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\right) = \frac{1}{2}\left(\overline{b}\times\overline{c}-\overline{b}\times\overline{a}-\overline{a}\times\overline{c}+\overline{a}\times\overline{a})\right)$$

$$= \frac{1}{2}\left(\overline{b}\times\overline{c}+\overline{a}\times\overline{b}+\overline{c}\times\overline{a}+\overline{0}\right)\left[\because \overline{a}\times\overline{a}=\overline{0}, \overline{a}\times\overline{b}=-\overline{b}\times\overline{a}, \overline{c}\times\overline{a}=-\overline{a}\times\overline{c}\right]$$

$$= \frac{1}{2}\left(\overline{b}\times\overline{c}+\overline{a}\times\overline{b}+\overline{c}\times\overline{a}\right)$$

Area of ΔABC is $\Delta = \left|\Delta\hat{n}\right| = \frac{1}{2}\left|\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right|$.

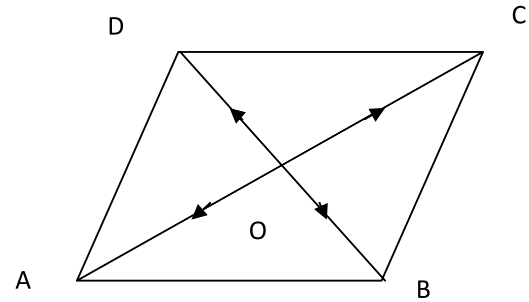
5.6.4 Note:

Since the vector area of a plane region D is a vector quantity perpendicular to the plane of D, it follows that, the vector $\left(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right)$ is perpendicular to the plane of the ΔABC where $\overline{a},\overline{b},\overline{c}$ are the position vectors of A, B, C respectively.

5.6.5 Theorem (Vector area of a parallelogram):

Let $ABCD$ be a parallelogram with vertices A, B, C and D described in anti clock wise sense. Then, vectors area of $ABCD$ in terms of the diagonals $\overline{AC}$ and $\overline{BD}$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$.

Proof: Given $ABCD$ be a parallelogram with vertices A, B, C and D described in anti clock wise sense.



$$\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)=\frac{1}{2}\left[\left(\overline{AB}+\overline{BC}\right)\times\left(\overline{BA}+\overline{AD}\right)\right]$$

$$=\frac{1}{2}\left[\overline{AB}\times\overline{BA}+\overline{AB}\times\overline{AD}+\overline{BC}\times\overline{BA}+\overline{BC}\times\overline{AD}\right]$$

$$=\frac{1}{2}\left[\overline{AB}\times\overline{AD}+(-\overline{CB})\times\overline{BA}\right]$$

$$=\frac{1}{2}\left[\overline{AB}\times\overline{AD}+(-\overline{CB})\times\overline{CD}\right]\left(\because \overline{BA}=\overline{CD}\right)$$

$$=\frac{1}{2}\left[\overline{AB}\times\overline{AD}+\overline{CD}\times\overline{CB}\right]$$

$$=\frac{1}{2}\left(\overline{AB}\times\overline{AD}\right)+\frac{1}{2}\left(\overline{CD}\times\overline{CB}\right)$$

$=$ vector area of ΔABD + vector area of ΔCDB

$=$ vector area of $ABCD$

5.6.6 Note:

(i) In fact, the vector area of any plane quadrilateral $ABCD$ in terms of the diagonals $\overline{AC}$ and $\overline{BD}$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$.

(ii) The area of quadrilateral $ABCD$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$.

(iii) The vector area of a parallelogram with $\overline{a}$ and $\overline{b}$ as adjacent sides is $\overline{a}\times\overline{b}$ and the area is $\left|\overline{a}\times\overline{b}\right|$.

5.6.7 Theorem:

Let $(\overline{a},\overline{b},\overline{c})$ be a non-coplanar vector triad, $\overline{\alpha}=l_1\overline{a}+l_2\overline{b}+l_3\overline{c}$ and $\overline{\beta}=m_1\overline{a}+m_2\overline{b}+m_3\overline{c}$. Then $\overline{\alpha}\times\overline{\beta}=\begin{vmatrix}\overline{a}\times\overline{b} & \overline{b}\times\overline{c} & \overline{c}\times\overline{a}\\ l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{vmatrix}$.

Proof: Using the distributive law of cross product over vector addition (Theorem5.4.8)we have $\overline{\alpha}\times\overline{\beta}=(l_1\overline{a}+l_2\overline{b}+l_3\overline{c})\times(m_1\overline{a}+m_2\overline{b}+m_3\overline{c})$

$$=\left[l_1m_1(\overline{a}\times\overline{a})+l_1m_2(\overline{a}\times\overline{b})+l_1m_3(\overline{a}\times\overline{c})\right]$$
$$+\left[l_2m_1(\overline{b}\times\overline{a})+l_2m_2(\overline{b}\times\overline{b})+l_2m_3(\overline{b}\times\overline{c})\right]$$
$$+\left[l_3m_1(\overline{c}\times\overline{a})+l_3m_2(\overline{c}\times\overline{b})+l_3m_3(\overline{c}\times\overline{c})\right]$$

$$=\left[l_1m_1(\overline{0})+l_1m_2(\overline{a}\times\overline{b})+l_1m_3(\overline{a}\times\overline{c})\right]$$
$$+\left[l_2m_1(\overline{b}\times\overline{a})+l_2m_2(\overline{0})+l_2m_3(\overline{b}\times\overline{c})\right]$$
$$+\left[l_3m_1(\overline{c}\times\overline{a})+l_3m_2(\overline{c}\times\overline{b})+l_3m_3(\overline{0})\right]$$

$$=\left[l_1m_2(\overline{a}\times\overline{b})+l_1m_3(\overline{a}\times\overline{c})\right]+\left[l_2m_1(\overline{b}\times\overline{a})+l_2m_3(\overline{b}\times\overline{c})\right]$$
$$+\left[l_3m_1(\overline{c}\times\overline{a})+l_3m_2(\overline{c}\times\overline{b})\right]$$

$$=\left[l_1m_2(\overline{a}\times\overline{b})+l_1m_3(-(\overline{c}\times\overline{a}))\right]+\left[l_2m_1(-(\overline{a}\times\overline{b}))+l_2m_3(\overline{b}\times\overline{c})\right]$$
$$+\left[l_3m_1(\overline{c}\times\overline{a})+l_3m_2(-(\overline{b}\times\overline{c}))\right]$$
$$\left(\because \overline{b}\times\overline{a}=-(\overline{a}\times\overline{b}),\overline{c}\times\overline{b}=-(\overline{b}\times\overline{c}),\overline{a}\times\overline{c}=-(\overline{c}\times\overline{a})\right)$$

$$=(\overline{b}\times\overline{c})(l_2m_3-l_3m_2)-(\overline{c}\times\overline{a})(l_1m_3-l_3m_1)+(\overline{a}\times\overline{b})(l_1m_2-l_2m_1)$$

$$=\begin{vmatrix}\overline{a}\times\overline{b} & \overline{b}\times\overline{c} & \overline{c}\times\overline{a}\\ l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{vmatrix}.$$

5.6.8 Note:

In the above theorem, if we take $\overline{a}=\overline{i},\overline{b}=\overline{j}$ and $\overline{c}=\overline{k}$ such that $(\overline{i},\overline{j},\overline{k})$ is a right handed system, then we obtain the formula for $\overline{\alpha}\times\overline{\beta}$ as in 5.5.2.

5.7 Scalar and vector triple products:

. In this section we introduce the concept of scalar triple product and vector triple product of three vectors and discuss its properties and its geometrical interpretation.

5.7.1 Definition:

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three vectors. We call $(\bar{a} \times \bar{b}).\bar{c},$ the scalar triple product of $\bar{a}, \bar{b}$ and $\bar{c}$ and denote this by $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

5.7.2 Note: $(\bar{a} \times \bar{b}).\bar{c} = \bar{0}$ when

(i) one of $\bar{a}, \bar{b}, \bar{c}$ is $\bar{0}$ or

(ii) $\bar{a}, \bar{b}$ or $\bar{b}, \bar{c}$ or $\bar{c}, \bar{a}$ are collinear vectors or

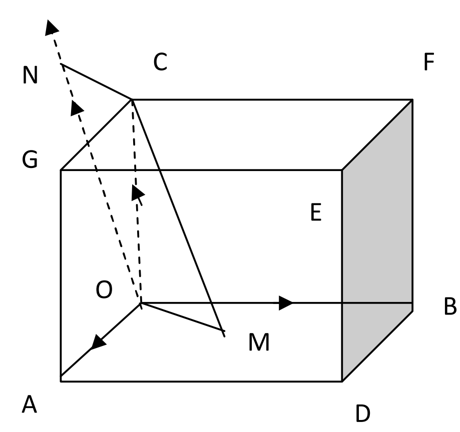
(iii) $\bar{c}$ is perpendicular to $(\bar{a} \times \bar{b})$.

5.7.3 Theorem:

Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three non-coplanar vectors and $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$. Let V be the volume of the paralleleopiped with $\overline{OA}, \overline{OB}$ and $\overline{OC}$ as coterminous edges. Then

(i) $(\bar{a} \times \bar{b}).\bar{c} = V$, if $(\bar{a}, \bar{b}, \bar{c})$ is a right handed system.

(ii) $(\bar{a} \times \bar{b}).\bar{c} = -V$, if $(\bar{a}, \bar{b}, \bar{c})$ is a left handed system.



Proof:(i) Consider the parallelepiped $OADBFCGE$ having $\overline{OA}, \overline{OB}$ and $\overline{OC}$ as coterminous edges . Assume that $\bar{a}, \bar{b}, \bar{c}$ is a right handed system. Draw $\overline{CM}$ perpendicular to the plane determined by $\overline{OA}$ and $\overline{OB}$ and N be the foot of the perpendicular to the support of $(\bar{a} \times \bar{b})$. Let $\hat{n}$ be the unit vector in the direction of $\bar{a} \times \bar{b}$ so that by definition of $\bar{a} \times \bar{b}$, we have $(\bar{a}, \bar{b}, \hat{n})$ is a right handed system. Let θ be the angle between $\bar{a} \times \bar{b}$ and $\bar{c}$.

i.e $\theta = \angle CON$.

$V =$ Area of the base parallelogram $OADB \times$ length of the vertex C from base

$= |\bar{a}\times\bar{b}|(\overline{CM}) = |\bar{a}\times\bar{b}|(\overline{ON})$

But from ΔOCN, $|\overline{ON}| = |\overline{OC}|\cos\theta$

$$\therefore V = |\bar{a}\times\bar{b}||\overline{OC}|\cos\theta = |\bar{a}\times\bar{b}||\bar{c}|\cos\theta = (\bar{a}\times\bar{b}).\bar{c}$$

Thus $(\bar{a}\times\bar{b}).\bar{c} = [\bar{a}\ \bar{b}\ \bar{c}]$.

(ii) Suppose $(\bar{a},\bar{b},\bar{c})$ is a left handed system.

$\therefore (\bar{a},\bar{b},-\bar{c})$ is a right handed system. But the volume of the corresponding parallelepiped are same.

$\therefore V = (\bar{a}\times\bar{b}).(-\bar{c}) = -(\bar{a}\times\bar{b}).\bar{c} \Rightarrow (\bar{a}\times\bar{b}).\bar{c} = -V$

5.7.4 Theorem:

For any three vectors $\bar{a},\bar{b}$ and $\bar{c}$, $(\bar{a}\times\bar{b}).\bar{c} = (\bar{b}\times\bar{c}).\bar{a} = (\bar{c}\times\bar{a}).\bar{b}$

$$i.e\ [\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}].$$

Proof: If one of $\bar{a},\bar{b}$ and $\bar{c}$ is $\bar{0}$ or any two are collinear then equality holds.

Assume that $(\bar{a},\bar{b},\bar{c}),(\bar{b},\bar{c},\bar{a})$ and $(\bar{c},\bar{a},\bar{b})$ forms the right handed systems.

$\therefore (\bar{a}\times\bar{b}).\bar{c} = (\bar{b}\times\bar{c}).\bar{a} = (\bar{c}\times\bar{a}).\bar{b} =$ volume of the parallelepiped $= V$.

If all the triads $(\bar{a},\bar{b},\bar{c}),(\bar{b},\bar{c},\bar{a})$ and $(\bar{c},\bar{a},\bar{b})$ forms the left handed systems, then

$(\bar{a}\times\bar{b}).\bar{c} = (\bar{b}\times\bar{c}).\bar{a} = (\bar{c}\times\bar{a}).\bar{b} = -V$.

Thus $(\bar{a}\times\bar{b}).\bar{c} = (\bar{b}\times\bar{c}).\bar{a} = (\bar{c}\times\bar{a}).\bar{b}$

5.7.5 Theorem:

If $\bar{a},\bar{b}$ and $\bar{c}$ are any three vectors, then $(\bar{a}\times\bar{b}).\bar{c} = \bar{a}.(\bar{b}\times\bar{c})$

i.e in a scalar triple product dot and cross are interchanged.

Proof: From Theorem 5.7.4, we have $\left(\bar{a}\times\bar{b}\right).\bar{c}=\left(\bar{b}\times\bar{c}\right).\bar{a}$

$$=\bar{a}.\left(\bar{b}\times\bar{c}\right)\ (\because \text{dot product is commutative})$$

5.7.6 Theorem:

If $\bar{a},\bar{b}$ and $\bar{c}$ are three vectors such that no two are collinear, then $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$ if and only if $\bar{a},\bar{b}$ and $\bar{c}$ are coplanar.

Proof: Suppose $\bar{a},\bar{b}$ and $\bar{c}$ are coplanar.

Since $\bar{a}\times\bar{b}$ is perpendicular to the plane determined by $\bar{a}$ and $\bar{b}$ it is also perpendicular to $\bar{c}$. Hence $\left(\bar{a}\times\bar{b}\right).\bar{c}=0$

$\therefore\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

Conversely assume that $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$ *i.e* $\left(\bar{a}\times\bar{b}\right).\bar{c}=0$

$\therefore\ \bar{a}\times\bar{b}$ is perpendicular to $\bar{c}$.

But $\bar{a}\times\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ it is also

$\therefore\ \bar{a}\times\bar{b}$ is perpendicular to $\bar{a},\bar{b}$ and $\bar{c}$.

$\therefore\ \bar{a},\bar{b}$ and $\bar{c}$ are coplanar.

5.7.7 Corollary:

Four distinct points A,B,C and D are coplanar if and only if $\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right]=0$

Proof: A,B,C and D are coplanar $\Leftrightarrow \overline{AB},\overline{AC},\overline{AD}$ are coplanar

$$\Leftrightarrow\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right]=0.$$

5.7.8 Theorem:

Let $(\bar{i},\bar{j},\bar{k})$ be orthogonal triad of unit vectors which is a right handed system.

Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k},\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ and $\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$.

Then, $\left[\overline{a}\ \overline{b}\ \overline{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

Proof: It is known that $\overline{a}\times\overline{b}=(a_2b_3-a_3b_2)\overline{i}-(a_1b_3-a_3b_1)\overline{j}+(a_1b_2-a_2b_1)\overline{k}.$

$$\text{Now } \left(\overline{a}\times\overline{b}\right).\overline{c}=\left((a_2b_3-a_3b_2)\overline{i}-(a_1b_3-a_3b_1)\overline{j}+(a_1b_2-a_2b_1)\overline{k}\right).\left(c_1\overline{i}+c_2\overline{j}+c_3\overline{k}\right)$$

$$=(a_2b_3-a_3b_2)c_1-(a_1b_3-a_3b_1)c_2+(a_1b_2-a_2b_1)c_3$$

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

5.7.9 Corollary:

Let $\overline{a}=a_1\overline{i}+a_2\overline{j}+a_3\overline{k},\overline{b}=b_1\overline{i}+b_2\overline{j}+b_3\overline{k}$ and $\overline{c}=c_1\overline{i}+c_2\overline{j}+c_3\overline{k}$ are coplanar if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

Proof: It follows from Theorems 5.7.6 and 5.7.8.

5.7.10 Corollary:

If $\overline{\alpha},\overline{\beta},\overline{\gamma}$ be three non-coplanar vectors and $\overline{a}=a_1\overline{\alpha}+a_2\overline{\beta}+a_3\overline{\gamma},\overline{b}=b_1\overline{\alpha}+b_2\overline{\beta}+b_3\overline{\gamma}$ and $\overline{c}=c_1\overline{\alpha}+c_2\overline{\beta}+c_3\overline{\gamma}.$ Then $\overline{a},\overline{b}$ and $\overline{c}$ are coplanar if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

Proof: From Theorems 5.6.7, $\overline{a}\times\overline{b}=\begin{vmatrix} \overline{\beta}\times\overline{\gamma} & \overline{\gamma}\times\overline{\alpha} & \overline{\alpha}\times\overline{\beta} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$=\left((a_2b_3-a_3b_2)(\overline{\beta}\times\overline{\gamma})-(a_1b_3-a_3b_1)(\overline{\gamma}\times\overline{\alpha})+(a_1b_2-a_2b_1)(\overline{\alpha}\times\overline{\beta})\right)$$

Now

$$\left(\overline{a}\times\overline{b}\right).\overline{c}=\left((a_2b_3-a_3b_2)(\overline{\beta}\times\overline{\gamma})-(a_1b_3-a_3b_1)(\overline{\gamma}\times\overline{\alpha})+(a_1b_2-a_2b_1)(\overline{\alpha}\times\overline{\beta})\right).\left(c_1\overline{\alpha}+c_2\overline{\beta}+c_3\overline{\gamma}\right)$$

$$=\left((a_2b_3-a_3b_2)c_1(\overline{\beta}\times\overline{\gamma}).\overline{\alpha}-(a_1b_3-a_3b_1)c_2(\overline{\gamma}\times\overline{\alpha}).\overline{\beta}+(a_1b_2-a_2b_1)c_3(\overline{\alpha}\times\overline{\beta}).\overline{\gamma}\right)$$

$$=\left((a_2b_3-a_3b_2)c_1\left[\overline{\beta}\ \overline{\gamma}\ \overline{\alpha}\right]-(a_1b_3-a_3b_1)c_2\left[\overline{\gamma}\ \overline{\alpha}\ \overline{\beta}\right]+(a_1b_2-a_2b_1)c_3\left[\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\right]\right)$$

$$=\left((a_2b_3-a_3b_2)c_1-(a_1b_3-a_3b_1)c_2+(a_1b_2-a_2b_1)c_3\right)\left[\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\right]$$

$$\left(\because \left[\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\right]=\left[\overline{\beta}\ \overline{\gamma}\ \overline{\alpha}\right]=\left[\overline{\gamma}\ \overline{\alpha}\ \overline{\beta}\right]\right)$$

$$\therefore \left[\overline{a}\ \overline{b}\ \overline{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\left[\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\right]$$

Since $\overline{\alpha},\overline{\beta},\overline{\gamma}$ are non-coplanar vectors, $\left[\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\right]\neq 0$

$\therefore \overline{a},\overline{b},\overline{c}$ are coplanar if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

5.7.11 Theorem:

The volume of a tetrahedron with $\overline{a},\overline{b}$ and $\overline{c}$ as coterminous edges is $\frac{1}{6}\left|\left[\overline{a}\ \overline{b}\ \overline{c}\right]\right|$.

Proof: Let $OABC$ be a tetrahedron and $\overline{OA}=\overline{a},\overline{OB}=\overline{b},\overline{OC}=\overline{c}.$

Let V be the volume of a tetrahedron $OABC$.

By definition, the volume V is given by

$V=\frac{1}{3}$ (area of the base ΔOAB)$\times$(length of the perpendicular from C to ΔOAB)

CN is the perpendicular from C to ΔOAB and CM is the perpendicular from C to the supporting line $\overline{a}\times\overline{b}$ so that $CN=OM=$ length of the projection of $\overline{c}$ onto $\overline{a}\times\overline{b}$

$$=\frac{\left|\left(\overline{a}\times\overline{b}\right).\overline{c}\right|}{\left|\overline{a}\times\overline{b}\right|}=\frac{\left|\left[\overline{a}\ \overline{b}\ \overline{c}\right]\right|}{\left|\overline{a}\times\overline{b}\right|}$$

Area of $\Delta OAB=\frac{1}{2}\left|\overline{a}\times\overline{b}\right|$

$\therefore$ Volume of the tetrahedron $OABC = \frac{1}{3}.\frac{1}{2}\left|\bar{a}\times\bar{b}\right|\frac{\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|}{\left|\bar{a}\times\bar{b}\right|} = \frac{1}{6}\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$.

5.7.12 Corollary:

The volume of the tetrahedron whose vertices are A, B, C and D is

$\frac{1}{6}\left|\left[\overline{DA}\ \ \overline{AB}\ \ \overline{AC}\right]\right|$.

Proof: Since $\overline{DA}, \overline{DB}, \overline{DC}$ are coterminous edges of the tetrahedron $ABCD$, from the above theorem it follows that its volume is $\frac{1}{6}\left|\left[\overline{DA}\ \ \overline{AB}\ \ \overline{AC}\right]\right|$.

5.7.13 Theorem:

The vector equation of a plane passing through the point $A(\bar{a})$ and parallel to the non-collinear vectors $\bar{b}$ and $\bar{c}$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Proof: Let $\bar{a}$ represent the point A and $P(\bar{r})$ be any point in the plane. We may assume that $A \neq P$.

P lies in the plane.

$\Rightarrow$ The vectors $\overline{AP}, \bar{b}$ and $\bar{c}$ are coplanar

$\Rightarrow \left[\overline{AP}\ \bar{b}\ \bar{c}\right] = 0$ (by Theorem 5.7.6)

$\Rightarrow \overline{AP}.\left(\bar{b}\times\bar{c}\right) = 0$

$\Rightarrow \left(\bar{r}-\bar{a}\right).\left(\bar{b}\times\bar{c}\right) = 0$

$\Rightarrow \bar{r}.\left(\bar{b}\times\bar{c}\right) = \bar{a}.\left(\bar{b}\times\bar{c}\right)$

$\Rightarrow \left[\bar{r}\ \bar{b}\ \bar{c}\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

Suppose $P(\bar{r})$ is any point in the space such that $\left[\bar{r}\ \bar{b}\ \bar{c}\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

In the above argument, if we replace the steps backwards, we will have $\left[\overline{AP}\ \bar{b}\ \bar{c}\right] = 0$

Thus the vectors $\overline{AP}, \bar{b}$ and $\bar{c}$ are coplanar

Hence P lies in the plane.

5.7.14 Theorem:

The vector equation of a plane passing through the point $A(\bar{a}), B(\bar{b})$ and parallel to the vector $\bar{c}$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Proof: Let $P(\bar{r})$ be any point in the plane. We may assume that $A \neq P$.

P lies in the plane $\Leftrightarrow$ The vector $\overline{AP}\times\overline{AB}$ is perpendicular to the plane

$\Leftrightarrow$ The vector $\overline{AP}\times\overline{AB}$ is perpendicular to the vector $\bar{c}$

$$\Leftrightarrow \left(\overline{AP}\times\overline{AB}\right).\bar{c}=0$$

$$\Leftrightarrow \overline{AP}.\left(\overline{AB}\times\bar{c}\right)=0 \text{ (by Theorem 5.7.5)}$$

$$\Leftrightarrow \left(\bar{r}-\bar{a}\right).\left((\bar{b}-\bar{a})\times\bar{c}\right)=0$$

$$\Leftrightarrow \left(\bar{r}-\bar{a}\right).\left(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}\right)=0$$

$$\Leftrightarrow \bar{r}.\left(\bar{b}\times\bar{c}\right)+\bar{r}.\left(\bar{c}\times\bar{a}\right)-\bar{a}.\left(\bar{b}\times\bar{c}\right)-\bar{a}.\left(\bar{c}\times\bar{a}\right)=0$$

$$\Leftrightarrow \left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]-\left[\bar{a}\ \bar{b}\ \bar{c}\right]-\left[\bar{a}\ \bar{c}\ \bar{a}\right]=0$$

$$\Leftrightarrow \left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]\left(\because \left[\bar{a}\ \bar{c}\ \bar{a}\right]=0\right)$$

5.7.15 Theorem:

The vector equation of a plane passing through three non-collinear points is $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]+\left[\bar{r}\ \bar{a}\ \bar{b}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Proof: Let $P(\bar{r})$ be any point in the plane. The four points A, B, C and P are coplanar.

$\Leftrightarrow$ The vectors $\overline{AP}, \overline{AB}$ and $\overline{AC}$ are coplanar

$\Leftrightarrow$ $\bar{r}-\bar{a}, \bar{r}-\bar{b}$ and $\bar{r}-\bar{c}$ are coplanar

$$\Leftrightarrow \left[\bar{r}-\bar{a}\quad \bar{b}-\bar{a}\quad \bar{c}-\bar{a}\right]=0$$

$$\Leftrightarrow \left(\overline{r}-\overline{a}\right).\left((\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\right)=0$$

$$\Leftrightarrow \left(\overline{r}-\overline{a}\right).\left(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right)=0$$

$$\Leftrightarrow \overline{r}.\left(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right)=\overline{a}.\left(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\right)$$

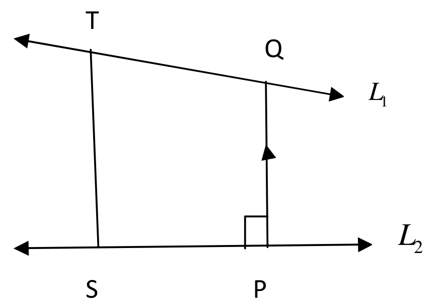
$$\Leftrightarrow \left[\overline{r}\ \overline{b}\ \overline{c}\right]+\left[\overline{r}\ \overline{c}\ \overline{a}\right]+\left[\overline{r}\ \overline{a}\ \overline{b}\right]=\left[\overline{a}\ \overline{b}\ \overline{c}\right]+\left[\overline{a}\ \overline{c}\ \overline{a}\right]+\left[\overline{a}\ \overline{a}\ \overline{b}\right]$$

$$\Leftrightarrow \left[\overline{r}\ \overline{b}\ \overline{c}\right]+\left[\overline{r}\ \overline{c}\ \overline{a}\right]+\left[\overline{r}\ \overline{a}\ \overline{b}\right]=\left[\overline{a}\ \overline{b}\ \overline{c}\right]\left(\because \left[\overline{a}\ \overline{c}\ \overline{a}\right]=\left[\overline{a}\ \overline{a}\ \overline{b}\right]=0\right)$$

5.7.16 Definition (skew lines):

In a space, there are pairs of lines which are neither intersecting nor parallel. Such a pair of lines is called a pair of *skew lines*. Thus, two lines are called *skew lines,* if there is no plane containing both the lines.

5.7.17 Distance between two skew lines:



Let L_1 and L_2 be two skew lines with equations $\overline{r}=\overline{a_1}+\lambda\overline{b_1}$ and $\overline{r}=\overline{a_2}+\mu\overline{b_2}$. Let S be the point on L_1 with position vector $\overline{a_1}$ and let T be the point on L_2 with position vector $\overline{a_2}$. Then the magnitude of the vector of shortest distance will be equal to that of the projection of $\overline{ST}$ along the direction of the line of shortest distance.

If $\overline{PQ}$ is the vector of shortest distance between L_1 and L_2, then it is perpendicular to both $\overline{b_1}$ and $\overline{b_2}$. The unit vector $\hat{n}$ along $\overline{PQ}$ would therefore be

$\hat{n}=\dfrac{\overline{b_1}\times\overline{b_2}}{\left|\overline{b_1}\times\overline{b_2}\right|}$. Then $\overline{PQ}=d\,\hat{n}$ where d is the magnitude of the shortest distance vector.

Let θ be the angle between $\overline{ST}$ and $\overline{PQ}$.

Then $\left|\overline{PQ}\right|=\left|\overline{ST}\right|\left|\cos\theta\right|$

But $\cos\theta = \left| \frac{\overline{PQ}.\overline{ST}}{|\overline{PQ}||\overline{ST}|} \right| = \left| \frac{d\,\hat{n}.(\overline{a_2} - \overline{a_1})}{d|\overline{ST}|} \right|$, since $\overline{ST} = \overline{a_2} - \overline{a_1}$.

$$= \left| \frac{(\overline{b_1} \times \overline{b_2}).(\overline{a_2} - \overline{a_1})}{|\overline{ST}||\overline{b_1} \times \overline{b_2}|} \right|, \text{ since } \hat{n} = \frac{\overline{b_1} \times \overline{b_2}}{|\overline{b_1} \times \overline{b_2}|}.$$

Hence the required shortest distance is

$$d = |\overline{PQ}| = |\overline{ST}||\cos\theta| = = \left| \frac{(\overline{b_1} \times \overline{b_2}).(\overline{a_2} - \overline{a_1})}{|\overline{b_1} \times \overline{b_2}|} \right|.$$

5.7.18 Cartesian form:

The shortest distance between the lines $L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $\dfrac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$.

5.7.19 Definition (Vector triple product):

Suppose $\overline{a}, \overline{b}, \overline{c}$ are three vectors. Then $\overline{a} \times (\overline{b} \times \overline{c})$ or $(\overline{a} \times \overline{b}) \times \overline{c}$ is called the *vector triple product* or *vector product of three vectors*.

5.7.20 Theorem:

Let $\overline{a}, \overline{b}, \overline{c}$ be three vectors. Then

(*i*) $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a}.\overline{c})\overline{b} - (\overline{b}.\overline{c})\overline{a}$

(*ii*) $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c}$

Proof: (i) With out loss of generality, we may assume that $\overline{a}$ and $\overline{b}$ are non-collinear vectors and $\overline{c}$ is not parallel to $\overline{a} \times \overline{b}$, as otherwise $(\overline{a} \times \overline{b}) \times \overline{c} = \overline{0} = (\overline{a}.\overline{c})\overline{b} - (\overline{b}.\overline{c})\overline{a}$. Fix the origin *O*. Let $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}$. We consider the plane *OAB* as XY^- plane. Let $\overline{i}$ be the unit vector in the direction of $\overline{OA}$ and $\overline{j}$ be unit vector perpendicular to $\overline{i}$ in the XY^- plane. Fix the unit vector $\overline{k}$ in the XY^- plane such that $(\overline{i}, \overline{j}, \overline{k})$ is an orthogonal triad of unit vectors forming a right handed system. Then we can write $\bar{a} = a_1\bar{i}, \bar{b} = b_1\bar{i} + b_2\bar{j}$ and $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$.

$$\left(\bar{a}\times\bar{b}\right)\times\bar{c} = \left(a_1b_2\bar{k}\right)\times\bar{c} = \left(a_1b_2\bar{k}\right)\times\left(c_1\bar{i} + c_2\bar{j} + c_3\bar{k}\right)$$

$$= a_1b_2c_1\bar{j} - a_1b_2c_2\bar{k}$$

$$(\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a} = a_1c_1(b_1\bar{i} + b_2\bar{j}) - (b_1c_1 + b_2c_2)(c_1\bar{i} + c_2\bar{j} + c_3\bar{k})$$

$$= a_1b_2c_1\bar{j} - a_1b_2c_2\bar{k}$$

$$\therefore \left(\bar{a}\times\bar{b}\right)\times\bar{c} = (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a}$$

(ii) $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = -\left((\bar{b}\times\bar{c})\times\bar{a}\right) = (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a})(\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

$$= -\left[(\bar{b}.\bar{a})\bar{c} - (\bar{c}.\bar{a})\bar{b}\right]$$

$$= (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$$

$$\therefore \bar{a}\times\left(\bar{b}\times\bar{c}\right) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$$

5.7.21 Note:

In general, the vector product of three vectors is not associative.

5.7.22 Theorem:

For any four vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$ and in particular

$(\bar{a}\times\bar{b})^2 = (\bar{a})^2(\bar{b})^2 - (\bar{a}.\bar{b})^2$.

Proof: $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = \bar{a}.\left(\bar{b}\times(\bar{c}\times\bar{d})\right)$

$$= \bar{a}.\left((\bar{b}.\bar{d})\bar{c} - (\bar{b}.\bar{c})\bar{d}\right)$$

$$= (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c}) = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$$

In the above formula if $\bar{c} = \bar{a}$ and $\bar{d} = \bar{b}$, then $(\bar{a}\times\bar{b}).(\bar{a}\times\bar{b}) = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} \end{vmatrix}$

$$= (\bar{a}.\bar{a})(\bar{b}.\bar{b}) - (\bar{a}.\bar{b})(\bar{b}.\bar{a}) = (\bar{a})^2(\bar{b})^2 - (\bar{a}.\bar{b})^2.$$

5.7.23 Solved Problems:

1. Problem: If $\bar{a} = 6\bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{b} = 2\bar{i} - 9\bar{j} + 6\bar{k}$ then find the angle between the vectors $\bar{a}$ and $\bar{b}$.

Solution: Given $\bar{a} = 6\bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{b} = 2\bar{i} - 9\bar{j} + 6\bar{k}$

We have $\bar{a}.\bar{b} = (6\bar{i} + 2\bar{j} + 3\bar{k}).(2\bar{i} - 9\bar{j} + 6\bar{k}) = 6.2 + 2(-9) + 3.6$

$$= 12 - 18 + 18 = 12$$

Also $|\bar{a}| = |6\bar{i} + 2\bar{j} + 3\bar{k}| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{36+4+9} = \sqrt{49} = 7$

$|\bar{b}| = |2\bar{i} - 9\bar{j} + 6\bar{k}| = \sqrt{2^2 + (-9)^2 + 6^2} = \sqrt{4+81+36} = \sqrt{121} = 11$

Let θ be the angle between the vectors $\bar{a}$ and $\bar{b}$ then $\theta = \cos^{-1}\left(\dfrac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{12}{7.11}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{12}{77}\right)$$

2. Problem: If $|\bar{a}| = 11,\ |\bar{b}| = 23$ and $|\bar{a} - \bar{b}| = 30$ then find the angle between the vectors $\bar{a}$ and $\bar{b}$ also find $|\bar{a} + \bar{b}|$.

Solution: Given $|\bar{a}| = 11,\ |\bar{b}| = 23$ and $|\bar{a} - \bar{b}| = 30$

We have $|\bar{a} - \bar{b}|^2 = (\bar{a} - \bar{b})^2 = |\bar{a}|^2 + |\bar{b}|^2 - 2\bar{a}.\bar{b}$

$$\Rightarrow 2\bar{a}.\bar{b} = |\bar{a}|^2 + |\bar{b}|^2 - |\bar{a} - \bar{b}|^2 \Rightarrow 2\bar{a}.\bar{b} = 11^2 + 23^2 - 30^2$$

$$\Rightarrow 2\bar{a}.\bar{b} = 121 + 529 - 900 \Rightarrow 2\bar{a}.\bar{b} = 650 - 900 \Rightarrow 2\bar{a}.\bar{b} = -250$$

$$\Rightarrow \bar{a}.\bar{b} = -125$$

Let θ be the angle between the vectors $\bar{a}$ and $\bar{b}$ then $\theta = \cos^{-1}\left(\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-125}{11.23}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{-125}{253}\right) \Rightarrow \theta = \pi - \cos^{-1}\left(\frac{125}{253}\right)$$

Also $|\bar{a}+\bar{b}|^2 = (\bar{a}+\bar{b})^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}.\bar{b} \Rightarrow |\bar{a}+\bar{b}|^2 = 11^2 + 23^2 + 2(-125)$

$$\Rightarrow |\bar{a}+\bar{b}|^2 = 121+529-250 \Rightarrow |\bar{a}+\bar{b}|^2 = 650-250 \Rightarrow |\bar{a}+\bar{b}|^2 = 400$$

$$\therefore |\bar{a}+\bar{b}| = 20$$

3. Problem: If the vectors $\lambda\bar{i} - 3\bar{j} + 5\bar{k}$ and $2\lambda\bar{i} - \lambda\bar{j} - \bar{k}$ are perpendicular to each other then find λ.

Solution: Let $\bar{a} = \lambda\bar{i} - 3\bar{j} + 5\bar{k}$ and $\bar{b} = 2\lambda\bar{i} - \lambda\bar{j} - \bar{k}$

Given that $\bar{a}$ and $\bar{b}$ are perpendicular implies we have $\bar{a}.\bar{b} = 0$

$$\Rightarrow (\lambda\bar{i} - 3\bar{j} + 5\bar{k}).(2\lambda\bar{i} - \lambda\bar{j} - \bar{k}) = 0 \Rightarrow \lambda.2\lambda + (-3)(-\lambda) + 5(-1) = 0$$

$$\Rightarrow 2\lambda^2 + 3\lambda - 5 = 0 \Rightarrow (2\lambda+5)(\lambda-1) = 0 \Rightarrow \lambda = 1, \lambda = \frac{-5}{2}$$

4. Problem: If $|\bar{a}| = 2$, $|\bar{b}| = 3$ and $|\bar{c}| = 4$ and each of the $\bar{a}, \bar{b}, \bar{c}$ is perpendicular to the sum of the other two then find the magnitude of $\bar{a}+\bar{b}+\bar{c}$.

Solution: Given $|\bar{a}| = 2$, $|\bar{b}| = 3$ and $|\bar{c}| = 4$

Also each of the $\bar{a}, \bar{b}, \bar{c}$ is perpendicular to the sum of the other two

i.e $\bar{a} \perp (\bar{b}+\bar{c})$, $\bar{b} \perp (\bar{c}+\bar{a})$, $\bar{c} \perp (\bar{a}+\bar{b})$

$$\Rightarrow \bar{a}.(\bar{b}+\bar{c}) = 0,\ \bar{b}.(\bar{c}+\bar{a}) = 0,\ \bar{c}.(\bar{a}+\bar{b}) = 0$$

$$\Rightarrow \bar{a}.\bar{b} + \bar{a}.\bar{c} = 0,\ \bar{b}.\bar{c} + \bar{b}.\bar{a} = 0,\ \bar{a}.\bar{b} + \bar{a}.\bar{c} = 0$$

We have $|\bar{a}+\bar{b}+\bar{c}|^2 = |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 + \bar{a}.\bar{b} + \bar{a}.\bar{c} + \bar{b}.\bar{c} + \bar{b}.\bar{a} + \bar{a}.\bar{b} + \bar{a}.\bar{c}$

$$\Rightarrow \left|\bar{a}+\bar{b}+\bar{c}\right|^2 = 2^2+3^2+4^2+0+0+0 \;\Rightarrow \left|\bar{a}+\bar{b}+\bar{c}\right|^2 = 4+9+16$$

$$\Rightarrow \left|\bar{a}+\bar{b}+\bar{c}\right|^2 = 29 \;\Rightarrow \left|\bar{a}+\bar{b}+\bar{c}\right| = \sqrt{29}$$

$\therefore$ The magnitude of $\bar{a}+\bar{b}+\bar{c}$ is $\left|\bar{a}+\bar{b}+\bar{c}\right| = \sqrt{29}$.

5. Problem: Find the area of the parallelogram for which the vectors $\bar{a} = 2\bar{i} - 3\bar{j}$ and $\bar{b} = 3\bar{i} - \bar{k}$ are adjacent sides.

Solution: Given $\bar{a} = 2\bar{i} - 3\bar{j}$ and $\bar{b} = 3\bar{i} - \bar{k}$

The vector area of the parallelogram for which the vectors $\bar{a}$ and $\bar{b}$ are adjacent sides is $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix}$

$$\Rightarrow \bar{a}\times\bar{b} = \bar{i}\begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix} - \bar{j}\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & -3 \\ 3 & 0 \end{vmatrix}$$

$$\Rightarrow \bar{a}\times\bar{b} = \bar{i}((-3)(-1)-0.0) - \bar{j}(2(-1)-0.3) + \bar{k}(2.0-(-3).3)$$

$$\Rightarrow \bar{a}\times\bar{b} = 3\bar{i} + 2\bar{j} + 9\bar{k}$$

The area of the parallelogram is $\left|\bar{a}\times\bar{b}\right| = \left|3\bar{i}+2\bar{j}+9\bar{k}\right|$

$$= \sqrt{3^2+2^2+9^2} = \sqrt{9+4+81} = \sqrt{94}$$

6. Problem: If the vector $4\bar{i} + \frac{2p}{3}\bar{j} + p\bar{k}$ is parallel to the vector $\bar{i}+2\bar{j}+3\bar{k}$ then find the value of p.

Solution: Let $\bar{a} = 4\bar{i} + \frac{2p}{3}\bar{j} + p\bar{k}$ and $\bar{b} = \bar{i}+2\bar{j}+3\bar{k}$

Given that $\bar{a}$ and $\bar{b}$ are parallel implies we have $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

$$\Rightarrow \frac{4}{1} = \frac{2p/3}{2} = \frac{p}{3} \;\Rightarrow \frac{4}{1} = \frac{2p/3}{2}, \frac{4}{1} = \frac{p}{3} \Rightarrow p = 12$$

7. Problem: If $|\bar{a}| = 13,\ |\bar{b}| = 5$ and $\bar{a}.\bar{b} = 60$ then find $|\bar{a}\times\bar{b}|$.

Solution: Given $|\bar{a}| = 13,\ |\bar{b}| = 5$ and $\bar{a}.\bar{b} = 60$

$$\text{We have } |\bar{a}.\bar{b}|^2 + |\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 \ \Rightarrow |\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - |\bar{a}.\bar{b}|^2$$

$$\Rightarrow |\bar{a}\times\bar{b}|^2 = 13^2.5^2 - 60^2 = 65^2 - 60^2 = 4225 - 3600 = 625$$

$$\therefore\ |\bar{a}\times\bar{b}| = \sqrt{625} = 25$$

8. Problem: If $\bar{a} = 7\bar{i} - 2\bar{j} + 3\bar{k},\ \bar{b} = 2\bar{i} + 8\bar{k}$ and $\bar{c} = \bar{i} + \bar{j} + \bar{k}$ then compute $\bar{a}\times\bar{b},\ \bar{a}\times\bar{c}$, and $\bar{a}\times(\bar{b}+\bar{c})$.

Solution: Given $\bar{a} = 7\bar{i} - 2\bar{j} + 3\bar{k},\ \bar{b} = 2\bar{i} + 8\bar{k}$ and $\bar{c} = \bar{i} + \bar{j} + \bar{k}$

$$\text{Now } \bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & -2 & 3 \\ 2 & 0 & 8 \end{vmatrix} = \bar{i}\begin{vmatrix} -2 & 3 \\ 0 & 8 \end{vmatrix} - \bar{j}\begin{vmatrix} 7 & 3 \\ 2 & 8 \end{vmatrix} + \bar{k}\begin{vmatrix} 7 & -2 \\ 2 & 0 \end{vmatrix}$$

$$= \bar{i}((-2)(8) - 3.0) - \bar{j}(7(8) - 2.3) + \bar{k}(7.0 - (-2).2)$$

$$= -16\bar{i} - 50\bar{j} + 4\bar{k}$$

$$\text{Now } \bar{a}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} - \bar{j}\begin{vmatrix} 7 & 3 \\ 1 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 7 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= \bar{i}((-2)(1) - 3.1) - \bar{j}(7(1) - 3.1) + \bar{k}(7.1 - (-2).1)$$

$$= -5\bar{i} - 4\bar{j} + 9\bar{k}$$

$$\text{We have } \bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b} + \bar{a}\times\bar{c} \ = -16\bar{i} - 50\bar{j} + 4\bar{k} - 5\bar{i} - 4\bar{j} + 9\bar{k}$$

$$= -21\bar{i} - 54\bar{j} + 13\bar{k}$$

9. Problem: If $\bar{a} = 3\bar{i} - \bar{j} + 2\bar{k},\ \bar{b} = -\bar{i} + 3\bar{j} + 2\bar{k},\ \bar{c} = 4\bar{i} + 5\bar{j} - 2\bar{k}$ and $\bar{d} = \bar{i} + 3\bar{j} + 5\bar{k}$ then compute the following $(i)\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$ $(ii)\left(\bar{a}\times\bar{b}\right).\bar{c} - \left(\bar{a}\times\bar{d}\right).\bar{b}$

Solution: Given $\bar{a} = 3\bar{i} - \bar{j} + 2\bar{k},\ \bar{b} = -\bar{i} + 3\bar{j} + 2\bar{k},\ \bar{c} = 4\bar{i} + 5\bar{j} - 2\bar{k}$ and $\bar{d} = \bar{i} + 3\bar{j} + 5\bar{k}$

Now $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & -1 & 2\\ -1 & 3 & 2\end{vmatrix}=\bar{i}\begin{vmatrix}-1 & 2\\ 3 & 2\end{vmatrix}-\bar{j}\begin{vmatrix}3 & 2\\ -1 & 2\end{vmatrix}+\bar{k}\begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}$

$$=\bar{i}((-1)(2)-2.3)-\bar{j}(3(2)-2(-1))+\bar{k}(3.3-(-1)(-1))$$

$$=\bar{i}(-2-6)-\bar{j}(6+2)+\bar{k}(9-1)=-8\bar{i}-8\bar{j}+8\bar{k}$$

Now $\bar{c}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 4 & 5 & -2\\ 1 & 3 & 5\end{vmatrix}=\bar{i}\begin{vmatrix}5 & -2\\ 3 & 5\end{vmatrix}-\bar{j}\begin{vmatrix}4 & -2\\ 1 & 5\end{vmatrix}+\bar{k}\begin{vmatrix}4 & 5\\ 1 & 3\end{vmatrix}$

$$=\bar{i}(5.5-(-2)3)-\bar{j}(4.5-(-2)1)+\bar{k}(4.3-1.5)$$

$$=\bar{i}(25+6)-\bar{j}(20+2)+\bar{k}(12-5)=31\bar{i}-22\bar{j}+7\bar{k}$$

Now $\bar{a}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & -1 & 2\\ 1 & 3 & 5\end{vmatrix}=\bar{i}\begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix}-\bar{j}\begin{vmatrix}3 & 2\\ 1 & 5\end{vmatrix}+\bar{k}\begin{vmatrix}3 & -1\\ 1 & 3\end{vmatrix}$

$$=\bar{i}((-1)(5)-2.3)-\bar{j}(3.5-2.1)+\bar{k}(3.3-(-1)1)$$

$$=\bar{i}(-5-6)-\bar{j}(15-2)+\bar{k}(9+1)=-11\bar{i}-13\bar{j}+10\bar{k}$$

We have $(\bar{a}\times\bar{b}).\bar{c}=(-8\bar{i}-8\bar{j}+8\bar{k}).(4\bar{i}+5\bar{j}-2\bar{k})$

$$=(-8)4+(-8)5+8(-2)=-32-40-16=-88$$

We have $(\bar{a}\times\bar{d}).\bar{b}=(-11\bar{i}-13\bar{j}+10\bar{k}).(-\bar{i}+3\bar{j}+2\bar{k})$

$$=(-11)(-1)+(-13)3+10.2=11-39+20=-8$$

$(i)\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -8 & -8 & 8\\ 31 & -22 & 7\end{vmatrix}=\bar{i}\begin{vmatrix}-8 & 8\\ -22 & 7\end{vmatrix}-\bar{j}\begin{vmatrix}-8 & 8\\ 31 & 7\end{vmatrix}+\bar{k}\begin{vmatrix}-8 & -8\\ 31 & -22\end{vmatrix}$

$$=\bar{i}((-8)7-8(-22))-\bar{j}((-8)7-8.31)+\bar{k}((-8)(-22)-(-8)31)$$

$$=\bar{i}(-56+176)-\bar{j}(-56-248)+\bar{k}(176+248)$$

$$=120\bar{i}+304\bar{j}+428\bar{k}$$

$(ii)\left(\bar{a}\times\bar{b}\right).\bar{c}-\left(\bar{a}\times\bar{d}\right).\bar{b}=-88+8=-80$

10. Problem: If $\bar{a}=2\bar{i}-\bar{j}+\bar{k},\ \bar{b}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{c}=3\bar{i}+p\bar{j}+5\bar{k}$ are coplanar then find p

Solution: Given $\bar{a}=2\bar{i}-\bar{j}+\bar{k},\ \bar{b}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{c}=3\bar{i}+p\bar{j}+5\bar{k}$

Since $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar then we have $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

$$\Rightarrow\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & -3\\ 3 & p & 5\end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right]=0\Rightarrow\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & -3\\ 3 & p & 5\end{vmatrix}.1=0\ \left(\because\left[\bar{i}\ \bar{j}\ \bar{k}\right]=1\right)$$

$$\Rightarrow 2\begin{vmatrix}2 & -3\\ p & 5\end{vmatrix}+1\begin{vmatrix}1 & -3\\ 3 & 5\end{vmatrix}+1\begin{vmatrix}1 & 2\\ 3 & p\end{vmatrix}=0$$

$$\Rightarrow 2(2.5-(-3)p)+1(1.5-(-3)3)+1(1.p-2.3)=0$$

$$\Rightarrow 2(10+3p)+1(5+9)+1(p-6)=0\Rightarrow 20+6p+14+p-6=0$$

$$\Rightarrow 7p+28=0\Rightarrow 7p=-28\Rightarrow p=-4$$

11. Problem: Simplify the following

$(i)\left(\bar{i}-2\bar{j}+3\bar{k}\right)\times\left(2\bar{i}+\bar{j}-\bar{k}\right).\left(\bar{j}+\bar{k}\right)$

$(ii)\left(2\bar{i}-3\bar{j}+\bar{k}\right).\left(\bar{i}-\bar{j}+2\bar{k}\right)\times\left(2\bar{i}+\bar{j}+\bar{k}\right)$

Solution: (i) Let $\bar{a}=\bar{i}-2\bar{j}+3\bar{k},\ \bar{b}=2\bar{i}+\bar{j}-\bar{k}$ and $\bar{c}=\bar{j}+\bar{k}$

We have for any vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ $\quad\bar{a}\times\bar{b}.\bar{c}=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$$\Rightarrow\left(\bar{i}-2\bar{j}+3\bar{k}\right)\times\left(2\bar{i}+\bar{j}-\bar{k}\right).\left(\bar{j}+\bar{k}\right)=\begin{vmatrix}1 & -2 & 3\\ 2 & 1 & -1\\ 0 & 1 & 1\end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right]$$

$$=\begin{vmatrix}1 & -2 & 3\\ 2 & 1 & -1\\ 0 & 1 & 1\end{vmatrix}.1\ \left(\because\left[\bar{i}\ \bar{j}\ \bar{k}\right]=1\right)=1\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}+2\begin{vmatrix}2 & -1\\ 0 & 1\end{vmatrix}+3\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}$$

$$=1(1.1-(-1)1)+2(2.1-(-1)0)+3(2.1-0.1)=2+4+6=12$$

(ii) Let $\bar{a} = 2\bar{i} - 3\bar{j} + \bar{k},\ \bar{b} = \bar{i} - \bar{j} + 2\bar{k}$ and $\bar{c} = 2\bar{i} + \bar{j} + \bar{k}$

We have for any vectors $\bar{a}, \bar{b}$ and $\bar{c}$ $\quad \bar{a}.\bar{b}\times\bar{c} = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$$\Rightarrow \left(2\bar{i} - 3\bar{j} + \bar{k}\right).\left(\bar{i} - \bar{j} + 2\bar{k}\right)\times\left(2\bar{i} + \bar{j} + \bar{k}\right) = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right]$$

$$= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}.1\ \left(\because \left[\bar{i}\ \bar{j}\ \bar{k}\right] = 1\right) = 2\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 2((-1)1 - 2.1) + 3(1.1 - 2.2) + 1(1.1 - (-1)2) \quad = -6 - 9 + 3 = -12$$

12. Problem: Find λ in order that the four points $A = (3,2,1),\ B = (4,\lambda,5),$ $C = (4,2,-2)$ and $D = (6,5,-1)$ be coplanar.

Solution: Given $A = (3,2,1),\ B = (4,\lambda,5),\ C = (4,2,-2)$ and $D = (6,5,-1)$

Let $O = (0,0,0)$ be the position vector of the origin.

$\therefore \overline{OA} = 3\bar{i} + 2\bar{j} + \bar{k},\ \overline{OB} = 4\bar{i} + \lambda\bar{j} + 5\bar{k},\ \overline{OC} = 4\bar{i} + 2\bar{j} - 2\bar{k},$

$\overline{OD} = 6\bar{i} + 5\bar{j} - \bar{k}$

We have $\overline{AB} = \overline{OB} - \overline{OA}$

$\Rightarrow \overline{AB} = (4\bar{i} + \lambda\bar{j} + 5\bar{k}) - (3\bar{i} + 2\bar{j} + \bar{k}) \Rightarrow \overline{AB} = \bar{i} + (\lambda - 2)\bar{j} + 4\bar{k}$

We have $\overline{AC} = \overline{OC} - \overline{OA}$

$\Rightarrow \overline{AC} = (4\bar{i} + 2\bar{j} - 2\bar{k}) - (3\bar{i} + 2\bar{j} + \bar{k}) \Rightarrow \overline{AC} = \bar{i} - 3\bar{k}$

We have $\overline{AD} = \overline{OD} - \overline{OA}$

$\Rightarrow \overline{AD} = (6\bar{i} + 5\bar{j} - \bar{k}) - (3\bar{i} + 2\bar{j} + \bar{k}) \Rightarrow \overline{AD} = 3\bar{i} + 3\bar{j} - 2\bar{k}$

Since A, B, C and D are coplanar then we have $\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$

$$\Rightarrow \begin{vmatrix} 1 & \lambda - 2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right] = 0 \Rightarrow \begin{vmatrix} 1 & \lambda - 2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}.1 = 0 \quad \left(\because \left[\bar{i}\ \bar{j}\ \bar{k}\right] = 1\right)$$

$$\Rightarrow 1\begin{vmatrix} 0 & -3 \\ 3 & -2 \end{vmatrix} - (\lambda - 2)\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 1(0(-2) - (-3)3) - (\lambda - 2)(1(-2) - (-3)3) + 4(1.3 - 0.3) = 0$$

$$\Rightarrow 1(0+9) - (\lambda - 2)(-2+9) + 4(3-0) = 0 \Rightarrow 9 - (\lambda - 2)7 + 12 = 0$$

$$\Rightarrow 21 - 7\lambda + 14 = 0 \Rightarrow 7\lambda = 35 \Rightarrow \lambda = 5$$

13. Problem: Find the volume of the tetrahedron having the edges $\bar{i} + \bar{j} + \bar{k}, \bar{i} - \bar{j}$ and $\bar{i} + 2\bar{j} + \bar{k}$

Solution: Let $\bar{a} = \bar{i} + \bar{j} + \bar{k}$, $\bar{b} = \bar{i} - \bar{j}$ and $\bar{c} = \bar{i} + 2\bar{j} + \bar{k}$

The volume of the tetrahedron having the edges $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $V = \frac{1}{6}\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$

$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right] = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix}.1 \ \left(\because \left[\bar{i}\ \bar{j}\ \bar{k}\right] = 1\right)$$

$$= \frac{1}{6}\left(1\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}\right)$$

$$= \frac{1}{6}\left(1((-1)1 - 0.2) - 1(1.1 - 0.1) + 1(1.2 - (-1)1)\right)$$

$$= \frac{1}{6}\left(-1 - 1 + 3\right) = \frac{1}{6}\ \textit{cubic units.}$$

14. Problem: Compute $\left[\bar{i} - \bar{j}\ \ \bar{j} - \bar{k}\ \ \bar{k} - \bar{i}\right]$

Solution: Let $\bar{a} = \bar{i} - \bar{j}$, $\bar{b} = \bar{j} - \bar{k}$ and $\bar{c} = \bar{k} - \bar{i}$

Now $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\bar{i} - \bar{j}\ \ \bar{i} - \bar{k}\ \ \bar{k} - \bar{i}\right]$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}\left[\bar{i}\ \bar{j}\ \bar{k}\right] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}.1 \ \left(\because \left[\bar{i}\ \bar{j}\ \bar{k}\right] = 1\right)$$

$$= 1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

$$= \left(1(1.1-(-1)0)+1(0.1-(-1)(-1))+0(0.0-(-1)1)\right)=1-1=0$$

15. Problem: If $\bar{a}=(1,-2,1)$, $\bar{b}=(2,1,1)$ and $\bar{c}=(1,2,-1)$ then find $\left|\bar{a}\times(\bar{b}\times\bar{c})\right|$ and $\left|(\bar{a}\times\bar{b})\times\bar{c}\right|$

Solution: Given $\bar{a}=(1,-2,1)$, $\bar{b}=(2,1,1)$ and $\bar{c}=(1,2,-1)$

i.e $\bar{a}=\bar{i}-2\bar{j}+\bar{k}$, $\bar{b}=2\bar{i}+\bar{j}+\bar{k}$ and $\bar{c}=\bar{i}+2\bar{j}-\bar{k}$

Now $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 1\\ 2 & 1 & 1\end{vmatrix}=\bar{i}\begin{vmatrix}-2 & 1\\ 1 & 1\end{vmatrix}-\bar{j}\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}+\bar{k}\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}$

$$=\bar{i}((-2)1-1.1)-\bar{j}(1.1-2.1)+\bar{k}(1.1-(-2)2)$$

$$=\bar{i}(-2-1)-\bar{j}(1-2)+\bar{k}(1+4)=-3\bar{i}+\bar{j}+5\bar{k}$$

Now $\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 1 & 1\\ 1 & 2 & -1\end{vmatrix}=\bar{i}\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}-\bar{j}\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}+\bar{k}\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}$

$$=\bar{i}(1(-1)-2.1)-\bar{j}(2(-1)-1.1)+\bar{k}(2.2-1.1)$$

$$=\bar{i}(-1-2)-\bar{j}(-2-1)+\bar{k}(4-1)=-3\bar{i}+3\bar{j}+3\bar{k}$$

Now $\bar{a}\times(\bar{b}\times\bar{c})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 1\\ -3 & 3 & 3\end{vmatrix}=\bar{i}\begin{vmatrix}-2 & 1\\ 3 & 3\end{vmatrix}-\bar{j}\begin{vmatrix}1 & 1\\ -3 & 3\end{vmatrix}+\bar{k}\begin{vmatrix}1 & -2\\ -3 & 3\end{vmatrix}$

$$=\bar{i}((-2)3-1.3)-\bar{j}(1.3-(-3)1)+\bar{k}(1.3-(-2)(-3))$$

$$=\bar{i}(-6-3)-\bar{j}(3+3)+\bar{k}(3-6)=-9\bar{i}-6\bar{j}-3\bar{k}$$

Now $(\bar{a}\times\bar{b})\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -3 & 1 & 5\\ 1 & 2 & -1\end{vmatrix}=\bar{i}\begin{vmatrix}1 & 5\\ 2 & -1\end{vmatrix}-\bar{j}\begin{vmatrix}-3 & 5\\ 1 & -1\end{vmatrix}+\bar{k}\begin{vmatrix}-3 & 1\\ 1 & 2\end{vmatrix}$

$$=\bar{i}(1(-1)-2.5)-\bar{j}((-3)(-1)-5.1)+\bar{k}((-3)2-1.1)$$

$$= \bar{i}(-1-10) - \bar{j}(3-5) + \bar{k}(-6-1) = -11\bar{i} + 2\bar{j} - 7\bar{k}$$

We have $\left| \bar{a}\times(\bar{b}\times\bar{c})\right| = \left|-9\bar{i} - 6\bar{j} - 3\bar{k}\right| = \sqrt{(-9)^2 + (-6)^2 + (-3)^2} = \sqrt{81+36+9} = \sqrt{126}$

We have $\left| (\bar{a}\times\bar{b})\times\bar{c}\right| = \left|-11\bar{i} + 2\bar{j} - 7\bar{k}\right| = \sqrt{(-11)^2 + (2)^2 + (-7)^2}$

$= \sqrt{121+4+49} = \sqrt{174}$

. **16. Problem:** If $\bar{a} = 2\bar{i} + 2\bar{j} - 3\bar{k},\ \bar{b} = 3\bar{i} - \bar{j} + 2\bar{k}$ then find the angle between $(2\bar{a}+\bar{b})$ and $(\bar{a}+2\bar{b})$

Solution: Given $\bar{a} = 2\bar{i} + 2\bar{j} - 3\bar{k},\ \bar{b} = 3\bar{i} - \bar{j} + 2\bar{k}$

$$2\bar{a}+\bar{b} = 2(2\bar{i} + 2\bar{j} - 3\bar{k}) + 3\bar{i} - \bar{j} + 2\bar{k} = 4\bar{i} + 4\bar{j} - 6\bar{k} + 3\bar{i} - \bar{j} + 2\bar{k} = 7\bar{i} + 3\bar{j} - 4\bar{k}$$

$$\bar{a}+2\bar{b} = 2\bar{i} + 2\bar{j} - 3\bar{k} + 2(3\bar{i} - \bar{j} + 2\bar{k}) = 2\bar{i} + 2\bar{j} - 3\bar{k} + 6\bar{i} - 2\bar{j} + 4\bar{k} = 8\bar{i} + \bar{k}$$

Now $(2\bar{a}+\bar{b})\times(\bar{a}+2\bar{b}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & 3 & -4 \\ 8 & 0 & 1 \end{vmatrix} = \bar{i}\begin{vmatrix} 3 & -4 \\ 0 & 1 \end{vmatrix} - \bar{j}\begin{vmatrix} 7 & -4 \\ 8 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 7 & 3 \\ 8 & 0 \end{vmatrix}$

$$= \bar{i}(3.1-(-4)0) - \bar{j}(7.1-(-4)8) + \bar{k}(7.0-8.3)$$

$$= \bar{i}(3-0) - \bar{j}(7+32) + \bar{k}(0-24) = 3\bar{i} - 49\bar{j} - 24\bar{k}$$

$$\left|(2\bar{a}+\bar{b})\times(\bar{a}+2\bar{b})\right| = \left|3\bar{i} - 49\bar{j} - 24\bar{k}\right| = \sqrt{(3)^2 + (-49)^2 + (-24)^2}$$

$$= \sqrt{9+2401+576} = \sqrt{2986}$$

$$\left|2\bar{a}+\bar{b}\right| = \left|7\bar{i} + 3\bar{j} - 4\bar{k}\right| = \sqrt{(7)^2 + (3)^2 + (-4)^2} = \sqrt{49+9+16} = \sqrt{74}$$

$$\left|\bar{a}+2\bar{b}\right| = \left|8\bar{i} + \bar{k}\right| = \sqrt{(8)^2 + (1)^2} = \sqrt{64+1} = \sqrt{65}$$

Let θ be the angle between the vectors $(2\bar{a}+\bar{b})$ and $(\bar{a}+2\bar{b})$ then

$$\theta = \sin^{-1}\left(\frac{\left|(2\bar{a}+\bar{b})\times(\bar{a}+2\bar{b})\right|}{\left|2\bar{a}+\bar{b}\right|\left|\bar{a}+2\bar{b}\right|}\right) \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{2986}}{\sqrt{74}\sqrt{65}}\right)$$

17. Problem: If $\bar{a}, \bar{b}, \bar{c}$ are non- coplanar vectors, then find the value of

$$\frac{(\bar{a}+2\bar{b}-\bar{c}).\left[(\bar{a}-\bar{b})\times(\bar{a}-\bar{b}-\bar{c})\right]}{\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]}$$

Solution: Given $\bar{a}, \bar{b}, \bar{c}$ are non- coplanar vectors $\Rightarrow \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] \neq 0$

We have $(\bar{a}+2\bar{b}-\bar{c}).\left[(\bar{a}-\bar{b})\times(\bar{a}-\bar{b}-\bar{c})\right] = \left[\bar{a}+2\bar{b}-\bar{c}\quad \bar{a}-\bar{b}\quad \bar{a}-\bar{b}-\bar{c}\right]$

$$= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{vmatrix} \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] \quad = \left(1\begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\right)\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

$$= \left(1((-1)(-1)-(-1)0) - 2(1(-1)-0.1) - 1(1(-1)-(-1)1)\right)\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

$$= \left(1(1-0) - 2(-1-0) - 1(-1+1)\right)\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = (1+2+0)\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = 3\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

Now $$\frac{(\bar{a}+2\bar{b}-\bar{c}).\left[(\bar{a}-\bar{b})\times(\bar{a}-\bar{b}-\bar{c})\right]}{\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]} = \frac{3\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]}{\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]} = 3$$

18. Problem: Find the equation of the plane passing through the points $A=(2,3,-1)$, $B=(4,5,2)$ and $C=(3,6,5)$.

Solution: Let O be the origin. Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ be the position vector of any point P in the plane of ΔABC. Then $\overline{AP}, \overline{AB}, \overline{AC}$ are coplanar.

$$\therefore \left[\overline{AP}\ \ \overline{AB}\ \ \overline{AC}\right] = 0.$$

Now $\overline{AP} = (x-2)\bar{i} + (y-3)\bar{j} + (z+1)\bar{k}$, $\overline{AB} = 2\bar{i} + 2\bar{j} + 3\bar{k}$, $\overline{AC} = \bar{i} + 3\bar{j} + 6\bar{k}$.

$$\therefore \left[\overline{AP}\ \ \overline{AB}\ \ \overline{AC}\right] = 0 \Rightarrow \begin{vmatrix} x-2 & y-3 & z+1 \\ 2 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}\left[\bar{i}\ \ \bar{j}\ \ \bar{k}\right] = 0$$

$$i.e \quad \begin{vmatrix} x-2 & y-3 & z+1 \\ 2 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 0 \quad \left(\because \left[\bar{i}\ \ \bar{j}\ \ \bar{k}\right] = 1\right)$$

$$i.e \quad (x-2)\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - (y-3)\begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix} + (z+1)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 0$$

i.e $(x-2)(12-9)-(y-3)(12-3)+(z+1)(6-2)=0$

i.e $3(x-2)-9(y-3)+4(z+1)=0$

i.e $3x-9y+4z+25=0$

Exercise 5

1. Find the cosine angle between the vectors $2\bar{i}-\bar{j}+\bar{k}$ and $3\bar{i}+4\bar{j}-\bar{k}.$
2. If the vectors $2\bar{i}+\lambda\bar{j}-\bar{k}$ and $4\bar{i}-2\bar{j}+2\bar{k}$ are perpendicular to each other, find $\lambda.$
3. If $\bar{a}+\bar{b}+\bar{c}=\bar{0},\ |\bar{a}|=3,\ |\bar{b}|=5$ and $|\bar{c}|=7,$ then find the cosine angle between the vectors $\bar{a}$ and $\bar{b}.$
4. If $|\bar{a}|=2,\ |\bar{b}|=3$ and $(\bar{a},\bar{b})=30^0,$ then find $|\bar{a}\times\bar{b}|^2.$
5. Find the unit vector perpendicular to both $\bar{i}+\bar{j}+\bar{k}$ and $2\bar{i}+\bar{j}+3\bar{k}.$
6. If is θ the angle between $\bar{i}+\bar{j}$ and $\bar{j}+\bar{k},$ then find $\sin\theta.$
7. Find the area of the parallelogram whose diagonals are $3\bar{i}+\bar{j}-2\bar{k}$ and $\bar{i}-3\bar{j}+4\bar{k}.$
8. Find the area of the triangle having $3\bar{i}+4\bar{j}$ and $-5\bar{i}+7\bar{j}$ as two of its edges.
9. Find the equation of the plane passing through $A(1,2,3), B(2,3,1)$ and $C(3,1,2).$
10. If $\bar{a}+\bar{b}+\bar{c}=\bar{0},$ then prove that $\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}.$
11. If $\bar{a}=2\bar{i}+\bar{j}-\bar{k},\ \bar{b}=-\bar{i}+2\bar{j}-4\bar{k}$ and $\bar{c}=\bar{i}+\bar{j}+\bar{k},$ then find $(\bar{a}\times\bar{b}).(\bar{b}\times\bar{c})$
12. If $\bar{a}=2\bar{i}+3\bar{j}+4\bar{k},\ \bar{b}=\bar{i}+\bar{j}-\bar{k}$ and $\bar{c}=\bar{i}-\bar{j}+\bar{k},$ then find $\bar{a}\times(\bar{b}\times\bar{c}).$
13. Prove that the vectors $\bar{a}=2\bar{i}-\bar{j}+\bar{k},\ \bar{b}=\bar{i}-3\bar{j}-5\bar{k}$ and $\bar{c}=3\bar{i}-4\bar{j}-4\bar{k}$ are coplanar.
14. If $\bar{a},\bar{b}$ and $\bar{c}$ are unit coplanar vectors, then find $\left[\overline{2a}-\bar{b}\quad 2\bar{b}-\bar{c}\quad 2\bar{c}-\bar{a}\right]$
15. Find the value of t if the vectors $\bar{a}=2\bar{i}-\overline{3j}+\bar{k},\ \bar{b}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{c}=\bar{j}-t\bar{k}$ are coplanar.
16. Simplify the following $(i)\left(\bar{i}-2\bar{j}+3\bar{k}\right)\times\left[\left(2\bar{i}+\bar{j}-\bar{k}\right)\times\left(\bar{j}+\bar{k}\right)\right]$

    $(ii)\left[\left(2\bar{i}-3\bar{j}+\bar{k}\right)\times\left(\bar{i}-\bar{j}+2\bar{k}\right)\right]\times\left(2\bar{i}+\bar{j}+\bar{k}\right)$
17. Find the shortest distance between the skew lines

    $\bar{r}=\left(6\bar{i}+2\bar{j}+2\bar{k}\right)+t\left(\bar{i}-2\bar{j}+2\bar{k}\right)$ and $\bar{r}=\left(-4\bar{i}-\bar{k}\right)+s\left(3\bar{i}-2\bar{j}-2\bar{k}\right)$

Key Concepts

1. Let $\bar{a}$ and $\bar{b}$ be two vectors. The *scalar* (*or dot*) *product* of $\bar{a}$ and $\bar{b}$ written as $\bar{a}.\bar{b},$

is defined by $\bar{a}.\bar{b} = \begin{cases} 0 \; if \; one\, of \; \bar{a}, \bar{b} \; is\, \bar{0} \\ |\bar{a}||\bar{b}|\cos\theta, if \;\; \bar{a} \neq \bar{0} \neq \bar{b} \;\; and \;\; \theta \; is\, the\, angle\, between\, \bar{a}\, and\, \bar{b}. \end{cases}$

2. For any two vectors $\bar{a}$ and $\bar{b}$, $\bar{a}.\bar{b}$ is a scalar.

3. If $\bar{a}, \bar{b}$ are non-zero vectors, then $\bar{a}.\bar{b}$ is positive or zero or negative according as the angle θ, is acute or right or obtuse angle.

   (i) If $\theta = 0^0$ then $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|$. In particular $\bar{a}.\bar{a} = |\bar{a}|^2 \;\; or\, (\bar{a})^2$.

   (ii) If $\theta = 180^0$ then $\bar{a}.\bar{b} = -|\bar{a}||\bar{b}|$. In particular $\bar{a}.\bar{a} = |\bar{a}||\bar{a}|\cos 180^0 = -|\bar{a}|^2$.

4. The projection vector of $\bar{b}$ on $\bar{a}$ is $\left(\dfrac{\bar{a}.\bar{b}}{|\bar{a}|^2}\right)\bar{a}$ and its magnitude is $\dfrac{|\bar{a}.\bar{b}|}{|\bar{a}|}$

5. The component of $\bar{b}$ perpendicular to $\bar{a}$ is $\bar{b} - \dfrac{(\bar{a}.\bar{b})}{|\bar{a}|^2}\bar{a}.$

6. Let $\bar{a}$ and $\bar{b}$ be two non-zero vectors and θ is the angle between $\bar{a}$ and $\bar{b}$ Then

   $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta, \; |\bar{a}.\bar{b}| = |\bar{a}||\bar{b}||\cos\theta|$

7. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three non-zero vectors. Then the projection $\bar{b} + \bar{c}$ of on $\bar{a}$ is equal to the sum of the projections of $\bar{b}$ and $\bar{c}$ on $\bar{a}$ and hence $\dfrac{\bar{a}.(\bar{b}+\bar{c})}{|\bar{a}|^2}\bar{a} = \dfrac{\bar{a}.\bar{b}}{|\bar{a}|^2}\bar{a} + \dfrac{\bar{a}.\bar{c}}{|\bar{a}|^2}\bar{a}$

8. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three vectors. Then

   $(i)\, \bar{a}.(\bar{b}+\bar{c}) = \bar{a}.\bar{b} + \bar{a}.\bar{c}, \; (ii)\, (\bar{b}+\bar{c}).\bar{a} = \bar{b}.\bar{a} + \bar{c}.\bar{a}, \; (iii)(\bar{a}+\bar{b})^2 = |\bar{a}|^2 + |\bar{b}|^2 + 2\bar{a}.\bar{b}$

9. Let $\bar{a}, \bar{b}$ be two vectors. Then

   (i) $\bar{a}.\bar{b} = \bar{b}.\bar{a}$ $(commutative\;\, law)$, (ii) $(l\bar{a}).\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}), \; l \in R,$

   (iii) $(l\bar{a}).(m\bar{b}) = lm(\bar{a}.\bar{b}), \; l\; and\; m \in R,$ (iv) $(-\bar{a}).\bar{b} = \bar{a}.(-\bar{b}) = -(\bar{a}.\bar{b}),$

   (v) $(-\bar{a}).(-\bar{b}) = \bar{a}.\bar{b}$

10. If $\bar{i}, \bar{j}, \bar{k}$ are mutually perpendicular unit vectors, then $\bar{i}.\bar{i} = \bar{j}.\bar{j} = \bar{k}.\bar{k} = 1$ and $\bar{i}.\bar{j} = \bar{j}.\bar{k} = \bar{k}.\bar{i} = 0.$

11. Let $(\bar{i}, \bar{j}, \bar{k})$ be the orthogonal unit triad. Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ be the vectors Then $\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

12. If θ is the angle between two non-zero $\bar{a}$ and $\bar{b}$, from the definition of $\bar{a}.\bar{b}$, we have $\theta = \cos^{-1}\left(\dfrac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$

13. If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ then

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

14. If $\bar{a}, \bar{b}$ are perpendicular to each other $\Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0.$

15. Angle in a semi circle is a right angle.

16. In a parallelogram, the sum of squares of lengths of the diagonals is equal to sum of squares of lengths of its ides.(Parallelogram law)

17. (i) If $(\bar{a}, \bar{b}, \bar{c})$ is a Right (Left) handed system, then the triads $(\bar{b}, \bar{c}, \bar{a})$ and $(\bar{c}, \bar{a}, \bar{b})$ also form Right (Left) handed systems.

(ii) If $(\bar{a}, \bar{b}, \bar{c})$ is a Right handed system and $\bar{a}, \bar{b}, \bar{c}$ are mutually perpendicular to each other, then $(\bar{a}, \bar{b}, \bar{c})$ is called an orthogonal triad. Thus the vector triad $(\bar{i}, \bar{j}, \bar{k})$ is an orthogonal triad.

(iii) If any two vectors in a triad are interchanged, then the system will change. For example $(\bar{a}, \bar{b}, \bar{c})$ and $(\bar{b}, \bar{a}, \bar{c})$ form opposite systems.

18. Let $\bar{a}$ and $\bar{b}$ be two non zero non collinear vectors. The cross (or vector) product of $\bar{a}$ and $\bar{b}$, is written as $\bar{a} \times \bar{b}$ (read as $\bar{a}$ cross $\bar{b}$) $= |\bar{a}||\bar{b}| \sin\theta\, \hat{n}$ where θ is the angle between the vectors $\bar{a}$ and $\bar{b}$ and $\hat{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ such that $(\bar{a}, \bar{b}, \hat{n})$ is a right handed system. If one of the vectors $\bar{a}, \bar{b}$ is the null vector or $\bar{a}, \bar{b}$ are collinear vectors then the cross product $\bar{a} \times \bar{b}$ is defined as the null vector.

19. If $\overline{a},\overline{b}$ are non-zero and non-collinear vectors, then $\overline{a}\times\overline{b}$ is a vector, perpendicular to the plane determined by $\overline{a}$ and $\overline{b}$, whose magnitude is $|\overline{a}||\overline{b}|\sin\theta$ defined as the null vector.

20. Let $\overline{a},\overline{b}$ be two vectors. Then $\overline{a}\times\overline{b} = -\overline{b}\times\overline{a}$

21. $|\overline{a}\times\overline{b}| = |\overline{b}\times\overline{a}| = |\overline{a}||\overline{b}|\sin\theta.$

22. Let $\overline{a},\overline{b}$ be two vectors and l,m be scalars. Then

$(i)\ \ (-\overline{a})\times\overline{b} = \overline{a}\times(-\overline{b}) = -(\overline{a}\times\overline{b}) = \overline{b}\times\overline{a},\ (ii)\ \ (-\overline{a})\times(-\overline{b}) = \overline{a}\times\overline{b},$

$(iii)\ \ (l\overline{a})\times\overline{b} = \overline{a}\times(l\overline{b}) = l(\overline{a}\times\overline{b}),\quad (iv)\ \ (l\overline{a})\times(m\overline{b}) = lm(\overline{a}\times\overline{b})$

23. Let $\overline{a},\overline{b}$ and $\overline{c}$ are vectors. Then

$(i)\ \ \overline{a}\times(\overline{b}+\overline{c}) = \overline{a}\times\overline{b}+\overline{a}\times\overline{c},\ (ii)\ \ (\overline{a}+\overline{b})\times\overline{c} = \overline{a}\times\overline{c}+\overline{b}\times\overline{c}.$

24. If $(\overline{i},\overline{j},\overline{k})$ is an orthogonal triad, then from the definition of the cross product of vectors, it is easy to see that $(i)\ \ \overline{i}\times\overline{i} = \overline{j}\times\overline{j} = \overline{k}\times\overline{k} = \overline{0},\ (ii)\ \ \overline{i}\times\overline{j} = \overline{k},\ \overline{j}\times\overline{k} = \overline{i},\ \overline{k}\times\overline{i} = \overline{j}.$

25. Let $\overline{a} = a_1\overline{i}+a_2\overline{j}+a_3\overline{k}$ and $\overline{b} = b_1\overline{i}+b_2\overline{j}+b_3\overline{k}$. Then

$$\overline{a}\times\overline{b} = (a_2b_3-a_3b_2)\overline{i}-(a_1b_3-a_3b_1)\overline{j}+(a_1b_2-a_2b_1)\overline{k}.$$

26. The above formula for $\overline{a}\times\overline{b}$ can now expressed as

$$\overline{a}\times\overline{b} = \begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \overline{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \overline{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \overline{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$= \overline{i}(a_2b_3-a_3b_2) - \overline{j}(a_1b_3-a_3b_1) + \overline{k}(a_1b_2-a_2b_1)$$

27. Let $\overline{a} = a_1\overline{i}+a_2\overline{j}+a_3\overline{k},\ \overline{b} = b_1\overline{i}+b_2\overline{j}+b_3\overline{k}$ and θ is the angle between $\overline{a}$ and $\overline{b}$,

then $\sin\theta = \dfrac{\sqrt{(a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}.$

28. For any two vectors $\overline{a}$ and $\overline{b}$, $|\overline{a}\times\overline{b}|^2 = (\overline{a}.\overline{a})(\overline{b}.\overline{b})-(\overline{a}.\overline{b})^2 = |\overline{a}|^2|\overline{b}|^2-(\overline{a}.\overline{b})^2.$

29. If $\overline{a}$ and $\overline{b}$ are non-collinear, then, unit vectors perpendicular to both $\overline{a}$ and $\overline{b}$

are $\pm\dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$.

30. The vectors area of ΔABC is $\frac{1}{2}\left(\overline{AB}\times\overline{AC}\right)=\frac{1}{2}\left(\overline{BC}\times\overline{BA}\right)=\frac{1}{2}\left(\overline{CA}\times\overline{CB}\right)$.

31. If $\bar{a},\bar{b},\bar{c}$ are the position vectors of the vertices A,B and C (described in counter clock wise sense) of ΔABC, then the vector area of ΔABC is

$$\frac{1}{2}\left(\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right) \text{ and its area is } \frac{1}{2}\left|\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}\right|.$$

32. Let $ABCD$ be a parallelogram with vertices A,B,C and D described in anti clock wise sense. Then, vectors area of $ABCD$ in terms of the diagonals $\overline{AC}$ and $\overline{BD}$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$.

33. The area of quadrilateral $ABCD$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$.

34. The vector area of a parallelogram with $\bar{a}$ and $\bar{b}$ as adjacent sides is $\bar{a}\times\bar{b}$ and the area is $|\bar{a}\times\bar{b}|$.

35. Let $(\bar{a},\bar{b},\bar{c})$ be a non-coplanar vector triad, $\bar{\alpha}=l_1\bar{a}+l_2\bar{b}+l_3\bar{c}$ and $\bar{\beta}=m_1\bar{a}+m_2\bar{b}+m_3\bar{c}$. Then $\bar{\alpha}\times\bar{\beta}=\begin{vmatrix}\bar{a}\times\bar{b} & \bar{b}\times\bar{c} & \bar{c}\times\bar{a}\\ l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{vmatrix}$.

36. $(\bar{a}\times\bar{b}).\bar{c}=\bar{0}$ when (i) one of $\bar{a},\bar{b},\bar{c}$ is $\bar{0}$ or (ii) $\bar{a},\bar{b}$ or $\bar{b},\bar{c}$ or $\bar{c},\bar{a}$ are collinear vectors or (iii) $\bar{c}$ is perpendicular to $(\bar{a}\times\bar{b})$.

37. Let $\bar{a},\bar{b}$ and $\bar{c}$ be three non-coplanar vectors and $\overline{OA}=\bar{a},\overline{OB}=\bar{b}$ and $\overline{OC}=\bar{c}$. Let V be the volume of the paralleleopiped with $\overline{OA},\overline{OB}$ and $\overline{OC}$ as coterminous edges. Then

(i) $(\bar{a}\times\bar{b}).\bar{c}=V$, if $(\bar{a},\bar{b},\bar{c})$ is a right handed system.

(ii) $(\bar{a}\times\bar{b}).\bar{c}=-V$, if $(\bar{a},\bar{b},\bar{c})$ is a left handed system.

38. For any three vectors $\bar{a},\bar{b}$ and $\bar{c}$, $\left(\bar{a}\times\bar{b}\right).\bar{c}=\left(\bar{b}\times\bar{c}\right).\bar{a}=\left(\bar{c}\times\bar{a}\right).\bar{b}$

$$i.e \left[\bar{a}\ \bar{b}\ \bar{c}\right]=\left[\bar{b}\ \bar{c}\ \bar{a}\right]=\left[\bar{c}\ \bar{a}\ \bar{b}\right].$$

39. If $\bar{a},\bar{b}$ and $\bar{c}$ are any three vectors, then $\left(\bar{a}\times\bar{b}\right).\bar{c}=\bar{a}.\left(\bar{b}\times\bar{c}\right)$

40. If $\bar{a},\bar{b}$ and $\bar{c}$ are three vectors such that no two are collinear, then

$\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0 \Leftrightarrow \bar{a},\bar{b}$ and $\bar{c}$ are coplanar.

41. Four distinct points A,B,C and D are coplanar $\Leftrightarrow \left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right]=0$

42. Let $(\bar{i},\bar{j},\bar{k})$ be orthogonal triad of unit vectors which is a right handed system.

Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k},\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ and $\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}.$

$$\text{Then, } \left[\bar{a}\ \bar{b}\ \bar{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

42. Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k},\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ and $\bar{c}=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$ are coplanar if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$

43. If $\bar{\alpha},\bar{\beta},\bar{\gamma}$ be three non-coplanar vectors and $\bar{a}=a_1\bar{\alpha}+a_2\bar{\beta}+a_3\bar{\gamma},$

$\bar{b}=b_1\bar{\alpha}+b_2\bar{\beta}+b_3\bar{\gamma}$ and $\bar{c}=c_1\bar{\alpha}+c_2\bar{\beta}+c_3\bar{\gamma}.$ Then $\bar{a},\bar{b}$ and $\bar{c}$ are coplanar

$$\Leftrightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

43. The volume of a tetrahedron with $\bar{a},\bar{b}$ and $\bar{c}$ as coterminous edges is $\frac{1}{6}\left\|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right\|.$

44. The volume of the tetrahedron whose vertices are A,B,C and D is $\frac{1}{6}\left\|\left[\overline{DA}\ \overline{AB}\ \overline{AC}\right]\right\|$

45. The vector equation of a plane passing through the point $A(\bar{a})$ and parallel to the non-collinear vectors $\bar{b}$ and $\bar{c}$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right].$

46. The vector equation of a plane passing through the point $A(\bar{a}), B(\bar{b})$ and parallel to the vector $\bar{c}$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

47. The vector equation of a plane passing through three non-collinear points is $A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is $\left[\bar{r}\ \bar{b}\ \bar{c}\right]+\left[\bar{r}\ \bar{c}\ \bar{a}\right]+\left[\bar{r}\ \bar{a}\ \bar{b}\right]=\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

48. Let L_1 and L_2 be two skew lines then the of shortest distance $d=\left|\frac{(\bar{b}_1\times\bar{b}_2).(\bar{a}_2-\bar{a}_1)}{\left|\bar{b}_1\times\bar{b}_2\right|}\right|$.

49. The Cartesian form of the shortest distance between the lines

$L_1:\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $L_2:\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}.$$

50. Let $\bar{a},\bar{b},\bar{c}$ are three vectors. Then $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ or $\left(\bar{a}\times\bar{b}\right)\times\bar{c}$ is called the *vector triple product* or *vector product of three vectors*.

51. Let $\bar{a},\bar{b},\bar{c}$ be three vectors. Then

$(i)\ \left(\bar{a}\times\bar{b}\right)\times\bar{c}=(\bar{a}.\bar{c})\bar{b}-(\bar{b}.\bar{c})\bar{a},\ (ii)\ \bar{a}\times\left(\bar{b}\times\bar{c}\right)=(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}$

52. For any four vectors $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d}\end{vmatrix}$

53. $(\bar{a}\times\bar{b})^2=(\bar{a})^2(\bar{b})^2-(\bar{a}.\bar{b})^2$.

Answers

Exercise 5

(1) $\theta=\cos^{-1}\left(\frac{1}{\sqrt{156}}\right)$ (2) $\lambda=3$ (3) 60^0 (4) 9 (5) $\pm\frac{1}{\sqrt{6}}(2\bar{i}-\bar{j}-\bar{k})$

(6) $\frac{\sqrt{3}}{2}$ (7) $5\sqrt{3}$ (8) $\frac{41}{2}$ (9) $x+y+z=6$ (11) -54 (12) $2\bar{i}+4\bar{j}-4\bar{k}$

(14) 0 (15) 1 (16) (i) $\bar{r}=2\bar{i}+8\bar{j}-6\bar{k}$ (ii) $\bar{r}=-4\bar{i}+7\bar{j}+\bar{k}$ (17) 9

6. TRIGONOMETRIC RATIOS UPTO TRANSFORMATIONS

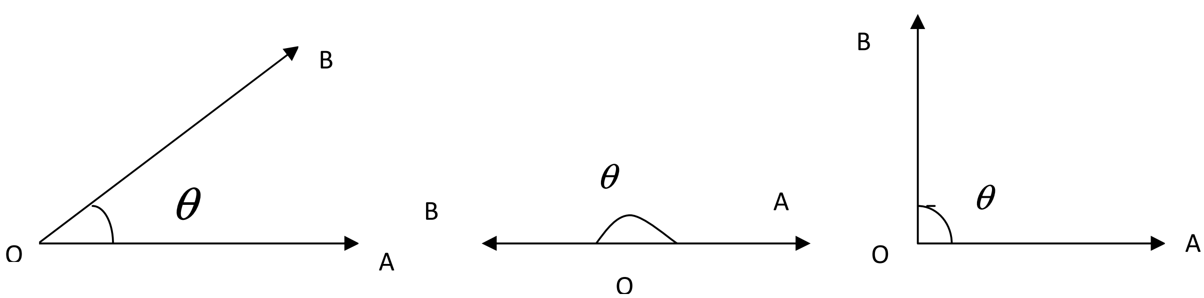
6.1 Trigonometric ratios-variation

Introduction:

The word ‘trigonon’ means a triangle and the word ‘metron’ means a measure. Thus trigonometry is the science that deals with measurement of triangles. Trigonometry has great use in measurement of areas, heights, distances etc.

It has many applications in almost all branches of science in general and in Physics and Engineering in particular.

An *angle* is the union of two rays having a common end point in a plane. The amount of rotation in the plane that is necessary to bring one ray into the position of the other ray is called *magnitude of the angle,* An angle is actually denoted by θ, α etc.



The acute angle, straight angle and the right angles are shown in the above figures. We have learnt, in previous classes, that there are three systems for the measurement of angles.

1. Sexagesimal system or British system
2. Centisimal system or French system
3. Circular measurement

In the Sexagesimal system

1 right angle = 90 degrees (90^0)

1 degree = 60 minutes $(60')$

1 minute = 60 seconds $(60'')$

In the Centisimal system

1 right angle = 100 grades (100^g)

1 grade = 100 minutes $(100')$

1 minute = 100 seconds $(100'')$

In the Circular measurement system, one *radian* is defined as the amount of the angle subtended by an arc of length r units of a circle of radius r units at the centre of that circle. This angle is independent of the size of the circle (i.e., the radius of the circle). One radian is denoted by 1^c. In this measurement 2 right angles $= \pi^c$.

1 minute in the Sexagesimal system $= \frac{1}{90\times 60}th$ of a right angle where as

1 minute in the Centisimal system $= \frac{1}{100\times 100}th$ of a right angle.

The conversion from one system to the other can be easily done using the equation :

$$\frac{180}{\mathrm{D}} = \frac{200}{\mathrm{G}} = \frac{\pi}{\mathrm{R}}.$$

where $\mathrm{D}, \mathrm{G}, \mathrm{R}$ respectively denote degrees, grades and radians.

For example, to convert 30^0 into grades and radians, put $\mathrm{D} = 30^0$ in the above equation

and G, R as follows :

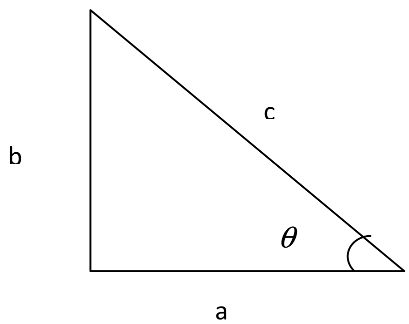
$$\frac{180}{30} = \frac{200}{\mathrm{G}} = \frac{\pi}{\mathrm{R}}. \text{ Hence } \mathrm{G} = \frac{100}{3}, \mathrm{R} = \frac{30\pi}{180} = \frac{\pi}{6}.$$

Thus $30^0 = \frac{100^g}{3} = \frac{\pi^c}{6}$.

6.1 Trigonometric ratios- variation:

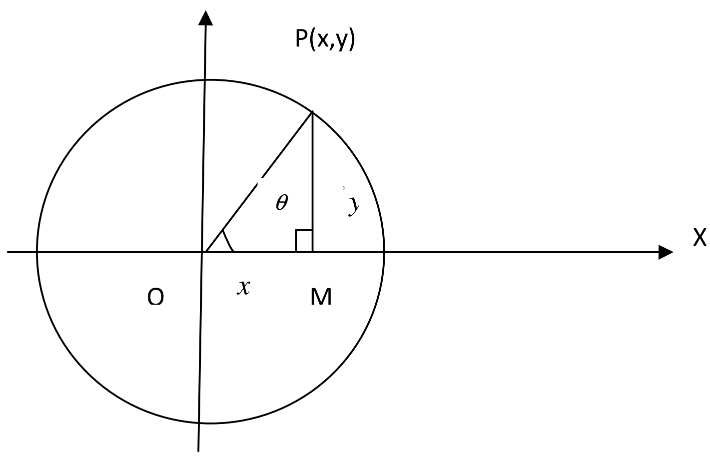
A ratio is $\frac{a}{b}$ where a, b are two real numbers and b is non-zero. If we take a right angled triangle with θ as one of its acute angle, using the lengths a, b, c of the three sides of the triangle we can form six ratios, namely $\frac{b}{c}, \frac{a}{c}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}, \frac{a}{b}$.

These six ratios are called the trigonometric ratios of the angle θ.



For example, $\frac{b}{c}$ is called sine θ, $\frac{a}{c}$ is called cosine θ, $\frac{b}{a}$ is called tangent θ, $\frac{a}{b}$ is called cotangent θ, $\frac{c}{a}$ is called secant θ, and $\frac{c}{b}$ is called cosecant θ.

6.1.1 Definition: Let θ be a real number and $0 \le \theta \le 2\pi$ and $r > 0$. Consider a rectangular co-ordinate system with OX, OY as axes. Draw a circle with centre O and radius r. Choose a point on the circle such that the line OP makes an angle θ radians with $\overrightarrow{OX}$ (positive $X-$axis) measured in anti-clock wise direction (positive direction). θ



We define the six trigonometric ratios of θ as follows:

Sine of $\theta = \frac{y}{r}$

Cosine of $\theta = \frac{x}{r}$

Tangent of $\theta = \frac{y}{x}$

Cotangent of $\theta = \frac{x}{y}$

Secant of $\theta = \frac{r}{x}$

Cosecant of $\theta = \frac{r}{y}$

The six trigonometric ratios of θ defined above are briefly written as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ respectively. From these definitions we can observe the following:

6.1.2 Note:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$.

2. $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$.

3. $\cos^2\theta + \sin^2\theta = 1$.

4. $\sec^2\theta - \tan^2\theta = 1$.

5. $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.

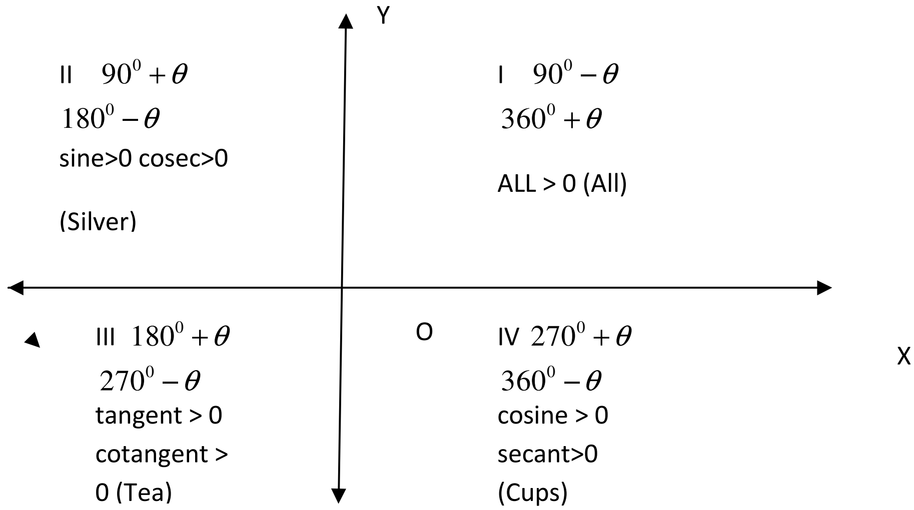
6. From the definitions of the six trigonometric ratios we can make the following observations.

If $P(x, y)$ is in the first quadrant, that is, if $0 < \theta < \frac{\pi}{2}$, then $x > 0$ and $y > 0$. Hence all the six trigonometric ratios are positive. If $P(x, y)$ is in the second quadrant, that is, if $\frac{\pi}{2} < \theta < \pi$, then $x < 0$ and $y > 0$. Hence $\sin\theta$, $\operatorname{cosec}\theta$ are positive and the other trigonometric ratios are negative. If $P(x, y)$ is in the third quadrant, that is, if $\pi < \theta < \frac{3\pi}{2}$, then $x < 0$ and $y < 0$. Hence $\tan\theta$, $\cot\theta$ are positive and the other trigonometric ratios are negative. If $P(x, y)$ is in the fourth quadrant, that is, if $\frac{3\pi}{2} < \theta < 2\pi$, then $x > 0$ and

$y < 0$. Hence $\cos\theta$, $\sec\theta$ are positive and the other trigonometric ratios are negative.

The six trigonometric ratios which are positive in various quadrants can also be remembered as follows.

I	II	III	IV
All	Silver	Tea	Cups
(all	sine	tan	cos)



Now we can write the properties of the six trigonometric functions as follows:

$\sin(90^0-\theta)=\cos\theta$ $\cos(90^0-\theta)=\sin\theta$ $\tan(90^0-\theta)=\cot\theta$ $\cot(90^0-\theta)=\tan\theta$ $\sec(90^0-\theta)=\operatorname{cosec}\theta$ $\operatorname{cosec}(90^0-\theta)=\sec\theta$	$\sin(90^0+\theta)=\cos\theta$ $\cos(90^0+\theta)=-\sin\theta$ $\tan(90^0+\theta)=-\cot\theta$ $\cot(90^0+\theta)=-\tan\theta$ $\sec(90^0+\theta)=-\operatorname{cosec}\theta$ $\operatorname{cosec}(90^0+\theta)=\sec\theta$	$\sin(180^0-\theta)=\sin\theta$ $\cos(180^0-\theta)=-\cos\theta$ $\tan(180^0-\theta)=-\tan\theta$ $\cot(180^0-\theta)=-\cot\theta$ $\sec(180^0-\theta)=-\sec\theta$ $\operatorname{cosec}(180^0-\theta)=\operatorname{cosec}\theta$
$\sin(180^0+\theta)=-\sin\theta$ $\cos(180^0+\theta)=-\cos\theta$ $\tan(180^0+\theta)=\tan\theta$ $\cot(180^0+\theta)=\cot\theta$ $\sec(180^0+\theta)=-\sec\theta$ $\operatorname{cosec}(180^0+\theta)=-\operatorname{cosec}\theta$	$\sin(270^0-\theta)=-\cos\theta$ $\cos(270^0-\theta)=-\sin\theta$ $\tan(270^0-\theta)=\cot\theta$ $\cot(270^0-\theta)=\tan\theta$ $\sec(270^0-\theta)=-\operatorname{cosec}\theta$ $\operatorname{cosec}(270^0-\theta)=-\sec\theta$	$\sin(270^0+\theta)=-\cos\theta$ $\cos(270^0+\theta)=\sin\theta$ $\tan(270^0+\theta)=-\cot\theta$ $\cot(270^0+\theta)=-\tan\theta$ $\sec(270^0+\theta)=\operatorname{cosec}\theta$ $\operatorname{cosec}(270^0+\theta)=-\sec\theta$
$\sin(360^0-\theta)=-\sin\theta$ $\cos(360^0-\theta)=\cos\theta$ $\tan(360^0-\theta)=-\tan\theta$ $\cot(360^0-\theta)=-\cot\theta$ $\sec(360^0-\theta)=\sec\theta$ $\operatorname{cosec}(360^0-\theta)=-\operatorname{cosec}\theta$	$\sin(360^0+\theta)=\sin\theta$ $\cos(360^0+\theta)=\cos\theta$ $\tan(360^0+\theta)=\tan\theta$ $\cot(360^0+\theta)=\cot\theta$ $\sec(360^0+\theta)=\sec\theta$ $\operatorname{cosec}(360^0+\theta)=\operatorname{cosec}\theta$	$\sin(n.360^0+\theta)=\sin\theta$ $\cos(n.360^0+\theta)=\cos\theta$ $\tan(n.360^0+\theta)=\tan\theta$ $\cot(n.360^0+\theta)=\cot\theta$ $\sec(n.360^0+\theta)=\sec\theta$ $\operatorname{cosec}(n.360^0+\theta)=\operatorname{cosec}\theta$

6.1.3 Definition: The angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ have their terminal side along either X^- axis or Y^- axis. Hence these angles are called *Quadrant angles*

We have learnt the values of the trigonometric ratios of the angles $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ in earlier classes. The values of the trigonometric ratios and the quadrant angles are given in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞

6.1.4 Definition: If θ is any angle then $\frac{\pi}{2} - \theta$ is called its *complement angle* and $\pi - \theta$ is its *supplement angle*. In other words, two angles θ, ϕ are said to be *complementary angles* if $\theta + \phi = \frac{\pi}{2}$ and *supplementary angles* if $\theta + \phi = \pi$.

For example, the angles $\frac{\pi}{6}, \frac{\pi}{3}$ are *complementary angles* and $\frac{\pi}{6}, \frac{5\pi}{6}$ are *supplementary angles*.

6.1.5 Solved Problems:

1. Problem: Find the value of $\cos 225^0 - \sin 225^0 + \tan 495^0 - \cot 495^0$.

Solution: We have $\cos 225^0 - \sin 225^0 + \tan 495^0 - \cot 495^0$

$$= \cos(180^0 + 45^0) - \sin(180^0 + 45^0) + \tan(360^0 + 135^0) - \cot(360^0 + 135^0)$$

$$= -\cos 45^0 + \sin 45^0 + \tan 135^0 - \cot 135^0$$

$$\left[\begin{array}{c} \because \cos(180^0 + \theta) = -\cos\theta, \sin(180^0 + \theta) = -\sin\theta \\ \tan(360^0 + \theta) = \tan\theta, \cot(360^0 + \theta) = \cot\theta \end{array}\right]$$

$$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \tan(90^0 + 45^0) - \cot(90^0 + 45^0)$$

$$= -\cot 45^0 + \tan 45^0$$

$$\left[\because \tan(90^0 + \theta) = -\cot\theta, \cot(90^0 + \theta) = -\tan\theta\right]$$

$$= -1 + 1 = 0$$

2. Problem: Find the value of $\sin^2\frac{\pi}{10} + \sin^2\frac{4\pi}{10} + \sin^2\frac{6\pi}{10} + \sin^2\frac{9\pi}{10}$

Solution: We have $\sin^2\frac{\pi}{10} + \sin^2\frac{4\pi}{10} + \sin^2\frac{6\pi}{10} + \sin^2\frac{9\pi}{10}$

$$= \sin^2 18^0 + \sin^2 72^0 + \sin^2 108^0 + \sin^2 162^0$$

$$= \sin^2 18^0 + \sin^2 72^0 + \sin^2(180^0 - 72^0) + \sin^2(180^0 - 18^0)$$

$$= \sin^2 18^0 + \sin^2 72^0 + \sin^2 72^0 + \sin^2 18^0$$

$$\left[\because \sin(180^0 - \theta) = \sin\theta\right]$$

$$= 2(\sin^2 18^0 + \sin^2 72^0) = 2(\sin^2 18^0 + \sin^2(90^0 - 18^0))$$

$$= 2(\sin^2 18^0 + \cos^2 18^0)$$

$$\left[\because \sin(90^0 - \theta) = \cos\theta\right]$$

$$= 2(1) = 2$$

3. Problem: Find the value of $\cos^2 45^0 + \cos^2 135^0 + \cos^2 225^0 + \cos^2 315^0$

Solution: We have $\cos^2 45^0 + \cos^2 135^0 + \cos^2 225^0 + \cos^2 315^0$

$$= \cos^2 45^0 + \cos^2(180^0 - 45^0) + \cos^2(180^0 + 45^0) + \cos^2(360^0 - 45^0)$$

$$= \cos^2 45^0 + \cos^2 45^0 + \cos^2 45^0 + \cos^2 45^0$$

$$\left[\because \cos(180^0 - \theta) = -\cos\theta, \cos(180^0 + \theta) = -\cos\theta, \cos(360^0 - \theta) = \cos\theta\right]$$

$$= 4(\cos^2 45^0) = 4(\frac{1}{\sqrt{2}})^2 = 4(\frac{1}{2}) = 2$$

4. Problem: Find the value of $\sin^2 \frac{2\pi}{3} + \cos^2 \frac{5\pi}{6} - \tan^2 \frac{3\pi}{4}$

Solution: We have $\sin^2 \frac{2\pi}{3} + \cos^2 \frac{5\pi}{6} - \tan^2 \frac{3\pi}{4}$

$$= \sin^2 120^0 + \cos^2 150^0 - \tan^2 135^0$$

$$= \sin^2(90^0 + 30^0) + \cos^2(180^0 - 30^0) - \tan^2(180^0 - 45^0)$$

$$= \cos^2 30^0 + \cos^2 30^0 - \tan^2 45^0$$

$$\left[\because \cos(180^0 - \theta) = -\cos\theta, \tan(180^0 - \theta) = -\tan\theta, \sin(90^0 + \theta) = -\cos\theta\right]$$

$$= (\frac{\sqrt{3}}{2})^2 + (\frac{\sqrt{3}}{2})^2 - (1)^2 = \frac{3}{4} + \frac{3}{4} - 1 = \frac{1}{2}$$

5. Problem: Show that $\cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20} = 1$

Solution: $\text{L.H.S} = \cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20}$

$$= \cot 9^0 \cot 27^0 \cot 45^0 \cot 63^0 \cot 81^0$$

$$= \cot 9^0 \cot 27^0 \cot 45^0 \cot(90^0 - 27^0)\cot(90^0 - 9^0)$$

$$= \cot 9^0 \cot 27^0 \cot 45^0 \tan 27^0 \tan 9^0 \quad \left[\because \cot(90^0 - \theta) = \tan\theta\right]$$

$$= \left(\cot 9^0 \tan 9^0\right)\left(\cot 27^0 \tan 27^0\right)\cot 45^0 \quad \left[\because \tan\theta = \frac{1}{\cot\theta}, \cot 45^0 = 1\right]$$

$$= 1 = \text{R.H.S}$$

6. Problem: If $\tan 20^0 = p$ then prove that $\frac{\tan 610^0 + \tan 700^0}{\tan 560^0 - \tan 470^0} = \frac{1-p^2}{1+p^2}$

Solution: $\text{L.H.S} = \frac{\tan 610^0 + \tan 700^0}{\tan 560^0 - \tan 470^0}$

$$= \frac{\tan(360^0 + 250^0) + \tan(360^0 + 340^0)}{\tan(360^0 + 200^0) - \tan(360^0 + 110^0)}$$

$$=\frac{\tan 250^0+\tan 340^0}{\tan 200^0-\tan 110^0} \qquad \left[\because \tan(360^0+\theta)=\tan\theta\right]$$

$$=\frac{\tan(270^0-20^0)+\tan(360^0-20^0)}{\tan(180^0+20^0)-\tan(90^0+20^0)}$$

$$=\frac{\cot 20^0-\tan 20^0}{\tan 20^0-\cot+20^0} \qquad \left[\begin{array}{l}\because \tan(360^0-\theta)=-\tan\theta, \tan(180^0+\theta)=\tan\theta, \\ \tan(90^0+\theta)=-\cot\theta, \tan(270^0-\theta)=\cot\theta\end{array}\right]$$

$$=\frac{\frac{1}{p}-p}{p-\frac{1}{p}} \qquad \left[\because \tan 20^0=p, \cot 20^0=\frac{1}{p}\right]$$

$$=\frac{\frac{1-p^2}{p}}{\frac{1+p^2}{p}} \quad =\frac{1-p^2}{1+p^2}=\text{R.H.S}$$

7. Problem: If $\tan 20^0=\lambda$ then prove that $\frac{\tan 160^0-\tan 110^0}{1+\tan 160^0\tan 110^0}=\frac{1-\lambda^2}{2\lambda}$

Solution: L.H.S= $\frac{\tan 160^0-\tan 110^0}{1+\tan 160^0\tan 110^0}$

$$=\frac{\tan(180^0-20^0)-\tan(90^0+20^0)}{1+\tan(180^0-20^0)\tan(90^0+20^0)}$$

$$=\frac{-\tan 20^0+\cot 20^0}{1+\tan 20^0\cot 20^0} \quad \left[\because \tan(180^0-\theta)=-\tan\theta, \tan(90^0+\theta)=-\cot\theta\right]$$

$$=\frac{-\lambda+\frac{1}{\lambda}}{1+\lambda\frac{1}{\lambda}} \quad \left[\because \tan 20^0=\lambda, \cot 20^0=\frac{1}{\lambda}\right]$$

$$=\frac{\frac{1-\lambda^2}{\lambda}}{1+1} \quad =\frac{1-\lambda^2}{2\lambda} \quad =\text{R.H.S}$$

8. Problem: Prove that $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2-(\tan^2\theta+\cot^2\theta)=7$

Solution: L.H.S= $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2-(\tan^2\theta+\cot^2\theta)$

$$=(\sin^2\theta+\operatorname{cosec}^2\theta+2\sin\theta\operatorname{cosec}\theta)+(\cos^2\theta+\sec^2\theta+2\cos\theta\sec\theta)$$
$$-(\tan^2\theta+\cot^2\theta)$$

$$=(\sin^2\theta+\cos^2\theta)+(\sec^2\theta-\tan^2\theta)+(\operatorname{cosec}^2\theta-\cot^2\theta)$$
$$+(2\sin\theta.\frac{1}{\sin\theta})+(2\cos\theta.\frac{1}{\cos\theta})$$

$$=(1)+(1)+(1)+(2)+(2)$$
$$\left[\begin{array}{l}\because \sin^2\theta+\cos^2\theta=1,\sec^2\theta-\tan^2\theta=1,\\ \operatorname{cosec}^2\theta-\cot^2\theta=1,\sin\theta.\frac{1}{\sin\theta}=1,\cos\theta.\frac{1}{\cos\theta}=1\end{array}\right]$$

$=7$ = R.H.S

9. Problem: Prove that $\frac{(1+\sin\theta-\cos\theta)^2}{(1+\sin\theta+\cos\theta)^2}=\frac{1-\cos\theta}{1+\cos\theta}$

Solution: L.H.S= $\frac{(1+\sin\theta-\cos\theta)^2}{(1+\sin\theta+\cos\theta)^2}$

$$=\frac{1+\sin^2\theta+\cos^2\theta+2\sin\theta-2\cos\theta-2\sin\theta\cos\theta}{1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}$$

$$=\frac{1+1+2\sin\theta-2\cos\theta-2\sin\theta\cos\theta}{1+1+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}\ \left[\because \sin^2\theta+\cos^2\theta=1\right]$$

$$=\frac{2+2\sin\theta-2\cos\theta-2\sin\theta\cos\theta}{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}=\frac{2(1+\sin\theta-\cos\theta-\sin\theta\cos\theta)}{2(1+\sin\theta+\cos\theta+\sin\theta\cos\theta)}$$

$$=\frac{(1+\sin\theta)(1-\cos\theta)}{(1+\sin\theta)(1+\cos\theta)}=\frac{1-\cos\theta}{1+\cos\theta}\quad=\text{R.H.S}$$

10. Problem: Prove that $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

Solution: L.H.S= $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1$

$$=2\left((\sin^2\theta)^3+(\cos^2\theta)^3\right)-3\left((\sin^2\theta)^2+(\cos^2\theta)^2\right)+1$$

$$=2\left((\sin^2\theta+\cos^2\theta)^3-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)\right)$$
$$-3\left((\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta\right)+1$$
$$\left[\because a^2+b^2=(a+b)^2-2ab,a^3+b^3=(a+b)^3-3ab(a+b)\right]$$

$$=2\left((1)^3-3\sin^2\theta\cos^2\theta(1)\right)-3\left((1)^2-2\sin^2\theta\cos^2\theta\right)+1$$

$$\left[\because \sin^2\theta+\cos^2\theta=1\right]$$

$$=2-6\sin^2\theta\cos^2\theta-3+6\sin^2\theta\cos^2\theta+1=0=\text{R.H.S}$$

11. Problem: If $\tan^2\theta=1-e^2$ then prove that $\sec\theta+\tan^3\theta\operatorname{cosec}\theta=(2-e^2)^{3/2}$.

Solution: Given $\tan^2\theta=1-e^2$

$$\text{L.H.S}=\sec\theta+\tan^3\theta\operatorname{cosec}\theta$$

$$=\frac{1}{\cos\theta}+\tan^2\theta\tan\theta\frac{1}{\sin\theta}=\frac{1}{\cos\theta}+\tan^2\theta\frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta}$$

$$=\frac{1}{\cos\theta}(1+\tan^2\theta)=\sec\theta(1+\tan^2\theta)$$

$$=\sqrt{(1+\tan^2\theta)}(1+\tan^2\theta)=(1+\tan^2\theta)^{3/2}$$

$$=(1+1-e^2)^{3/2}=(2-e^2)^{3/2}=\text{R.H.S}$$

12. Problem: Prove that $\cos^4\alpha+2\cos^2\alpha\left(1-\frac{1}{\sec^2\alpha}\right)=1-\sin^4\alpha$

Solution: $\text{L.H.S}=\cos^4\alpha+2\cos^2\alpha\left(1-\frac{1}{\sec^2\alpha}\right)$

$$=\cos^4\alpha+2\cos^2\alpha\left(\frac{\sec^2\alpha-1}{\sec^2\alpha}\right)=\cos^4\alpha+2\cos^2\alpha\left(\frac{\tan^2\alpha}{\sec^2\alpha}\right)$$

$$=\cos^4\alpha+2\cos^2\alpha\sin^2\alpha=\cos^2\alpha(\cos^2\alpha+2\sin^2\alpha)$$

$$=\cos^2\alpha(\cos^2\alpha+\sin^2\alpha+\sin^2\alpha)$$

$$=(1-\sin^2\alpha)(1+\sin^2\alpha)\quad\left[\because \sin^2\theta+\cos^2\theta=1,\cos^2\theta=1-\sin^2\theta\right]$$

$$=1-\sin^4\alpha=\text{R.H.S}$$

13. Problem: If $\frac{2\sin\theta}{1+\cos\theta+\sin\theta}=x$ then prove that $\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}=x$

Solution: Given $\frac{2\sin\theta}{1+\cos\theta+\sin\theta}=x$

$$\Rightarrow \frac{2\sin\theta}{(1+\cos\theta+\sin\theta)}\frac{(1-\cos\theta+\sin\theta)}{(1-\cos\theta+\sin\theta)} = x \Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{(1+\sin\theta)^2-\cos^2\theta} = x$$

$$\Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{1+\sin^2\theta+2\sin\theta-\cos^2\theta} = x \Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{1-\cos^2\theta+\sin^2\theta+2\sin\theta} = x$$

$$\Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{\sin^2\theta+\sin^2\theta+2\sin\theta} = x \Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{2\sin\theta+2\sin^2\theta} = x$$

$$\Rightarrow \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{2\sin\theta(1+\sin\theta)} = x \Rightarrow \frac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x$$

14. Problem: Prove that $(\tan\theta+\cot\theta)^2 = \sec^2\theta+\operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$

Solution: L.H.S= $(\tan\theta+\cot\theta)^2 = \tan^2\theta+\cot^2\theta+2\tan\theta\cot\theta$

$$= \tan^2\theta+\cot^2\theta+2(1) \qquad [\because \tan\theta\cot\theta = 1]$$

$$= \sec^2\theta-1+\operatorname{cosec}^2\theta-1+2 \quad [\because \tan^2\theta = \sec^2\theta-1, \cot^2\theta = \operatorname{cosec}^2\theta-1]$$

$$= \sec^2\theta+\operatorname{cosec}^2\theta$$

$$= \frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta} \quad \left[\because \sec\theta = \frac{1}{\cos\theta}, \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\sin^2\theta} = \frac{1}{\cos^2\theta\sin^2\theta} \quad [\because \sin^2\theta+\cos^2\theta = 1]$$

$$= \sec^2\theta\operatorname{cosec}^2\theta \quad \left[\because \sec\theta = \frac{1}{\cos\theta}, \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$

= R.H.S

15. Problem: If $3\sin\theta+4\cos\theta = 5$ then find the value of $4\sin\theta-3\cos\theta$

Solution: Given $3\sin\theta+4\cos\theta = 5$

Take $4\sin\theta-3\cos\theta = x$

Squaring and adding the above two equations we get

$$(3\sin\theta+4\cos\theta)^2+(4\sin\theta-3\cos\theta)^2 = 5^2+x^2$$

$$\Rightarrow 9\sin^2\theta+16\cos^2\theta+24\sin\theta\cos\theta+16\sin^2\theta$$
$$+9\cos^2\theta-24\sin\theta\cos\theta = 25+x^2$$

$\Rightarrow 25\sin^2\theta + 25\cos^2\theta = 25 + x^2$

$\Rightarrow 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$

$\Rightarrow 25(1) = 25 + x^2 \;\; (\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow 25 = 25 + x^2 \Rightarrow x^2 = 0$

$\therefore x = 0$

16. Problem: If $a\cos\theta + b\sin\theta = c$ then prove that $a\sin\theta - b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

Solution: Given $a\cos\theta + b\sin\theta = c$

Take $a\sin\theta - b\cos\theta = x$

Squaring and adding the above two equations we get

$(a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 = c^2 + x^2$

$$\Rightarrow a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2 + x^2$$

$\Rightarrow a^2(\cos^2\theta + \sin^2\theta) + b^2(\cos^2\theta + \sin^2\theta) = c^2 + x^2$

$\Rightarrow a^2(1) + b^2(1) = c^2 + x^2 \quad \left[\because \cos^2\theta + \sin^2\theta = 1\right]$

$\Rightarrow a^2 + b^2 = c^2 + x^2 \Rightarrow a^2 + b^2 - c^2 = x^2 \Rightarrow x = \pm\sqrt{a^2 + b^2 - c^2}$

17. Problem: If $x = a\cos\theta, y = b\sin\theta$ then eliminate θ

Solution: Given $x = a\cos\theta, y = b\sin\theta$

$$\Rightarrow \frac{x}{a} = \cos\theta, \frac{y}{b} = \sin\theta$$

Squaring and adding the above two equations we get

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = (\cos\theta)^2 + (\sin\theta)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \left[\because \cos^2\theta + \sin^2\theta = 1\right]$$

18. Problem: If $x = a\cos^3\theta, y = b\sin^3\theta$ then eliminate θ

Solution: Given $x = a\cos^3\theta, y = b\sin^3\theta$

$$\Rightarrow \frac{x}{a} = \cos^3\theta, \frac{y}{b} = \sin^3\theta$$

$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} = \left(\cos^3\theta\right)^{2/3}, \left(\frac{y}{b}\right)^{2/3} = \left(\sin^3\theta\right)^{2/3}$$

$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} = \cos^2\theta, \left(\frac{y}{b}\right)^{2/3} = \sin^2\theta$$

Now adding the above two equations we get

$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = \cos^2\theta + \sin^2\theta \Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1 \quad \left[\because \cos^2\theta + \sin^2\theta = 1\right]$$

Exercise 6(a)

1. Find the values of the following

$$i)\sin\frac{5\pi}{3} \quad ii)\sec\frac{13\pi}{3} \quad iii)\cos\left(-\frac{7\pi}{2}\right) \quad iv)\tan 855^0 \quad v)\sec 2100^0 \quad vi)\cot(-315^0)$$

2. Prove the following

$i)3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4(\sin^6\theta + \cos^6\theta) = 13$

$ii)\dfrac{(\tan\theta + \sec\theta - 1)}{(\tan\theta - \sec\theta + 1)} = \dfrac{1 + \sin\theta}{\cos\theta}$

$iii)\dfrac{\cos(\pi - \theta)\cot(\frac{\pi}{2} + \theta)\cos(-\theta)}{\tan(\pi + \theta)\tan(\frac{3\pi}{2} + \theta)\sin(2\pi - \theta)} = \cos\theta$

$iv)\dfrac{\sin(3\pi - \theta)\cos(\theta - \frac{\pi}{2})\tan(\frac{3\pi}{2} - \theta)}{\sec(3\pi + \theta)\cos ec(\frac{13\pi}{2} + \theta)\cot(\theta - \frac{\pi}{2})} = \cos^4\theta$

$v)\cot\frac{\pi}{16}\cot\frac{2\pi}{16}\cot\frac{3\pi}{16}...\cot\frac{7\pi}{16} = 1$

3. Simplify the following

$i)\sin 780^0 \sin 480^0 + \cos 240^0 \cos 300^0$

$ii)\sin 330^0 \cos 120^0 + \cos 210^0 \sin 300^0$

$iii)\cos 225^0 - \sin 225^0 + \tan 495^0 - \cot 495^0$

$iv)\cos^2 45^0 + \cos^2 135^0 + \cos^2 225^0 + \cos^2 315^0$

4. If $3\sin A + 5\cos A = 5$ then prove that $5\sin A - 3\cos A = \pm 3$

5. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$ then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

6. Eliminate θ from the following

   $i) x = a\cos^4\theta, y = a\sin^4\theta \quad ii) x = a(\sec\theta + \tan\theta), y = b(\sec\theta - \tan\theta)$

   $iii) x = \cot\theta + \tan\theta, y = \sec\theta - \cos\theta$

7. (i) If $\sin\alpha = \frac{-1}{3}$ and α does not lie in the third quadrant then find the values of $\cot\alpha$ and $\cos\alpha$.

   (ii) If $\sin\theta = \frac{4}{5}$ and θ does not lie in the first quadrant then find the value of $\cos\theta$.

6.2. Trigonometric ratios of compound angles:

In this section, we define a compound angle and give formulae to find the trigonometric ratios of compound angles.

6.2.1 Definition:

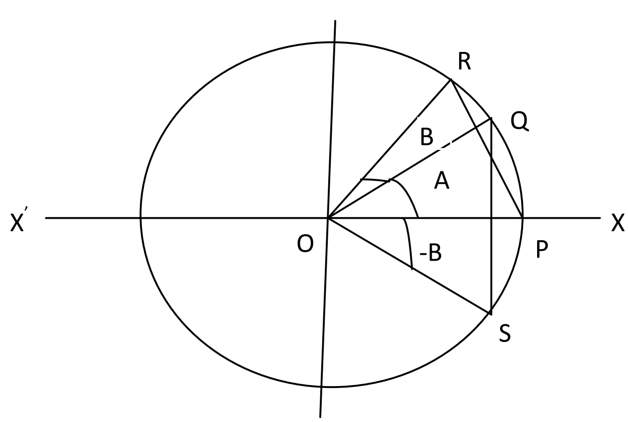
The algebraic sum of two or more angles is called a *compound angle*.

If A, B, C are three angles then $A+B.B+C, C+A, A-C, A+B-C...$ are compound angles.

6.2.2 Theorem:

If A, B are two real numbers then $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Proof: Consider a unit circle with centre at the origin O Let the terminal sides of the angles $A, A+B, -B$ in the standard position cuts the circle at Q, R, S respectively. Let OX cut the circle at P.



$\therefore P=(1,0), Q=(\cos A, \sin A),$
$R=(\cos(A+B), \sin(A+B)),$
$S=(\cos(-B), \sin(-B))=(\cos B, -\sin B).$

We have $\underline{|POR} = A+B = \underline{|QOS} \Rightarrow (PR)^2=(QS)^2$

$$(\cos(A+B)-1)^2+(\sin(A+B)-0)^2=(\cos A-\cos B)^2+(\sin A+\sin B)^2$$

$$\Rightarrow \cos^2(A+B)-2\cos(A+B)+1+\sin^2(A+B)$$
$$=\cos^2 A+\cos^2 B-2\cos A\cos B+\sin^2 A+\sin^2 B+2\sin A\sin B$$

$$\Rightarrow \cos^2(A+B)+\sin^2(A+B)-2\cos(A+B)+1$$
$$=\cos^2 A+\sin^2 A+\cos^2 B+\sin^2 B-2\cos A\cos B+2\sin A\sin B$$

$$\Rightarrow 1-2\cos(A+B)+1=1+1-2\cos A\cos B+2\sin A\sin B\left[\because \cos^2\theta+\sin^2\theta=1\right]$$

$$\Rightarrow -2\cos(A+B)=-2\cos A\cos B+2\sin A\sin B$$

$$\Rightarrow \cos(A+B)=\cos A\cos B-\sin A\sin B$$

6.2.3 Corollary:

If A, B are two real numbers then

$$(i)\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$(ii)\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$(iii)\sin(A-B)=\sin A\cos B-\cos A\sin B$$

Proof: **(i)** We have from theorem 6.2.2 $\cos(A+B)=\cos A\cos B-\sin A\sin B$

Now $\cos(A-B)=\cos(A+(-B))=\cos A\cos(-B)-\sin A\sin(-B)$

$$=\cos A\cos B+\sin A\sin B\left[\because \cos(-B)=\cos B, \sin(-B)=-\sin B\right]$$

(ii) $\sin(A+B)=\cos\left(\frac{\pi}{2}-(A+B)\right)=\cos\left(\left(\frac{\pi}{2}-A\right)-B\right)$

$$=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$$

$$= \sin A \cos B + \cos A \sin B$$

$$\left[\because \cos\left(\frac{\pi}{2} - A\right) = \sin A, \sin\left(\frac{\pi}{2} - A\right) = \cos A\right]$$

(iii) $\sin(A - B) = \sin\left(A + (-B)\right) = \sin A \cos(-B) + \cos A \sin(-B)$

$$= \sin A \cos B - \cos A \sin B \left[\because \cos(-B) = \cos B, \sin(-B) = -\sin B\right]$$

6.2.4 Theorem:

(i) If none of $A, B \, and \, (A + B)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) If none of $A, B \, and \, (A + B)$ is an integral multiple of π, then

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

Proof: (i) Since none of $A, B \, and \, (A + B)$ is an odd multiple of $\frac{\pi}{2}$, none of

$\cos A, \cos B \, and \, \cos(A + B)$ is zero.

Now $\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)} = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

Dividing the numerator and denominator in R.H.S by $\cos A \cos B$, we get

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \dfrac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) Since none of $A, B \, and \, (A + B)$ is an integral multiple of π, none of

$\sin A, \sin B \, and \, \sin(A + B)$ is zero.

Now $\cot(A + B) = \dfrac{\cos(A + B)}{\sin(A + B)} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

Dividing the numerator and denominator in R.H.S by $\sin A \sin B$, we get

$$\cot(A+B) = \frac{\frac{\cos A \cos B - \sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B + \cos A \sin B}{\sin A \sin B}} = \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}}$$

$$= \frac{\frac{\cos A}{\sin A}\frac{\cos B}{\sin B} - 1}{\frac{\cos B}{\sin B} + \frac{\cos A}{\sin A}} = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

6.2.5 Corollary:

(i) If none of $A, B\ and\ (A-B)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(ii) If none of $A, B\ and\ (A-B)$ is an integral multiple of π, then

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Proof: (i) Since none of $A, B\ and\ (A-B)$ is an odd multiple of $\frac{\pi}{2}$, none of

$\cos A, \cos B\ and\ \cos(A-B)$ is zero.

Now $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

Dividing the numerator and denominator in R.H.S by $\cos A \cos B$, we get

$$\tan(A-B) = \frac{\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B}} = \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}{1 + \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(ii) Since none of $A, B\ and\ (A-B)$ is an integral multiple of π, none of

$\sin A, \sin B\ and\ \sin(A-B)$ is zero.

Now $\cot(A-B)=\dfrac{\cos(A-B)}{\sin(A-B)}=\dfrac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$

Dividing the numerator and denominator in R.H.S by $\sin A\sin B$, we get

$$\cot(A-B)=\frac{\dfrac{\cos A\cos B+\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B-\cos A\sin B}{\sin A\sin B}}=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}+\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}-\dfrac{\cos A\sin B}{\sin A\sin B}}$$

$$=\frac{\dfrac{\cos A}{\sin A}\dfrac{\cos B}{\sin B}+1}{\dfrac{\cos B}{\sin B}-\dfrac{\cos A}{\sin A}}=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

6.2.6 Theorem:

If A,B are two real numbers then

$$(i)\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$$

$$(ii)\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$$

Proof: (i) We have $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

Now $\sin(A+B)\sin(A-B)=\left(\sin A\cos B+\cos A\sin B\right)\left(\sin A\cos B-\cos A\sin B\right)$

$$=\left(\sin A\cos B\right)^2-\left(\cos A\sin B\right)^2=\left(\sin^2 A\cos^2 B\right)-\left(\cos^2 A\sin^2 B\right)$$

$$=\sin^2 A\left(1-\sin^2 B\right)-\left(1-\sin^2 A\right)\sin^2 B$$

$$=\sin^2 A-\sin^2 A\sin^2 B-\sin^2 B+\sin^2 A\sin^2 B=\sin^2 A-\sin^2 B$$

$$=\left(1-\cos^2 A\right)-\left(1-\cos^2 B\right)=\cos^2 B-\cos^2 A$$

(ii) We have $\cos(A+B)=\cos A\cos B-\sin A\sin B$ and

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

Now $\cos(A+B)\cos(A-B)=\left(\cos A\cos B-\sin A\sin B\right)\left(\cos A\cos B+\sin A\sin B\right)$

$$=\left(\cos A\cos B\right)^2-\left(\sin A\sin B\right)^2=\left(\cos^2 A\cos^2 B\right)-\left(\sin^2 A\sin^2 B\right)$$

$$=\cos^2 A\left(1-\sin^2 B\right)-\left(1-\cos^2 A\right)\sin^2 B$$

$$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B \;= \cos^2 A - \sin^2 B$$

$$= \left(1 - \sin^2 A\right) - \left(1 - \cos^2 B\right) = \cos^2 B - \sin^2 A$$

Now we derive the formulae for $\sin(A+B+C), \cos(A+B+C), \tan(A+B+C)$ and $\cot(A+B+C)$ as follows.

6.2.7 Theorem:

If A, B, C are three real numbers then

$$(i) \sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

$$(ii) \cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \cos C$$

(iii) If none of $A, B, C \; and \; (A+B+C)$ is an odd multiple of $\frac{\pi}{2}$, and at least one

of $A+B, B+C, \; C+A$ is not an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

(iv) If none of $A, B, C \; and \; (A+B+C)$ is an integral multiple of π, then

$$\cot(A+B+C) = \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$$

Proof: (i) $\sin(A+B+C) = \sin\left((A+B)+C\right) = \sin(A+B)\cos C + \cos(A+B)\sin C$

$$= \left(\sin A \cos B + \cos A \sin B\right)\cos C + \left(\cos A \cos B - \sin A \sin B\right)\sin C$$

$$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

(ii) $\cos(A+B+C) = \cos\left((A+B)+C\right) = \cos(A+B)\cos C - \sin(A+B)\sin C$

$$= \left(\cos A \cos B - \sin A \sin B\right)\cos C - \left(\sin A \cos B + \cos A \sin B\right)\cos C$$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C$$
$$- \sin A \cos B \sin C - \cos A \sin B \cos C$$

(iii) Since none of A, B, C *and* $(A+B+C)$ is an odd multiple of $\frac{\pi}{2}$, and at least one of $A+B, B+C, \; C+A$ is not an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A+B+C) = \tan\left((A+B)+C\right) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1-\tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1-\tan A \tan B}\tan C} = \frac{\dfrac{\tan A + \tan B + \tan C(1-\tan A \tan B)}{1-\tan A \tan B}}{\dfrac{1(1-\tan A \tan B) - (\tan A + \tan B)\tan C}{1-\tan A \tan B}}$$

$$= \frac{\tan A + \tan B + \tan C(1-\tan A \tan B)}{1(1-\tan A \tan B) - (\tan A + \tan B)\tan C}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1-\tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\therefore \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1-\tan A \tan B - \tan B \tan C - \tan C \tan A}$$

(iv) Since none of A, B, C *and* $(A+B+C)$ is an integral multiple of π then

$$\cot(A+B+C) = \cot\left((A+B)+C\right) = \frac{\cot(A+B)\tan C - 1}{\cot(A+B)+\cot C}$$

$$= \frac{\dfrac{\cot A \cot B - 1}{\cot B + \cot A}\cot C - 1}{\dfrac{\cot A \cot B - 1}{\cot B + \cot A} + \cot C} = \frac{\dfrac{(\cot A \cot B - 1)\cot C - 1(\cot B + \cot A)}{\cot B + \cot A}}{\dfrac{\cot A \cot B - 1 + \cot C(\cot B + \cot A)}{\cot B + \cot A}}$$

$$= \frac{(\cot A \cot B - 1)\cot C - 1(\cot B + \cot A)}{\cot A \cot B - 1 + \cot C(\cot B + \cot A)}$$

$$= \frac{\cot A \cot B \cot C - \cot C - \cot B - \cot A}{\cot A \cot B - 1 + \cot C \cot B + \cot C \cot A}$$

$$\therefore \cot(A+B+C) = \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1-\cot A \cot B - \cot B \cot C - \cot C \cot A}$$

6.2.8 Solved Problems:

1. Problem: Find the values of $\sin 75^0, \cos 75^0, \tan 75^0$.

Solution: (i) We have $\sin(A+B)=\sin A\cos B+\cos A\sin B$

Take $A=45^0, B=30^0$

$\sin(45^0+30^0)=\sin 45^0\cos 30^0+\cos 45^0\sin 30^0$

$$\Rightarrow \sin 75^0=\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2} \Rightarrow \sin 75^0=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) We have $\cos(A+B)=\cos A\cos B-\sin A\sin B$

Take $A=45^0, B=30^0$

$\cos(45^0+30^0)=\cos 45^0\cos 30^0-\sin 45^0\sin 30^0$

$$\Rightarrow \cos 75^0=\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$\Rightarrow \cos 75^0=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(iii) We have $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

Take $A=45^0, B=30^0$

$$\tan(45^0+30^0)=\frac{\tan 45^0+\tan 30^0}{1-\tan 45^0\tan 30^0}$$

$$\Rightarrow \tan 75^0=\frac{1+\frac{1}{\sqrt{3}}}{1-1.\frac{1}{\sqrt{3}}} \Rightarrow \tan 75^0=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} \Rightarrow \tan 75^0=\frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$\Rightarrow \tan 75^0=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

2. Problem: Find the values of $\sin 15^0, \cos 15^0, \tan 15^0$.

Solution: (i) We have $\sin(A-B)=\sin A\cos B-\cos A\sin B$

Take $A=45^0, B=30^0$

$\sin(45^0-30^0)=\sin 45^0\cos 30^0-\cos 45^0\sin 30^0$

$$\Rightarrow \sin 15^0 = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$\Rightarrow \sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) We have $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Take $A = 45^0, B = 30^0$

$$\cos(45^0 - 30^0) = \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$$

$$\Rightarrow \cos 15^0 = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$\Rightarrow \cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) We have $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Take $A = 45^0, B = 30^0$

$$\tan(45^0 - 30^0) = \frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0}$$

$$\Rightarrow \tan 15^0 = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} \Rightarrow \tan 15^0 = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \Rightarrow \tan 15^0 = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$\Rightarrow \tan 15^0 = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

3. Problem: Find the values of $\sin 105^0, \cos 105^0, \tan 105^0$.

Solution: (i) We have $\sin(180^0 - \theta) = \sin\theta$

Take $\theta = 75^0$

$$\sin 105^0 = \sin(180^0 - 75^0) = \sin 75^0$$

$$\Rightarrow \sin 105^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) We have $\cos(180^0-\theta)=-\cos\theta$

Take $\theta=75^0$

$\cos 105^0=\cos(180^0-75^0)=-\cos 75^0$

$$\Rightarrow \cos 105^0=-\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\Rightarrow \cos 105^0=\frac{1-\sqrt{3}}{2\sqrt{2}}$$

(iii) We have $\tan(180^0-\theta)=-\tan\theta$

Take $\theta=75^0$

$\tan 105^0=\tan(180^0-75^0)=-\tan 75^0$

$$\Rightarrow \tan 105^0=-\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\Rightarrow \tan 105^0=\frac{1+\sqrt{3}}{1-\sqrt{3}}$$

4. Problem: Prove that $\tan 75^0+\cot 75^0=4$

Solution: We have $\tan 75^0=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

$$\Rightarrow \tan 75^0=\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)\Rightarrow \tan 75^0=\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$\Rightarrow \tan 75^0=\frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}\Rightarrow \tan 75^0=\frac{(\sqrt{3})^2+2.\sqrt{3}.1+1^2}{(\sqrt{3})^2-1^2}$$

$$\Rightarrow \tan 75^0=\frac{4+2\sqrt{3}}{2}\Rightarrow \tanh^{-1}\frac{1}{2}=\frac{1}{2}\log 3$$

$$\therefore \tan 75^0=2+\sqrt{3}$$

We have $\cot 75^0=\dfrac{1}{\tan 75^0}=\dfrac{1}{2+\sqrt{3}}=\dfrac{1}{2+\sqrt{3}}\times\dfrac{2-\sqrt{3}}{2-\sqrt{3}}$

$$= \frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$$

$$\therefore \tan 75^0 + \cot 75^0 = 2+\sqrt{3}+2-\sqrt{3} = 4$$

Hence $\tan 75^0 + \cot 75^0 = 4$

5. Problem: Show that $\cos 100^0 \cos 40^0 + \sin 100^0 \sin 40^0 = \frac{1}{2}$

Solution: We have $\cos A \cos B + \sin A \sin B = \cos(A-B)$

Take $A = 100^0, B = 40^0$

$$\cos 100^0 \cos 40^0 + \sin 100^0 \sin 40^0 = \cos(100^0 - 40^0) = \cos 60^0 = \frac{1}{2}$$

6. Problem: Show that $\cos 42^0 + \cos 78^0 + \cos 162^0 = 0$

Solution: L.H.S= $\cos 42^0 + \cos 78^0 + \cos 162^0$

$$= \cos(60^0 - 18^0) + \cos(60^0 + 18^0) + \cos(180^0 - 18^0)$$

$$= \cos 60^0 \cos 18^0 + \sin 60^0 \sin 18^0$$
$$+ \cos 60^0 \cos 18^0 - \sin 60^0 \sin 18^0 - \cos 18^0$$

$$= 2\cos 60^0 \cos 18^0 - \cos 18^0 = 2.\frac{1}{2}.\cos 18^0 - \cos 18^0$$

$$= \cos 18^0 - \cos 18^0 = 0 = \text{R.H.S}$$

$$\therefore \cos 42^0 + \cos 78^0 + \cos 162^0 = 0$$

7. Problem: Find the value of $\tan 20^0 + \tan 40^0 + \sqrt{3}\tan 20^0 \tan 40^0$

Solution: We have $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Take $A = 20^0, B = 40^0$

$$\tan(20^0 + 40^0) = \frac{\tan 20^0 + \tan 40^0}{1 - \tan 20^0 \tan 40^0}$$

$$\Rightarrow \tan 60^0 = \frac{\tan 20^0 + \tan 40^0}{1 - \tan 20^0 \tan 40^0} \Rightarrow \sqrt{3} = \frac{\tan 20^0 + \tan 40^0}{1 - \tan 20^0 \tan 40^0}$$

$$\Rightarrow \tan 20^0 + \tan 40^0 = \sqrt{3}\left(1 - \tan 20^0 \tan 40^0\right)$$

$$\Rightarrow \tan 20^0 + \tan 40^0 = \sqrt{3} - \sqrt{3}\tan 20^0 \tan 40^0$$

$$\Rightarrow \tan 20^0 + \tan 40^0 + \sqrt{3}\tan 20^0 \tan 40^0 = \sqrt{3}$$

$$\therefore \tan 20^0 + \tan 40^0 + \sqrt{3}\tan 20^0 \tan 40^0 = \sqrt{3}$$

8. Problem: Find the value of $\tan 56^0 - \tan 11^0 - \tan 56^0 \tan 11^0$

Solution: We have $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Take $A = 56^0, B = 11^0$

$$\tan(56^0 - 11^0) = \frac{\tan 56^0 - \tan 11^0}{1 + \tan 56^0 \tan 11^0}$$

$$\Rightarrow \tan 45^0 = \frac{\tan 56^0 - \tan 11^0}{1 + \tan 56^0 \tan 11^0} \Rightarrow 1 = \frac{\tan 56^0 - \tan 11^0}{1 + \tan 56^0 \tan 11^0}$$

$$\Rightarrow \tan 56^0 - \tan 11^0 = 1\left(1 + \tan 56^0 \tan 11^0\right)$$

$$\Rightarrow \tan 56^0 - \tan 11^0 = 1 + \tan 56^0 \tan 11^0$$

$$\Rightarrow \tan 56^0 - \tan 11^0 - \tan 56^0 \tan 11^0 = 1$$

$$\therefore \tan 56^0 - \tan 11^0 - \tan 56^0 \tan 11^0 = 1$$

9. Problem: Find the value of $\sin^2 82\frac{1}{2}^0 - \sin^2 22\frac{1}{2}^0$

Solution: We have $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

Take $A = 82\frac{1}{2}^0, B = 22\frac{1}{2}^0$

$$\sin^2 82\frac{1}{2}^0 - \sin^2 22\frac{1}{2}^0 = \sin(82\frac{1}{2}^0 + 22\frac{1}{2}^0)\sin(82\frac{1}{2}^0 - 22\frac{1}{2}^0)$$

$$= \sin 105^0 . \sin 60^0 = \frac{\sqrt{3}+1}{2\sqrt{2}} . \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

10. Problem: Find the value of $\cos^2 112\frac{1}{2}^0 - \sin^2 52\frac{1}{2}^0$

Solution: We have $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$

Take $A = 112\frac{1}{2}^0, B = 52\frac{1}{2}^0$

$$\cos^2 112\frac{1}{2}^0 - \sin^2 52\frac{1}{2}^0 = \cos(112\frac{1}{2}^0 + 52\frac{1}{2}^0)\cos(112\frac{1}{2}^0 - 52\frac{1}{2}^0)$$

$$= \cos 165^0 . \cos 60^0 = -\frac{\sqrt{3}+1}{2\sqrt{2}} . \frac{1}{2} = -\frac{\sqrt{3}+1}{4\sqrt{2}}$$

11. Problem: If $A + B = 45^0$ then prove that $(1+\tan A)(1+\tan B) = 2.$

Solution: We have $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\Rightarrow \tan 45^0 = \frac{\tan A + \tan B}{1-\tan A \tan B} \quad \Rightarrow 1 = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow 1 - \tan A \tan B = \tan A + \tan B \;\Rightarrow 1 = \tan A + \tan B + \tan A \tan B$$

$$\Rightarrow 1 + 1 = 1 + \tan A + \tan B + \tan A \tan B \;\Rightarrow (1+\tan A)(1+\tan B) = 2$$

12. Problem: If $A - B = 135^0$ then prove that $(1-\tan A)(1+\tan B) = 2.$

Solution: We have $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\Rightarrow \tan 135^0 = \frac{\tan A - \tan B}{1+\tan A \tan B} \quad \Rightarrow -1 = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow -1 + \tan A \tan B = \tan A - \tan B \;\Rightarrow -1 = \tan A - \tan B - \tan A \tan B$$

$$\Rightarrow 1 - \tan A + \tan B - \tan A \tan B = 1 + 1 \;\Rightarrow (1-\tan A)(1+\tan B) = 2$$

13. Problem: If $A + B = 225^0$ then prove that $\dfrac{\cot A \cot B}{(1+\cot A)(1+\cot B)} = \dfrac{1}{2}$

Solution: We have $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

$$\Rightarrow \cot 225^0 = \frac{\cot A \cot B - 1}{\cot B + \cot A} \quad \Rightarrow 1 = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\Rightarrow \cot B + \cot A = \cot A \cot B - 1 \;\Rightarrow 1 + \cot B + \cot A + \cot A \cot B = 2\cot A \cot B$$

$$\Rightarrow (1+\cot A)(1+\cot B) = 2\cot A\cot B \Rightarrow \frac{\cot A\cot B}{(1+\cot A)(1+\cot B)} = \frac{1}{2}.$$

14. Problem: If $A+B+C=180^0$ then prove that

$$\tan A+\tan B+\tan C = \tan A\tan B\tan C.$$

Solution: Given $A+B+C=180^0$

$$\Leftrightarrow A+B=180^0-C$$

Apply **tan** on both sides we get

$$\Leftrightarrow \tan(A+B)=\tan(180^0-C) \Leftrightarrow \frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$$

$$\Leftrightarrow \tan A+\tan B = -\tan C\left(1-\tan A\tan B\right)$$

$$\Leftrightarrow \tan A+\tan B = -\tan C+\tan A\tan B\tan C$$

$$\Leftrightarrow \tan A+\tan B+\tan C = \tan A\tan B\tan C$$

15. Problem: If $A+B+C=180^0$ then prove that

$$\cot A\cot B+\cot B\cot C+\cot C\cot A=1.$$

Solution: Given $A+B+C=180^0$

$$\Leftrightarrow A+B=180^0-C$$

Apply **cot** on both sides we get

$$\Leftrightarrow \cot(A+B)=\cot(180^0-C) \Leftrightarrow \frac{\cot A\cot B-1}{\cot B+\cot A} = -\cot C$$

$$\Leftrightarrow \cot A\cot B-1 = -\cot C\left(\cot B+\cot A\right)$$

$$\Leftrightarrow \cot A\cot B-1 = -\cot B\cot C-\cot C\cot A$$

$$\Leftrightarrow \cot A\cot B+\cot B\cot C+\cot C\cot A=1$$

16. Problem: If $A+B+C=90^0$ then prove that

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1.$$

Solution: Given $A+B+C=90^0$

$$\Leftrightarrow A+B=90^0-C$$

Apply **tan** on both sides we get

$$\Leftrightarrow \tan(A+B)=\tan(90^0-C) \Leftrightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\cot C$$

$$\Leftrightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{1}{\tan C}$$

$$\Leftrightarrow (\tan A+\tan B)\tan C=1-\tan A\tan B$$

$$\Leftrightarrow \tan C\tan A+\tan B\tan C=1-\tan A\tan B$$

$$\Leftrightarrow \tan A\tan B+\tan B\tan C+\tan C\tan A=1$$

17. Problem: If $A+B+C=90^0$ then prove that

$$\cot A+\cot B+\cot C=\cot A\cot B\cot C$$

Solution: Given $A+B+C=90^0$

$$\Leftrightarrow A+B=90^0-C$$

Apply **cot** on both sides we get

$$\Leftrightarrow \cot(A+B)=\cot(90^0-C) \Leftrightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=\tan C$$

$$\Leftrightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=\frac{1}{\cot C}$$

$$\Leftrightarrow (\cot A\cot B-1)\cot C=\cot A+\cot B$$

$$\Leftrightarrow \cot A\cot B\cot C-\cot C=\cot A+\cot B$$

$$\Leftrightarrow \cot A+\cot B+\cot C=\cot A\cot B\cot C$$

18. Problem: If $\sin(A+B)=\frac{24}{25}$ and $\cos(A-B)=\frac{4}{5}$ where $0<A<B<\frac{\pi}{4}$ then find $\tan 2A$

Solution: Given $\sin(A+B)=\frac{24}{25} \Rightarrow \cos(A+B)=\frac{7}{25}, \tan(A+B)=\frac{24}{7}$

Also $\cos(A-B)=\frac{4}{5} \Rightarrow \sin(A-B)=-\frac{3}{5}, \tan(A-B)=-\frac{3}{4}\left[\because A-B<0\right]$

We have $\tan 2A = \tan\left(\overline{A+B} + \overline{A-B}\right) = \dfrac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$

$$\Rightarrow \tan 2A = \dfrac{\dfrac{24}{7}-\dfrac{3}{4}}{1-\left(\dfrac{24}{7}\right)\left(\dfrac{-3}{4}\right)} \Rightarrow \tan 2A = \dfrac{\dfrac{96-21}{28}}{\dfrac{28+72}{28}}$$

$$\Rightarrow \tan 2A = \dfrac{75}{28}\times\dfrac{28}{100} = \dfrac{3}{4}$$

19. Problem: If $\sin\alpha = \dfrac{1}{\sqrt{10}}$ and $\sin\beta = \dfrac{1}{\sqrt{5}}$ and $0 < \alpha, \beta < \dfrac{\pi}{2}$ then

show that $\alpha + \beta = \dfrac{\pi}{4}$

Solution: Given $\sin\alpha = \dfrac{1}{\sqrt{10}} \Rightarrow \tan\alpha = \dfrac{1}{3}$

Also $\sin\beta = \dfrac{1}{\sqrt{5}} \Rightarrow \tan\beta = \dfrac{1}{2}$

We have $\tan\left(\alpha+\beta\right) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$$\Rightarrow \tan(\alpha+\beta) = \dfrac{\dfrac{1}{3}+\dfrac{1}{2}}{1-\dfrac{1}{3}.\dfrac{1}{2}} \Rightarrow \tan(\alpha+\beta) = \dfrac{\dfrac{2+3}{6}}{\dfrac{6-1}{6}} \Rightarrow \tan(\alpha+\beta) = \dfrac{\dfrac{5}{6}}{\dfrac{5}{6}}$$

$$\Rightarrow \tan(\alpha+\beta) = 1$$

$$\therefore \alpha + \beta = \dfrac{\pi}{4}$$

Exercise 6(b)

1. Find the values of the following

$i)\cos^2 52\dfrac{1}{2}^0 - \sin^2 22\dfrac{1}{2}^0 \quad ii)\cos^2 22\dfrac{1}{2}^0 - \cos^2 82\dfrac{1}{2}^0$

$iii)\tan(\dfrac{\pi}{4}+\theta)\tan(\dfrac{\pi}{4}-\theta) \quad iv)\dfrac{\cot 55^0 \cot 35^0 - 1}{\cot 55^0 + \cot 35^0}$

$v)\sin 1140^0 \cos 390^0 - \cos 780^0 \sin 750^0$

2. Prove that

$i) \cos 35^0 + \cos 85^0 + \cos 155^0 = 0$ $ii) \sin 750^0 \cos 480^0 + \cos 120^0 \cos 60^0 = -\frac{1}{2}$

$iii) \cos\theta + \cos(\frac{4\pi}{3} + \theta) + \cos(\frac{4\pi}{3} - \theta) = 0$

$iv) \cos^2\theta + \cos^2(\frac{2\pi}{3} + \theta) + \cos^2(\frac{2\pi}{3} - \theta) = \frac{3}{2}$

$v) \sin^2\theta + \sin^2(\theta + \frac{\pi}{3}) + \cos^2(\theta - \frac{\pi}{3}) = \frac{3}{2}$

3. If $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{3}{5}$ and neither α nor β lie in the first quadrant then find the quadrant in which $\alpha + \beta$ lies.

4. (i) If $\cos\alpha = \frac{-3}{5}$ and $\sin\beta = \frac{7}{25}$ where $\frac{\pi}{2} < \alpha < \pi$ and $0 < \beta < \frac{\pi}{2}$ then find the values of $\tan(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

(ii) If $\sin(\theta + \alpha) = \cos(\theta + \alpha)$ then find $\tan\theta$ in terms of $\tan\alpha$

5. (i) If $A - B = \frac{3\pi}{4}$ then prove that $(1 - \tan A)(1 + \tan B) = 2$

(ii) If $A + B + C = \frac{\pi}{2}$ then prove that $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

(iii) If $A + B + C = \frac{\pi}{2}$ then prove that $\cot A + \cot B + \cot C = \cot A \cot B \cot C$

(iv) If $A + B + C = \pi$ then prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(v) If $A + B + C = \pi$ then prove that $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vi) Prove that $\frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0} = \cot 36^0$

6.3 Trigonometric ratios of multiple and sub-multiple angles:

In this section we derive formulae for the trigonometric ratios of multiple angles $2A, 3A, ...$ in terms of those of A. Also we discuss about the trigonometric ratios of sub-multiple angles $\frac{A}{2}, \frac{A}{3}, ...$ of A.

6.3.1 Definition:

If A is an angle, then its integral multiples $2A, 3A, ...$ are called *Multiple angles of* A and $\frac{A}{2}, \frac{A}{3}, ...$ are called *Sub-multiple angles of* A.

6.3.2 Theorem:

If A is any real number, then

$$i)\sin 2A = 2\sin A\cos A,$$
$$ii)\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

Proof: (i) We know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\Rightarrow \sin(A+A) = \sin A\cos A + \cos A\sin A$$

$$\Rightarrow \sin 2A = 2\sin A\cos A$$

$$\therefore \sin 2A = 2\sin A\cos A$$

(ii) We know that $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\Rightarrow \cos(A+A) = \cos A\cos A - \sin A\sin A$$

$$= \cos^2 A - \sin^2 A$$

$$= \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$$

$$2(1-\sin^2 A) - 1 = 1 - 2\sin^2 A$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

6.3.3 Theorem:

If A is any real number, which is not an odd multiple of $\frac{\pi}{2}$ then

$$i)\sin 2A = \frac{2\tan A}{1+\tan^2 A}, \quad ii)\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A},$$

$$iii)\tan 2A = \frac{2\tan A}{1-\tan^2 A} \quad (A \text{ and } 2A \text{ are not odd multiple of } \frac{\pi}{2})$$
$$iv)\cot 2A = \frac{\cot^2 A - 1}{2\cot A} \quad (2A \text{ is not an integral multiple of } \pi)$$

Proof: (i) We have from Theorem 6.3.2

$$\sin 2A = 2\sin A\cos A = \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A}$$

$$= \frac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A+\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1+\tan^2 A}$$

$$\therefore \sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

(ii) We have from Theorem 6.3.2

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$$

$$= \frac{\frac{\cos^2 A-\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A+\sin^2 A}{\cos^2 A}} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\therefore \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(iii) We know that $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

$$\Rightarrow \tan(A+A) = \frac{\tan A+\tan A}{1-\tan A\tan A}$$

$$\Rightarrow \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\therefore \tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

(iv) We know that $\cot(A+B) = \frac{\cot A\cot B-1}{\cot A+\cot B}$

$$\Rightarrow \cot(A+A) = \frac{\cot A\cot A-1}{\cot A+\cot A}$$

$$\Rightarrow \cot 2A = \frac{\cot^2 A-1}{2\cot A}$$

$$\therefore \cot 2A = \frac{\cot^2 A-1}{2\cot A}$$

6.3.4 Corollary:

If $\frac{A}{2}$ is not an odd multiple of $\frac{\pi}{2}$ then

$$(i)\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$(ii)\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$(iii)\ \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

$$(iv)\ \cot A = \frac{\cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}$$

Now we derive the formulae for $\sin 3A, \cos 3A, \tan 3A$ and $\cot 3A.$

6.3.5 Theorem:

If A is any real number, then

$$i)\sin 3A = 3\sin A - 4\sin^3 A, ii)\cos 3A = 4\cos^3 A - 3\cos A,$$

$$iii)\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A} \quad (3A\ \textit{is not odd multiple of}\ \frac{\pi}{2})$$

$$iv)\cot 3A = \frac{3\cot A - \cot^3 A}{1-3\cot^2 A} \quad (3A\ \textit{is not an integral multiple of}\ \pi)$$

Proof: (i) $\sin 3A = \sin(2A+A) = \sin 2A\cos A + \cos 2A\sin A$

$$= (2\sin A\cos A)\cos A + (1-2\sin^2 A)\sin A$$

$$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$$

$$= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A = 3\sin A - 4\sin^3 A$$

$$\therefore \sin 3A = 3\sin A - 4\sin^3 A$$

(ii) $\cos 3A = \cos(2A + A) = \cos 2A\cos A - \sin 2A\sin A$

$$= \left(2\cos^2 A - 1\right)\cos A - \left(2\sin A\cos A\right)\sin A$$

$$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$$

$$= 2\cos^3 A - \cos A - 2\left(1 - \cos^2 A\right)\cos A = 4\cos^3 A - 3\cos A$$

$$\therefore \cos 3A = 4\cos^3 A - 3\cos A$$

(iii) $\tan 3A = \tan\left(2A + A\right) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

$$= \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}.\tan A} = \frac{\dfrac{2\tan A + \tan A\left(1 - \tan^2 A\right)}{1 - \tan^2 A}}{\dfrac{\left(1 - \tan^2 A\right) - 2\tan^2 A}{1 - \tan^2 A}}$$

$$= \frac{2\tan A + \tan A\left(1 - \tan^2 A\right)}{\left(1 - \tan^2 A\right) - 2\tan^2 A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\therefore \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

(iv) $\cot\left(2A + A\right) = \dfrac{\cot 2A\cot A - 1}{\cot A + \cot A}$

$$= \frac{\dfrac{\cot^2 A - 1}{2\cot A}.\cot A - 1}{\dfrac{\cot^2 A - 1}{2\cot A} + \cot A} = \frac{\dfrac{\cot^3 A - \cot A - 2\cot A}{2\cot A}}{\dfrac{\cot^2 A - 1 + 2\cot^2 A}{2\cot A}}$$

$$= \frac{\cot^3 A - \cot A - 2\cot A}{\cot^2 A - 1 + 2\cot^2 A} = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$$

$$\therefore \cot 3A = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$$

6.3.6 Corollary:

If A is any real number, then

$$i)\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}, ii)\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3},$$

$$iii)\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}}, iv)\cot A = \frac{3\cot\frac{A}{3} - \cot^3\frac{A}{3}}{1 - 3\cot^2\frac{A}{3}}.$$

6.3.7 Theorem:

If A is any real number, then

$$i)\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}, ii)\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}},$$

$$iii)\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}} \quad (A\, is\, not\, odd\, multiple\, of\, \frac{\pi}{2})$$

$$iv)\cot A = \pm\sqrt{\frac{1+\cos 2A}{1-\cos 2A}} \quad (A\, is\, not\, an\, integral\, multiple\, of\, \pi)$$

Proof: (i) We know that $\cos 2A = 1 - 2\sin^2 A$

$$\Rightarrow 2\sin^2 A = 1 - \cos 2A \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$$

Hence $\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$

(ii) We know that $\cos 2A = 2\cos^2 A - 1$

$$\Rightarrow 2\cos^2 A = 1 + \cos 2A \Rightarrow \cos^2 A = \frac{1+\cos 2A}{2}$$

Hence $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$

(iii) Assume that A is not an odd multiple of $\frac{\pi}{2}$

$$\tan^2 A = \frac{2\sin^2 A}{2\cos^2 A} = \frac{1-\cos 2A}{1+\cos 2A}$$

Hence $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$

(iv) Assume that A is not an integer multiple of π

$$\cot^2 A = \frac{2\cos^2 A}{2\sin^2 A} = \frac{1+\cos 2A}{1-\cos 2A}$$

Hence $\cot A = \pm\sqrt{\frac{1+\cos 2A}{1-\cos 2A}}$

6.3.8 Corollary:

If A is any real number, then

$$i)\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, ii)\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}},$$

$$iii)\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad (A\, is\, not\, odd\, multiple\, of\, \pi)$$

$$iv)\cot\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{1-\cos A}} \quad (A\, is\, not\, an\ integer\, multiple\, of\, 2\pi)$$

6.3.9 Example:

Prove that $(i)\ \sin 18^0 = \frac{\sqrt{5}-1}{4}$, $(ii)\ \cos 36^0 = \frac{\sqrt{5}+1}{4}$,

$(iii)\ \sin 36^0 = \frac{\sqrt{10-2\sqrt{5}}}{4}$, $(iii)\ \cos 18^0 = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

Solution: (i) If $A = 18^0$ then $5A = 90^0 \Rightarrow 2A = 90^0 - 3A$

$$\Rightarrow \sin 2A = \sin(90^0 - 3A) = \cos 3A$$

$$\Rightarrow 2\sin A\cos A = 4\cos^3 A - 3\cos A$$

$$\Rightarrow 2\sin A = 4\cos^2 A - 3 \Rightarrow 2\sin A = 4(1-\sin^2 A) - 3$$

$$\Rightarrow 4 - 4\sin^2 A - 2\sin A - 3 = 0 \Rightarrow 4\sin^2 A + 2\sin A - 1 = 0$$

Which is a quadratic equation in $\sin A$

$$\sin A \ = \frac{-2 \pm \sqrt{4+16}}{8} \ = \frac{-1 \pm \sqrt{5}}{4}$$

Since A lies in first quadrant so that $\sin A \ = \frac{\sqrt{5}-1}{4}$

$$\therefore \sin 18^0 = \frac{\sqrt{5}-1}{4}$$

(ii) We have $\cos 2A = 1 - 2\sin^2 A$

$$\cos 36^0 = 1 - 2\sin^2 18^0 \quad = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \left(\frac{6-2\sqrt{5}}{8}\right) = \frac{\sqrt{5}+1}{4}$$

(iii) We have $\sin 36^0 = \sqrt{1-\cos^2 36^0}$

$$= \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \sqrt{\frac{10-2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

(iv) We have $\cos 18^0 = \sqrt{1-\sin^2 18^0}$

$$= \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

6.3.10 Solved Problems:

1. Problem: Find the values of $(i)\sin 22\frac{1}{2}^0, (ii)\cos 22\frac{1}{2}^0, (iii)\tan 22\frac{1}{2}^0, (iv)\cot 22\frac{1}{2}^0$.

Solution: If $A = 22\frac{1}{2}^0$ then $2A = 45^0$

(i) We have $\sin A = \sqrt{\frac{1-\cos 2A}{2}}$

$$\sin 22\frac{1}{2}^0 = \sqrt{\frac{1-\cos 45^0}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

(ii) We have $\cos A = \sqrt{\frac{1+\cos 2A}{2}}$

$$\cos 22\frac{1}{2}^0 = \sqrt{\frac{1+\cos 45^0}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(iii) We have $\tan 22\frac{1}{2}^0 = \dfrac{\sin 22\frac{1}{2}^0}{\cos 22\frac{1}{2}^0} = \dfrac{\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$

$$= \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}} = \sqrt{\frac{\left(\sqrt{2}-1\right)^2}{(\sqrt{2})^2 - 1^2}} = \sqrt{\frac{\left(\sqrt{2}-1\right)^2}{1}} = \sqrt{2}-1$$

(iv) We have $\cot 22\frac{1}{2}^0 = \dfrac{\cos 22\frac{1}{2}^0}{\sin 22\frac{1}{2}^0} = \dfrac{\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}}{\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}} = \sqrt{\dfrac{\sqrt{2}+1}{\sqrt{2}-1}}$

$$= \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}} = \sqrt{\frac{\left(\sqrt{2}+1\right)^2}{(\sqrt{2})^2 - 1^2}} = \sqrt{\frac{\left(\sqrt{2}+1\right)^2}{1}} = \sqrt{2}+1$$

2. Problem: Find the values of $(i) \sin 67\frac{1}{2}^0, (ii) \cos 67\frac{1}{2}^0, (iii) \tan 67\frac{1}{2}^0, (iv) \cot 67\frac{1}{2}^0$.

Solution: We have $67\frac{1}{2}^0 = 90^0 - 22\frac{1}{2}^0$

(i) We have $\sin 67\frac{1}{2}^0 = \sin\left(90^0 - 22\frac{1}{2}^0\right) = \cos 22\frac{1}{2}^0 = \sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$

(ii) We have $\cos 67\frac{1}{2}^0 = \cos\left(90^0 - 22\frac{1}{2}^0\right) = \sin 22\frac{1}{2}^0 = \sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$

(iii) We have $\tan 67\frac{1}{2}^0 = \tan\left(90^0 - 22\frac{1}{2}^0\right) = \cot 22\frac{1}{2}^0 = \sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}} = \sqrt{2}+1$

(iv) We have $\cot 67\frac{1}{2}^0 = \cot\left(90^0 - 22\frac{1}{2}^0\right) = \tan 22\frac{1}{2}^0 = \sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}} = \sqrt{2}-1$

3. Problem: Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta.$

Solution: L.H.S$= \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{1+2\cos^2\theta - 1}$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{R.H.S}$$

$$\therefore \frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$$

4. Problem: Prove that $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$.

Solution: L.H.S= $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{R.H.S}$$

$$\therefore \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$$

5. Problem: Prove that $\dfrac{\cos 3A+\sin 3A}{\cos A-\sin A} = 1+2\sin 2A$.

Solution: L.H.S= $\dfrac{\cos 3A+\sin 3A}{\cos A-\sin A} = \dfrac{\left(4\cos^3 A-3\cos A\right)-\left(3\sin A-4\sin^3 A\right)}{\cos A-\sin A}$

$$= \frac{4\left(\cos^3 A-\sin^3 A\right)-3\left(\cos A-\sin A\right)}{\cos A-\sin A}$$

$$= \frac{\left(\cos A-\sin A\right)\left[4\left(\cos^2 A+\sin^2 A+\cos A\sin A\right)-3\right]}{\cos A-\sin A}$$

$$= 4+4\cos A\sin A-3 = 1+2\sin 2A = \text{R.H.S}$$

$$\therefore \frac{\cos 3A+\sin 3A}{\cos A-\sin A} = 1+2\sin 2A.$$

6. Problem: Prove that $\dfrac{3\cos\theta+\cos 3\theta}{3\sin\theta-\sin 3\theta} = \cot^3\theta$.

Solution: L.H.S= $\dfrac{3\cos\theta+\left(4\cos^3\theta-3\cos\theta\right)}{3\sin\theta-\left(3\sin\theta-4\sin^3\theta\right)}$

$$= \frac{4\cos^3\theta}{4\sin^3\theta} = \cot^3\theta = \text{R.H.S}$$

$$\therefore \frac{3\cos\theta+\cos 3\theta}{3\sin\theta-\sin 3\theta} = \cot^3\theta.$$

7. Problem: Prove that $\dfrac{\sin 2A}{1-\cos 2A}.\dfrac{1-\cos A}{\cos A} = \tan\dfrac{A}{2}$

Solution: L.H.S= $\dfrac{\sin 2A}{1-\cos 2A}.\dfrac{1-\cos A}{\cos A} = \dfrac{2\sin A\cos A}{1-\left(1-2\sin^2 A\right)}.\dfrac{1-\cos A}{\cos A}$

$$= \frac{2\sin A\cos A}{2\sin^2 A}.\frac{1-\cos A}{\cos A} = \frac{1-\cos A}{\sin A} = \frac{1-\left(1-2\sin^2\frac{A}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \tan\frac{A}{2}$$

$$\therefore \frac{\sin 2A}{1-\cos 2A}.\frac{1-\cos A}{\cos A} = \tan\frac{A}{2}$$

8. Problem: Prove that $\dfrac{\cos^3\theta-\cos 3\theta}{\cos\theta}+\dfrac{\sin^3\theta+\sin 3\theta}{\sin\theta}=3.$

Solution: L.H.S= $\dfrac{\cos^3\theta-\cos 3\theta}{\cos\theta}+\dfrac{\sin^3\theta+\sin 3\theta}{\sin\theta}$

$$= \frac{\cos^3\theta-\left(4\cos^3\theta-3\cos\theta\right)}{\cos\theta}+\frac{\sin^3\theta+\left(3\sin\theta-4\sin^3\theta\right)}{\sin\theta}$$

$$= \frac{3\cos\theta-3\cos^3\theta}{\cos\theta}+\frac{3\sin\theta-3\sin^3\theta}{\sin\theta}$$

$$= \frac{3\cos\theta\left(1-\cos^2\theta\right)}{\cos\theta}+\frac{3\sin\theta\left(1-\sin^2\theta\right)}{\sin\theta}$$

$$= 3\left(1-\cos^2\theta\right)+3\left(1-\sin^2\theta\right) = 3\sin^2\theta+3\cos^2\theta = 3 = \text{R.H.S}$$

$$\therefore \frac{\cos^3\theta-\cos 3\theta}{\cos\theta}+\frac{\sin^3\theta+\sin 3\theta}{\sin\theta}=3.$$

9. Problem: Prove that $\sin\theta\sin(60^0-\theta)\sin(60^0+\theta)=\dfrac{1}{4}\sin 3\theta.$

Solution: L.H.S= $\sin\theta\sin(60^0-\theta)\sin(60^0+\theta)$

$$= \sin\theta\left(\sin^2 60^0-\sin^2\theta\right) \quad \left[\because \sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B\right]$$

$$= \sin\theta\left(\left(\frac{\sqrt{3}}{2}\right)^2-\sin^2\theta\right) \quad \left[\because \sin 60^0=\frac{\sqrt{3}}{2}\right]$$

$$= \sin\theta\left(\frac{3}{4} - \sin^2\theta\right) \quad = \frac{3\sin\theta - 4\sin^3\theta}{4}$$

$$= \frac{\sin 3\theta}{4} \quad = \text{R.H.S}$$

$$\therefore \sin\theta \sin(60^0 - \theta)\sin(60^0 + \theta) = \frac{1}{4}\sin 3\theta.$$

10. Problem: Prove that $\cos\theta\cos(60^0 - \theta)\cos(60^0 + \theta) = \frac{1}{4}\cos 3\theta.$

Solution: L.H.S$=\cos\theta\cos(60^0 - \theta)\cos(60^0 + \theta)$

$$= \cos\theta\left(\cos^2\theta - \sin^2 60^0\right) \quad \left[\because \cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A\right]$$

$$= \cos\theta\left(\cos^2\theta - \left(\frac{\sqrt{3}}{2}\right)^2\right) \quad \left[\because \sin 60^0 = \frac{\sqrt{3}}{2}\right]$$

$$= \cos\theta\left(\cos^2\theta - \frac{3}{4}\right) \quad = \frac{4\cos^3\theta - 3\cos\theta}{4}$$

$$= \frac{\cos 3\theta}{4} \quad = \text{R.H.S}$$

$$\therefore \cos\theta\cos(60^0 - \theta)\cos(60^0 + \theta) = \frac{1}{4}\cos 3\theta$$

11. Problem: Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}.$

Solution: L.H.S$=\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos(\pi - \frac{3\pi}{10})\right)\left(1+\cos(\pi - \frac{\pi}{10})\right)$$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$\left[\because \cos(\pi - \theta) = -\cos\theta\right]$$

$$= \left(1-\cos^2\frac{3\pi}{10}\right)\left(1-\cos^2\frac{\pi}{10}\right) \quad = \sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10} \quad = \sin^2 18^0 \sin^2 54^0$$

$$=\left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2=\frac{\left((\sqrt{5}-1)(\sqrt{5}+1)\right)^2}{256}=\frac{16}{256}=\frac{1}{16}=\text{R.H.S}$$

$$\therefore\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$$

12. Problem: Prove that $\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\frac{3\pi}{5}+\cos^2\frac{9\pi}{10}=2.$

Solution: L.H.S$=\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\frac{3\pi}{5}+\cos^2\frac{9\pi}{10}$

$$=\cos^2 18^0+\cos^2 72^0+\cos^2 108^0+\cos^2 162^0$$

$$=\cos^2 18^0+\cos^2(90^0-18^0)+\cos^2(90^0+18^0)+\cos^2(180^0-18^0)$$

$$=\cos^2 18^0+\sin^2 18^0+\sin^2 18^0+\cos^2 18^0$$

$$\left[\because\cos(90^0-\theta)=\sin\theta,\cos(90^0+\theta)=-\sin\theta,\cos(180^0-\theta)=-\cos\theta\right]$$

$$=2\cos^2 18^0+2\sin^2 18^0=2\left(\cos^2 18^0+\sin^2 18^0\right)$$

$$=2(1)=2=\text{R.H.S}$$

$$\therefore\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\frac{3\pi}{5}+\cos^2\frac{9\pi}{10}=2$$

13. Problem: Prove that $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}.$

Solution: L.H.S$=\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$

$$=\cos^4\frac{\pi}{8}+\cos^4\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{2}+\frac{\pi}{8}\right)+\cos^4\left(\pi-\frac{\pi}{8}\right)$$

$$=\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}$$

$$\left[\because\cos(\frac{\pi}{2}-\theta)=\sin\theta,\cos(\frac{\pi}{2}+\theta)=-\sin\theta,\cos(\pi-\theta)=-\cos\theta\right]$$

$$=2\cos^4\frac{\pi}{8}+2\sin^4\frac{\pi}{8}=2\left(\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}\right)$$

$$=2\left[\left(\cos^2\frac{\pi}{8}+\sin^2\frac{\pi}{8}\right)^2-2\cos^2\frac{\pi}{8}\sin^2\frac{\pi}{8}\right]$$

$$\left[\because a^4+b^4=\left(a^2+b^2\right)^2-2a^2b^2\right]$$

$$=2\left[(1)^2-2\cos^2\frac{\pi}{8}\sin^2\frac{\pi}{8}\right]=2-4\cos^2\frac{\pi}{8}\sin^2\frac{\pi}{8}$$

$$=2-\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2=2-\left(\sin\frac{\pi}{4}\right)^2=2-\left(\frac{1}{\sqrt{2}}\right)^2=2-\frac{1}{2}=\frac{3}{2}=\text{R.H.S}$$

$$\therefore \cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$$

14. Problem: Prove that $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8}$.

Solution: L.H.S$=\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$

$$=\frac{1}{2\sin\frac{2\pi}{7}}\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$$

$$=\frac{1}{2\sin\frac{2\pi}{7}}\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{4\sin\frac{2\pi}{7}}\left(2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}\right)\cos\frac{6\pi}{7}$$

$$=\frac{1}{4\sin\frac{2\pi}{7}}\sin\frac{8\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8\sin\frac{2\pi}{7}}\left(2\sin\frac{8\pi}{7}\cos\frac{6\pi}{7}\right)$$

$$=\frac{1}{8\sin\frac{2\pi}{7}}\left(2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\right)\left[\because \sin\frac{8\pi}{7}=-\sin\frac{\pi}{7},\cos\frac{6\pi}{7}=-\cos\frac{6\pi}{7}\right]$$

$$=\frac{1}{8\sin\frac{2\pi}{7}}\left(\sin\frac{2\pi}{7}\right)=\frac{1}{8}=\text{R.H.S}$$

$$\therefore \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}=\frac{1}{8}$$

15. Problem: Prove that $\cos^6 A+\sin^6 A=1-\frac{3}{4}\sin^2 2A$

Solution: L.H.S= $\cos^6 A + \sin^6 A = \left(\cos^2 A\right)^3 + \left(\sin^2 A\right)^3$

$$= \left[\left(\cos^2 A + \sin^2 A\right)^3 - 3\cos^2 A \sin^2 A\left(\cos^2 A + \sin^2 A\right)\right]$$

$$\left[\because a^3 + b^3 = \left(a+b\right)^3 - 3ab\left(a+b\right)\right]$$

$$= \left[\left(1\right)^3 - 3\cos^2 A \sin^2 A\left(1\right)\right]$$

$$= 1 - \frac{3}{4}\left(2\sin A\cos A\right)^2 = 1 - \frac{3}{4}\left(\sin 2A\right)^2$$

$$= 1 - \frac{3}{4}\sin^2 2A = \text{R.H.S}$$

$$\therefore \cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2 2A$$

16. Problem: Prove that $\dfrac{1-\cos A+\sin A}{1+\cos A+\sin A} = \tan\dfrac{A}{2}$

Solution: L.H.S= $\dfrac{1-\cos A+\sin A}{1+\cos A+\sin A} = \dfrac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}$

$$= \frac{2\sin\frac{A}{2}\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)}{2\cos\frac{A}{2}\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)} \quad = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \tan\frac{A}{2} = \text{R.H.S}$$

$$\therefore \frac{1-\cos A+\sin A}{1+\cos A+\sin A} = \tan\frac{A}{2}$$

17. Problem: If $\dfrac{\sin\alpha}{a} = \dfrac{\cos\alpha}{b}$ then prove that $a\sin 2\alpha + b\cos 2\alpha = b$

Solution: Given $\dfrac{\sin\alpha}{a} = \dfrac{\cos\alpha}{b} \Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{a}{b} \Rightarrow \tan\alpha = \dfrac{a}{b}$

$$\text{L.H.S} = a\sin 2\alpha + b\cos 2\alpha = a\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) + b\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)$$

$$= a\left(\frac{2\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2}\right)+b\left(\frac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2}\right) \quad = \frac{2a^2b}{b^2+a^2}+b\left(\frac{b^2-a^2}{b^2+a^2}\right)$$

$$= \frac{2a^2b+b(b^2-a^2)}{b^2+a^2} = \frac{a^2b+b^3}{b^2+a^2} = b = \text{R.H.S}$$

$\therefore a\sin 2\alpha + b\cos 2\alpha = b$

Exercise 6(c)

1. Prove the following

   $i)\cos 10^0 \cos 30^0 \cos 50^0 \cos 70^0 = \frac{3}{16}$ $ii)\cos 24^0 \cos 48^0 \cos 96^0 \cos 192^0 = \frac{1}{16}$

   $iii)\tan 6^0 \tan 42^0 \tan 66^0 \tan 78^0 = 1$ $iv)\sin 20^0 \sin 40^0 \sin 60^0 \sin 80^0 = \frac{3}{16}$

2. Prove the following

   $i)\tan\theta \tan(60^0-\theta)\tan(60^0+\theta) = \tan 3\theta.$

   $ii)\sin^2\theta + \sin^2(60^0+\theta)+\sin^2(60^0-\theta) = \frac{3}{2}$

   $iii)\cos^2\theta + \cos^2(120^0+\theta)+\cos^2(120^0-\theta) = \frac{3}{2}$

   $iv)\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$

   $v)\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8} = \frac{3}{2}$ $vi)\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{16}$

   $vii)\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$

3 Prove the following

$$i)\frac{\sin 3A}{1+2\cos 2A} = \sin A \quad ii)\frac{\cos 3A}{2\cos 2A-1} = \cos A \quad iii)\frac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$$

6.4 Sum and product transformations:

In this section we give the sum and difference of trigonometric functions can be transformed into their product vice-versa.

6.4.1 Theorem:

For $A, B \in R$ we have

$(i)\ \sin(A+B)+\sin(A-B)\ =2\sin A\cos B$
$(ii)\ \sin(A+B)-\sin(A-B)\ =2\cos A\sin B$
$(iii)\cos(A+B)+\cos(A-B)\ =2\cos A\cos B$
$(iv)\cos(A+B)-\cos(A-B)\ =-2\sin A\sin B$

Proof: **(i)** We have $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and

$$\sin(A-B)\ =\sin A\cos B-\cos A\sin B$$

By adding we get $\sin(A+B)+\sin(A-B)$

$$=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$$
$$=2\sin A\cos B$$

(ii) By subtracting we get $\sin(A+B)-\sin(A-B)$

$$=\left(\sin A\cos B+\cos A\sin B\right)-\left(\sin A\cos B-\cos A\sin B\right)$$
$$=2\cos A\sin B$$

(iii) We have $\cos(A+B)\ =\cos A\cos B-\sin A\sin B$ and

$$\cos(A-B)\ =\cos A\cos B+\sin A\sin B$$

By adding we get $\cos(A+B)+\cos(A-B)$

$$=\cos A\cos B-\sin A\sin B\ +\cos A\cos B+\sin A\sin B$$
$$=2\cos A\cos B$$

(iv) By subtracting we get $\cos(A+B)+\cos(A-B)$

$$=\left(\cos A\cos B-\sin A\sin B\right)-\left(\cos A\cos B+\sin A\sin B\right)$$
$$=-2\sin A\sin B$$

6.4.2 Note:

The above four identities can be rewrite as follows

$(i)\quad \sin(sum)+\sin(difference)\ =2\sin A\cos B$
$(ii)\quad \sin(sum)-\sin(difference)\ =2\cos A\sin B$
$(iii)\ \cos(sum)+\cos(difference)=2\cos A\cos B$
$(iv)\ \ \cos(sum)-\cos(difference)=-2\sin A\sin B$

6.4.3 Theorem:

For any $C,D\in R$ we have

$(i)\quad \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$(ii)\quad \sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$(iii)\quad \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$(iv)\quad \cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Proof: Take $A+B=C, A-B=D$ in theorem 6.4.1 we get the above 4 transformations.

6.4.4 Solved Problems:

1. Problem: Express $\sin 6\theta+\sin 4\theta$ as product.

Solution: We have $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

now $\sin 6\theta+\sin 4\theta=2\sin\left(\frac{6\theta+4\theta}{2}\right)\cos\left(\frac{6\theta-4\theta}{2}\right)$

$\Rightarrow \sin 6\theta+\sin 4\theta=2\sin 5\theta\cos\theta.$

2. Problem: Express $2\cos 48^0\cos 12^0$ into sum.

Solution: We have $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

Take $A=48^0, B=12^0$ we get $2\cos 48^0\cos 12^0=\cos(48^0+12^0)+\cos(48^0-12^0)$

$\Rightarrow 2\cos 48^0\cos 12^0=\cos 60^0+\cos 36^0$

$\Rightarrow 2\cos 48^0\cos 12^0=\frac{1}{2}+\frac{\sqrt{5}+1}{4} \Rightarrow 2\cos 48^0\cos 12^0=\frac{\sqrt{5}+3}{4}$

3. Problem: Prove that $\cos 40^0+\cos 80^0+\cos 160^0=0$.

Solution: L.H.S $=\cos 40^0+\cos 80^0+\cos 160^0$

$=\cos 160^0+\cos 40^0+\cos 80^0$

$=2\cos\left(\frac{160^0+40^0}{2}\right)\cos\left(\frac{160^0-40^0}{2}\right)+\cos 80^0$

$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$

$$= 2\cos 100^0 \cos 60^0 + \cos 80^0$$

$$= 2\left(\cos 100^0\right)\frac{1}{2} + \cos 80^0 \left[\because \cos 60^0 = \frac{1}{2}\right]$$

$$= \cos 100^0 + \cos 80^0 \left[\because \cos 60^0 = \frac{1}{2}\right]$$

$$= -\cos 80^0 + \cos 80^0 \left[\because \cos 100^0 = \cos\left(180^0 - 80^0\right) = -\cos 80^0\right]$$

$$= 0 = \text{R.H.S}$$

$$\therefore \cos 40^0 + \cos 80^0 + \cos 160^0 = 0$$

4. Problem: Prove that $\sin 34^0 + \cos 64^0 - \cos 4^0 = 0$.

Solution: L.H.S = $\sin 34^0 + \cos 64^0 - \cos 4^0$

$$= \sin 34^0 - 2\sin\left(\frac{64^0 + 4^0}{2}\right)\sin\left(\frac{64^0 - 4^0}{2}\right)$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \sin 34^0 - 2\sin 34^0 \sin 30^0$$

$$= \sin 34^0 - 2\left(\sin 34^0\right)\frac{1}{2}\left[\because \sin 30^0 = \frac{1}{2}\right]$$

$$= \sin 34^0 - \sin 34^0$$

$$= 0 = \text{R.H.S}$$

$$\therefore \sin 34^0 + \cos 64^0 - \cos 4^0 = 0$$

5. Problem: Prove that $\sin 78^0 + \cos 132^0 = \frac{\sqrt{5}-1}{4}$.

Solution: L.H.S = $\sin 78^0 + \cos 132^0$

$$= \sin 78^0 + \cos\left(90^0 + 42^0\right)$$

$$= \sin 78^0 - \sin 42^0 \quad \left[\because \cos\left(90^0 + \theta\right) = -\sin\theta\right]$$

$$= 2\cos\left(\frac{78^0 + 42^0}{2}\right)\sin\left(\frac{78^0 - 42^0}{2}\right)$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= 2\cos 60^0 \sin 18^0$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{5}-1}{4}\right)\left[\because \cos 60^0 = \frac{1}{2}, \sin 18^0 = \frac{\sqrt{5}-1}{4}\right]$$

$$= \frac{\sqrt{5}-1}{4} = \text{R.H.S}$$

$$\therefore \sin 78^0 + \cos 132^0 = \frac{\sqrt{5}-1}{4}$$

6. Problem: Prove that $\cos^2 76^0 + \cos^2 16^0 - \cos 76^0 \cos 16^0 = \frac{3}{4}$.

Solution: L.H.S $= \cos^2 76^0 + \cos^2 16^0 - \cos 76^0 \cos 16^0$

$$= \cos^2 76^0 + \left(1 - \sin^2 16^0\right) - \cos 76^0 \cos 16^0 \left[\because \cos^2\theta = 1 - \sin^2\theta\right]$$

$$= 1 + \cos\left(76^0 + 16^0\right)\cos\left(76^0 - 16^0\right) - \cos 76^0 \cos 16^0$$

$$\left[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)\right]$$

$$= 1 + \cos 92^0 \cos 60^0 - \frac{1}{2}\left(2\cos 76^0 \cos 16^0\right)$$

$$= 1 + \cos 92^0 \cos 60^0 - \frac{1}{2}\left(\cos\left(76^0 + 16^0\right) + \cos\left(76^0 - 16^0\right)\right)$$

$$\left[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)\right]$$

$$= 1 + \cos 92^0 \cos 60^0 - \frac{1}{2}\left(\cos 92^0 + \cos 60^0\right)$$

$$= 1 + \left(\cos 92^0\right)\frac{1}{2} - \frac{1}{2}\left(\cos 92^0 + \frac{1}{2}\right)\left[\because \cos 60^0 = \frac{1}{2}\right]$$

$$= 1 + \left(\cos 92^0\right)\frac{1}{2} - \frac{1}{2}\left(\cos 92^0\right) - \frac{1}{4}$$

$$=1-\frac{1}{4}$$

$$=\frac{3}{4}=\text{R.H.S}$$

$$\therefore \cos^2 76^0+\cos^2 16^0-\cos 76^0\cos 16^0=\frac{3}{4}$$

7. Problem: Prove that $\cos^2\theta+\cos^2\left(\frac{2\pi}{3}+\theta\right)+\cos^2\left(\frac{2\pi}{3}-\theta\right)=\frac{3}{2}$.

Solution: L.H.S $=\cos^2\theta+\cos^2\left(\frac{2\pi}{3}+\theta\right)+\cos^2\left(\frac{2\pi}{3}-\theta\right)$

$$=\frac{1+\cos 2\theta}{2}+\frac{1+\cos 2\left(\frac{2\pi}{3}+\theta\right)}{2}+\frac{1+\cos 2\left(\frac{2\pi}{3}-\theta\right)}{2}\left[\because \cos^2\theta=\frac{1+\cos 2\theta}{2}\right]$$

$$=\frac{3}{2}+\frac{1}{2}\left(\cos 2\theta+\cos 2\left(\frac{2\pi}{3}+\theta\right)+\cos 2\left(\frac{2\pi}{3}-\theta\right)\right)$$

$$=\frac{3}{2}+\frac{1}{2}\left(\cos 2\theta+\cos\left(\frac{4\pi}{3}+2\theta\right)+\cos\left(\frac{4\pi}{3}-2\theta\right)\right)$$

$$=\frac{3}{2}+\frac{1}{2}\left(\cos 2\theta+2\cos\left(\frac{4\pi}{3}\right)\cos 2\theta\right)$$

$$[\because \cos(A+B)+\cos(A-B)=2\cos A\cos B]$$

$$=\frac{3}{2}+\frac{1}{2}\left(\cos 2\theta+2\left(-\frac{1}{2}\right)\cos 2\theta\right)\left[\because \cos\frac{4\pi}{3}=-\frac{1}{2}\right]$$

$$=\frac{3}{2}+\frac{1}{2}(\cos 2\theta-\cos 2\theta)$$

$$=\frac{3}{2}+\frac{1}{2}(0)$$

$$=\frac{3}{2}+0$$

$$=\frac{3}{2}=\text{R.H.S}$$

$$\therefore \cos^2\theta + \cos^2\left(\frac{2\pi}{3}+\theta\right) + \cos^2\left(\frac{2\pi}{3}-\theta\right) = \frac{3}{2}$$

8. Problem: Prove that $\dfrac{\sin(n+1)\alpha - \sin(n-1)\alpha}{\cos(n+1)\alpha + 2\cos n\alpha + \cos(n-1)\alpha} = \tan\dfrac{\alpha}{2}$.

Solution: L.H.S $= \dfrac{\sin(n+1)\alpha - \sin(n-1)\alpha}{\cos(n+1)\alpha + 2\cos n\alpha + \cos(n-1)\alpha}$

$$= \frac{\sin(n\alpha+\alpha) - \sin(n\alpha-\alpha)}{\cos(n\alpha+\alpha) + \cos(n\alpha-\alpha) + 2\cos n\alpha}$$

$$= \frac{2\cos n\alpha \sin\alpha}{2\cos n\alpha\cos\alpha + 2\cos n\alpha} \begin{bmatrix} \because \sin(A+B) - \sin(A-B) = 2\cos A\sin B, \\ \cos(A+B) + \cos(A-B) = 2\cos A\cos B \end{bmatrix}$$

$$= \frac{2\cos n\alpha \sin\alpha}{2\cos n\alpha(\cos\alpha + 1)}$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} \left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}, \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \tan\frac{\alpha}{2} = \text{R.H.S}$$

$$\therefore \frac{\sin(n+1)\alpha - \sin(n-1)\alpha}{\cos(n+1)\alpha + 2\cos n\alpha + \cos(n-1)\alpha} = \tan\frac{\alpha}{2}$$

9. Problem: If $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{a+b}{a-b}$ then prove that $a\tan\beta = b\tan\alpha$

Solution: Given $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{a+b}{a-b}$

$$\Rightarrow \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)} = \frac{(a+b)+(a-b)}{(a+b)-(a-b)} [\because bycomponendoanddividendo]$$

$$\Rightarrow \frac{2\sin\alpha\cos\beta}{2\cos\alpha\sin\beta} = \frac{2a}{2b} \begin{bmatrix} \because \sin(A+B)-\sin(A-B) = 2\cos A\sin B, \\ \sin(A+B)+\sin(A-B) = 2\sin A\cos B \end{bmatrix}$$

$$\Rightarrow \frac{\tan\alpha}{\tan\beta} = \frac{a}{b}$$

$$\therefore a\tan\beta = b\tan\alpha$$

9. Problem: If $\cos x + \cos y = \frac{4}{5}, \cos x - \cos y = \frac{2}{7}$ then prove that

$$14\tan\left(\frac{x+y}{2}\right) + 5\cot\left(\frac{x-y}{2}\right) = 0$$

Solution: Given $\cos x + \cos y = \frac{4}{5}, \cos x - \cos y = \frac{2}{7}$

$$\cos x + \cos y = \frac{4}{5} \Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{4}{5} \qquad (1)$$

$$\cos x - \cos y = \frac{2}{7} \Rightarrow -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{2}{7} \qquad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} = \frac{4/5}{2/7}$$

$$\Rightarrow \frac{\cot\left(\frac{x+y}{2}\right)}{-\tan\left(\frac{x-y}{2}\right)} = \frac{4}{5} \times \frac{7}{2}$$

$$\Rightarrow 5\cot\left(\frac{x+y}{2}\right) = -14\tan\left(\frac{x-y}{2}\right)$$

$$\therefore 14\tan\left(\frac{x+y}{2}\right) + 5\cot\left(\frac{x-y}{2}\right) = 0$$

9. Problem: If $\cos x + \cos y = \frac{4}{5}, \cos x - \cos y = \frac{2}{7}$ then prove that

$$14\tan\left(\frac{x+y}{2}\right)+5\cot\left(\frac{x-y}{2}\right)=0$$

Solution: Given $\cos x+\cos y=\frac{4}{5},\cos x-\cos y=\frac{2}{7}$

$$\cos x+\cos y=\frac{4}{5}\Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=\frac{4}{5} \qquad (1)$$

$$\cos x-\cos y=\frac{2}{7}\Rightarrow -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)=\frac{2}{7} \qquad (2)$$

$$\frac{(1)}{(2)}\Rightarrow\frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}=\frac{4/5}{2/7}$$

$$\Rightarrow\frac{\cot\left(\frac{x+y}{2}\right)}{-\tan\left(\frac{x-y}{2}\right)}=\frac{4}{5}\times\frac{7}{2}\Rightarrow 5\cot\left(\frac{x+y}{2}\right)=-14\tan\left(\frac{x-y}{2}\right)$$

$$\therefore 14\tan\left(\frac{x+y}{2}\right)+5\cot\left(\frac{x-y}{2}\right)=0$$

Exercise 6(d)

1. Prove that $\cos 55^0+\cos 65^0+\cos 175^0=0$.
2. Prove that $\cos 35^0+\cos 85^0+\cos 155^0=0$.
3. Prove that $4\left(\sin 78^0+\cos 6^0\right)=\sqrt{15}+\sqrt{13}$.
4. Prove that $\sin 50^0-\sin 70^0+\sin 10^0=0$.
5. Prove that $\sin 10^0+\sin 20^0+\sin 40^0+\sin 50^0=\sin 70^0+\sin 80^0$.
6. Prove that $\sin^2\left(\alpha-45^0\right)+\sin^2\left(\alpha+15^0\right)-\sin^2\left(\alpha-15^0\right)=\frac{1}{2}$.
7. Prove that $\sin^2\left(\alpha-45^0\right)+\sin^2\left(\alpha+15^0\right)-\sin^2\left(\alpha-15^0\right)=\frac{1}{2}$.
8. If $m\sin B=n\sin\left(2A+B\right)$ then prove that $(m+n)\tan A=(m-n)\tan B$
9. If $\tan\left(A+B\right)=\lambda\tan\left(A-B\right)$ then prove that $(\lambda+1)\sin 2B=(\lambda-1)\sin 2A$
10. If $\sin x+\sin y=a,\cos x+\cos y=b$ then prove that

    $\tan\left(\frac{x+y}{2}\right)=\frac{a}{b}$

6.4.5 Solved Problems:

1. Problem: If $A+B+C=180^0$,

Prove that $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

Solution: L.H.S $=\sin 2A+\sin 2B+\sin 2C$

$$=2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\sin 2C$$

$$\left[\because \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C[\because \sin 2C=2\sin C\cos C]$$

$$=2\sin\left(180^0-C\right)\cos(A-B)+2\sin C\cos C\left[\because A+B=180^0-C\right]$$

$$=2\sin C\cos(A-B)+2\sin C\cos C\left[\because \sin\left(180^0-C\right)=\sin C\right]$$

$$=2\sin C\left[\cos(A-B)+\cos C\right]$$

$$=2\sin C\left[\cos(A-B)+\cos\left(180^0-(A+B)\right)\right]\left[\because C=180^0-(A+B)\right]$$

$$=2\sin C\left[\cos(A-B)-\cos(A+B)\right]\left[\because \cos\left(180^0-\theta\right)=-\cos\theta\right]$$

$$=2\sin C\left[2\sin A\sin B\right]\left[\because \cos(A-B)-\cos(A+B)=2\sin A\sin B\right]$$

$$=4\sin A\sin B\sin C=\text{R.H.S}$$

$\therefore \sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

2. Problem: If $A+B+C=180^0$,

Prove that $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

Solution: L.H.S $=\sin 2A+\sin 2B-\sin 2C$

$$=2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)-\sin 2C$$

$$\left[\because \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\sin(A+B)\cos(A-B)-2\sin C\cos C[\because \sin 2C=2\sin C\cos C]$$

$$=2\sin\left(180^0-C\right)\cos\left(A-B\right)-2\sin C\cos C\left[\because A+B=180^0-C\right]$$

$$=2\sin C\cos\left(A-B\right)-2\sin C\cos C\left[\because \sin\left(180^0-C\right)=\sin C\right]$$

$$=2\sin C\left[\cos\left(A-B\right)-\cos C\right]$$

$$=2\sin C\left[\cos\left(A-B\right)-\cos\left(180^0-\left(A+B\right)\right)\right]\left[\because C=180^0-\left(A+B\right)\right]$$

$$=2\sin C\left[\cos\left(A-B\right)+\cos\left(A+B\right)\right]\left[\because \cos\left(180^0-\theta\right)=-\cos\theta\right]$$

$$=2\sin C\left[2\cos A\cos B\right]\left[\because \cos\left(A-B\right)+\cos\left(A+B\right)=2\cos A\cos B\right]$$

$$=4\cos A\cos B\sin C=\text{R.H.S}$$

$$\therefore \sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C.$$

3. Problem: If $A+B+C=180^0$,

Prove that $\cos 2A+\cos 2B+\cos 2C+1=-4\cos A\cos B\cos C.$

Solution: L.H.S = $\cos 2A+\cos 2B+\cos 2C+1$

$$=2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\cos 2C+1$$

$$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\cos\left(A+B\right)\cos\left(A-B\right)+2\cos^2 C\left[\because 1+\cos 2C=2\cos^2 C\right]$$

$$=2\cos\left(180^0-C\right)\cos\left(A-B\right)+2\cos^2 C\left[\because A+B=180^0-C\right]$$

$$=-2\cos C\cos\left(A-B\right)+2\cos^2 C\left[\because \cos\left(180^0-C\right)=-\cos C\right]$$

$$=-2\cos C\left[\cos\left(A-B\right)-\cos C\right]$$

$$=-2\cos C\left[\cos\left(A-B\right)-\cos\left(180^0-\left(A+B\right)\right)\right]\left[\because C=180^0-\left(A+B\right)\right]$$

$$=-2\cos C\left[\cos\left(A-B\right)+\cos\left(A+B\right)\right]\left[\because \cos\left(180^0-\theta\right)=-\cos\theta\right]$$

$$=-2\cos C\left[2\cos A\cos B\right]\left[\because \cos\left(A-B\right)+\cos\left(A+B\right)=2\cos A\cos B\right]$$

$= -4\cos A\cos B\cos C =$ R.H.S

$\therefore \cos 2A + \cos 2B + \cos 2C + 1 = -4\cos A\cos B\cos C.$

4. Problem: If $A + B + C = 180^0$,

Prove that $\cos 2A + \cos 2B - \cos 2C - 1 = -4\sin A\sin B\cos C.$

Solution: L.H.S = $\cos 2A + \cos 2B - \cos 2C - 1$

$$= 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) - \cos 2C - 1$$

$$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= 2\cos(A+B)\cos(A-B) - 2\cos^2 C\left[\because 1 + \cos 2C = 2\cos^2 C\right]$$

$$= 2\cos(180^0 - C)\cos(A-B) - 2\cos^2 C\left[\because A + B = 180^0 - C\right]$$

$$= -2\cos C\cos(A-B) - 2\cos^2 C\left[\because \cos(180^0 - C) = -\cos C\right]$$

$$= -2\cos C\left[\cos(A-B) + \cos C\right]$$

$$= -2\cos C\left[\cos(A-B) + \cos(180^0 - (A+B))\right]\left[\because C = 180^0 - (A+B)\right]$$

$$= -2\cos C\left[\cos(A-B) - \cos(A+B)\right]\left[\because \cos(180^0 - \theta) = -\cos\theta\right]$$

$$= -2\cos C\left[2\sin A\sin B\right]\left[\because \cos(A-B) - \cos(A+B) = 2\sin A\sin B\right]$$

$= -4\sin A\sin B\cos C =$ R.H.S

$\therefore \cos 2A + \cos 2B - \cos 2C - 1 = -4\sin A\sin B\cos C.$

5. Problem: If $A + B + C = 180^0$,

Prove that $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

Solution: L.H.S = $\sin A + \sin B + \sin C$

$$=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin C$$

$$\left[\because \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$=2\sin\left(\frac{180^0-C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because A+B=180^0-C\right]$$

$$=2\sin\left(90^0-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because \sin\left(90^0-\frac{C}{2}\right)=\cos\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{180^0-(A+B)}{2}\right)\right]\left[\because C=180^0-(A+B)\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90^0-\frac{(A+B)}{2}\right)\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{(A+B)}{2}\right)\right]\left[\because \sin(90^0-\theta)=\cos\theta\right]$$

$$=2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]\left[\because \cos(A-B)+\cos(A+B)=2\cos A\cos B\right]$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\text{R.H.S}$$

$$\therefore \sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

6. Problem: If $A+B+C=180^0$,

Prove that $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

Solution: L.H.S $= \cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin C$$

$$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}\left[\because \cos\theta = 1 - 2\sin^2\frac{\theta}{2}\right]$$

$$= 2\cos\left(\frac{180^0 - C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}\left[\because A+B = 180^0 - C\right]$$

$$= 2\cos\left(90^0 - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}\ \left[\because \cos\left(90^0 - \theta\right) = \sin\theta\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{180^0 - (A+B)}{2}\right)\right]\left[\because C = 180^0 - (A+B)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(90^0 - \frac{(A+B)}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{(A+B)}{2}\right)\right]\left[\because \sin\left(90^0 - \theta\right) = \cos\theta\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]\left[\because \cos(A-B) - \cos(A+B) = 2\sin A\sin B\right]$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \text{R.H.S}$$

$$\therefore \cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

7. Problem: If $A + B + C = 180^0$,

Prove that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

Solution: L.H.S $=\sin^2 A+\sin^2 B+\sin^2 C$

$$=\frac{1-\cos 2A}{2}+\frac{1-\cos 2B}{2}+\frac{1-\cos 2C}{2}\left[\sin^2\theta=\frac{1-\cos 2\theta}{2}\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[\cos 2A+\cos 2B+\cos 2C\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[-1-4\cos A\cos B\cos C\right]$$

$$\left[\because \cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C\right]$$

$$=\frac{3}{2}+\frac{1}{2}+2\cos A\cos B\cos C=\text{R.H.S}$$

$\therefore \sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

8. Problem: If $A+B+C=180^0$,

Prove that $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$

Solution: L.H.S $=\cos^2 A+\cos^2 B-\cos^2 C$

$$=\frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}-\frac{1+\cos 2C}{2}\left[\cos^2\theta=\frac{1+\cos 2\theta}{2}\right]$$

$$=\frac{1}{2}+\frac{1}{2}\left[\cos 2A+\cos 2B-\cos 2C\right]$$

$$=\frac{1}{2}+\frac{1}{2}\left[1-4\sin A\sin B\cos C\right]$$

$$\left[\because \cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C\right]$$

$$=1-2\sin A\sin B\cos C=\text{R.H.S}$$

$\therefore \cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$

9. Problem: If A,B,C are the angles of a triangle ,

Prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution: L.H.S $=\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$

$$=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}\left[\sin^2\theta=\frac{1-\cos 2\theta}{2}\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[\cos A+\cos B+\cos C\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$\left[\because \cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$=\frac{3}{2}-\frac{1}{2}-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\text{R.H.S}$$

$$\therefore\ \sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

10. Problem: If $A+B+C=90^0$,

Prove that $\cos 2A+\cos 2B+\cos 2C=1+4\sin A\sin B\sin C.$

Solution: L.H.S = $\cos 2A+\cos 2B+\cos 2C$

$$=2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\cos 2C$$

$$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\cos(A+B)\cos(A-B)+\cos 2C$$

$$=2\cos(90^0-C)\cos(A-B)+\cos 2C\left[\because A+B=90^0-C\right]$$

$$=2\sin C\cos(A-B)+1-2\sin^2 C\left[\because \cos(90^0-C)=\sin C,\cos 2C=1-2\sin^2 C\right]$$

$$=1+2\sin C\left[\cos(A-B)-\sin C\right]$$

$$=1+2\sin C\left[\cos(A-B)-\sin(90^0-(A+B))\right]\left[\because C=90^0-(A+B)\right]$$

$$=1+2\sin C\left[\cos(A-B)-\cos(A+B)\right]\left[\because \sin(90^0-\theta)=\cos\theta\right]$$

$$=1+2\sin C\left[2\sin A\sin B\right]\left[\because \cos(A-B)-\cos(A+B)=2\sin A\sin B\right]$$

$$=1+4\sin A\sin B\sin C=\text{R.H.S}$$

$\therefore\ \cos 2A+\cos 2B+\cos 2C=1+4\sin A\sin B\sin C.$

11. Problem: If $A+B+C=0^0$,

Prove that $\sin 2A+\sin 2B+\sin 2C=-4\sin A\sin B\sin C.$

Solution: L.H.S $=\sin 2A+\sin 2B+\sin 2C$

$$=2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\sin 2C$$

$$\left[\because \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C[\because \sin 2C=2\sin C\cos C]$$

$$=2\sin(-C)\cos(A-B)+2\sin C\cos C[\because A+B=-C]$$

$$=-2\sin C\cos(A-B)+2\sin C\cos C[\because \sin(-C)=-\sin C]$$

$$=-2\sin C[\cos(A-B)-\cos C]$$

$$=-2\sin C[\cos(A-B)-\cos(-(A+B))][\because C=-(A+B)]$$

$$=-2\sin C[\cos(A-B)-\cos(A+B)][\because \cos(-\theta)=\cos\theta]$$

$$=-2\sin C[2\sin A\sin B][\because \cos(A-B)-\cos(A+B)=2\sin A\sin B]$$

$$=-4\sin A\sin B\sin C=\text{R.H.S}$$

$\therefore \sin 2A+\sin 2B+\sin 2C=-4\sin A\sin B\sin C.$

12. Problem: If $A+B+C=\frac{3\pi}{2}$,

Prove that $\cos 2A+\cos 2B+\cos 2C=1-4\sin A\sin B\sin C.$

Solution: L.H.S $=\cos 2A+\cos 2B+\cos 2C$

$$=2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\cos 2C$$

$$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\cos(A+B)\cos(A-B)+\cos 2C$$

$$=2\cos\left(\frac{3\pi}{2}-C\right)\cos(A-B)+\cos 2C\left[\because A+B=\frac{3\pi}{2}-C\right]$$

$$=-2\sin C\cos(A-B)+1-2\sin^2 C$$

$$\left[\because \cos\left(\frac{3\pi}{2}-C\right)=-\sin C, \cos 2C=1-2\sin^2 C\right]$$

$$=1-2\sin C\left[\cos(A-B)+\sin C\right]$$

$$=1-2\sin C\left[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)\right]\left[\because C=\frac{3\pi}{2}-(A+B)\right]$$

$$=1-2\sin C\left[\cos(A-B)-\cos(A+B)\right]\left[\because \sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta\right]$$

$$=1-2\sin C\left[2\sin A\sin B\right]\left[\because \cos(A-B)-\cos(A+B)=2\sin A\sin B\right]$$

$$=1-4\sin A\sin B\sin C=\text{R.H.S}$$

$\therefore\ \cos 2A+\cos 2B+\cos 2C=1-4\sin A\sin B\sin C.$

13. Problem: If $A+B+C=2S$,

Prove that $\cos(S-A)+\cos(S-B)+\cos(S-C)+\cos S=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

Solution: L.H.S $=\cos(S-A)+\cos(S-B)+\cos(S-C)+\cos S$

$$=2\cos\left(\frac{2S-A-B}{2}\right)\cos\left(\frac{B-A}{2}\right)+2\cos\left(\frac{2S-C}{2}\right)\cos\left(\frac{-C}{2}\right)$$

$$\left[\because \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$=2\cos\left(\frac{C}{2}\right)\cos\left(\frac{B-A}{2}\right)+2\cos\left(\frac{A+B}{2}\right)\cos\frac{C}{2}$$

$$\left[\because 2S-A-B=C, 2S-C=A+B, \cos\left(\frac{-C}{2}\right)=\cos\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]\left[\because \cos(-\theta)=\cos\theta\right]$$

$$=2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]\left[\because \cos(A-B)+\cos(A+B)=2\cos A\cos B\right]$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\text{R.H.S}$$

$$\therefore \cos(S-A)+\cos(S-B)+\cos(S-C)+\cos S=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Exercise 6(e)

1. If $A+B+C=180^0$, Prove that

   $(i)\sin 2A-\sin 2B+\sin 2C=4\cos A\sin B\cos C.$

   $(ii)\sin 2A-\sin 2B-\sin 2C=-4\sin A\cos B\cos C.$

   $(iii)\cos 2A-\cos 2B+\cos 2C=1-4\sin A\cos B\sin C$

   $(iv)\cos 2A-\cos 2B-\cos 2C=-1-4\cos A\sin B\sin C$

2. If $A+B+C=180^0$, Prove that

   $(i)\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

   $(ii)\sin A-\sin B-\sin C=-4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

   $(iii)\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$

   $(iv)\cos A-\cos B-\cos C=1-4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

3. If $A+B+C=180^0$, Prove that

   $(i)\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$

   $(ii)\sin^2 A-\sin^2 B+\sin^2 C=2\sin A\cos B\sin C$

   $(iii)\sin^2 A-\sin^2 B-\sin^2 C=-2\cos A\sin B\sin C$

$(iv)\ \cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

$(v)\ \cos^2 A-\cos^2 B+\cos^2 C=1-2\sin A\cos B\sin C$

$(vi)\ \cos^2 A-\cos^2 B-\cos^2 C=-1+2\cos A\sin B\sin C$

4. If $A+B+C=180^0$, Prove that

$(i)\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$(ii)\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$(iii)\sin^2\frac{A}{2}-\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=-1+2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$(iv)\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

5. If $A+B+C=90^0$, Prove that

$(i)\sin^2 A+\sin^2 B+\sin^2 C=1-2\sin A\sin B\sin C$

$(ii)\sin 2A+\sin 2B+\sin 2C=4\cos A\cos B\cos C.$

6. If $A+B+C=0^0$, Prove that

$(i)\sin A+\sin B-\sin C=-4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$(ii)\ \cos^2 A+\cos^2 B+\cos^2 C=1+2\cos A\cos B\cos C$

7. If $A+B+C=\frac{3\pi}{2}$, Prove that

$(i)\cos^2 A+\cos^2 B-\cos^2 C=-2\cos A\cos B\sin C$

$(ii)\sin 2A+\sin 2B-\sin 2C=-4\sin A\sin B\cos C.$

8. If $A+B+C=2S$, Prove that

$(i)\sin(S-A)+\sin(S-B)+\sin C=4\cos\frac{S-A}{2}\cos\frac{S-B}{2}\sin\frac{C}{2}$

$(ii)(i)\cos(S-A)+\cos(S-B)+\cos C=4\cos\frac{S-A}{2}\cos\frac{S-B}{2}\cos\frac{C}{2}$

Key Concepts

1. In the Sexagesimal system1 right angle = 90^0, 1 degree = $60'$, 1 minute = $60''$.

2. In the Centisimal system1 right angle = 100^g, 1 grade = $100'$, 1 minute = $100''$.

3. The conversion from one system to the other can be easily done using the equation:

$\frac{180}{\text{D}} = \frac{200}{\text{G}} = \frac{\pi}{\text{R}}$, where D, G, R respectively denote degrees, grades and radians.

4. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ & $\text{cosec}\,\theta = \frac{1}{\sin\theta}$.,

$\cos^2\theta + \sin^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1,\ \text{cosec}^2\theta - \cot^2\theta = 1.$

5. If A, B are two real numbers then

$(i)\sin(A+B) = \sin A\cos B + \cos A\sin B,\ (ii)\sin(A-B) = \sin A\cos B - \cos A\sin B,$

$(iii)\cos(A+B) = \cos A\cos B - \sin A\sin B,\ (iv)\cos(A-B) = \cos A\cos B + \sin A\sin B.$

$$(v)\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B},\ (vi)\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B},$$

$$(vii)\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A},\ (viii)\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A},$$

$$(ix)\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A,$$

$$(x)\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

6. If A, B, C are three real numbers then

$$(i)\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C,$$

$$(ii)\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\cos C,$$

$$(iii)\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A},$$

$$(iv)\cot(A+B+C) = \frac{\cot A + \cot B + \cot C - \cot A\cot B\cot C}{1 - \cot A\cot B - \cot B\cot C - \cot C\cot A}.$$

7. $\sin 75^0 = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \cos 75^0 = \frac{\sqrt{3}-1}{2\sqrt{2}},\ \tan 75^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

8. $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}},\ \cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \tan 15^0 = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

9. $\sin 105^0 = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \cos 105^0 = \frac{\sqrt{3}-1}{2\sqrt{2}},\ \tan 105^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

10. If A is any real number, then

$$i) \sin 2A = 2\sin A\cos A,$$
$$ii) \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

11. If A is any real number, which is not an odd multiple of $\frac{\pi}{2}$ then

$$i) \sin 2A = \frac{2\tan A}{1+\tan^2 A},\ ii) \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A},$$

$$iii) \tan 2A = \frac{2\tan A}{1-\tan^2 A}, iv) \cot 2A = \frac{\cot^2 A - 1}{2\cot A}.$$

12. If $\frac{A}{2}$ is not an odd multiple of $\frac{\pi}{2}$ then

$$(i)\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}},\ (ii)\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}},$$

$$(iii)\ \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}},\ (iv)\ \cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}$$

13. If A is any real number, then

$$i) \sin 3A = 3\sin A - 4\sin^3 A, ii) \cos 3A = 4\cos^3 A - 3\cos A,$$
$$iii) \tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A} \quad (3A \text{ is not odd multiple of } \frac{\pi}{2})$$
$$iv) \cot 3A = \frac{3\cot A - \cot^3 A}{1-3\cot^2 A} \quad (3A \text{ is not an integral multiple of } \pi)$$

14. If A is any real number, then

$$i)\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}, ii)\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3},$$

$$iii)\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}}, iv)\cot A = \frac{3\cot\frac{A}{3} - \cot^3\frac{A}{3}}{1 - 3\cot^2\frac{A}{3}}.$$

15. If A is any real number, then

$$i)\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}, ii)\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}},$$

$$iii)\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}} \quad (A\, is\, not\, odd\, multiple\, of\, \frac{\pi}{2})$$

$$iv)\cot A = \pm\sqrt{\frac{1+\cos 2A}{1-\cos 2A}} \quad (A\, is\, not\, an\, integral\, multiple\, of\, \pi)$$

16. If A is any real number, then

$$i)\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, ii)\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}},$$

$$iii)\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad (A\, is\, not\, odd\, multiple\, of\, \pi)$$

$$iv)\cot\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{1-\cos A}} \quad (A\, is\, not\, an\, integer\, multiple\, of\, 2\pi)$$

17. $(i)\ \sin 18^0 = \frac{\sqrt{5}-1}{4},$ $\quad (ii)\ \cos 36^0 = \frac{\sqrt{5}+1}{4},$

$(iii)\ \sin 36^0 = \frac{\sqrt{10-2\sqrt{5}}}{4},$ $\quad (iv)\ \cos 18^0 = \frac{\sqrt{10+2\sqrt{5}}}{4}.$

18. $(i)\ \sin 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}},$ $\quad (ii)\ \cos 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}},$

$(iii)\ \tan 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{2}-1,$ $\quad (iv)\ \cot 22\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} = \sqrt{2}+1.$

19. $(i)\ \sin 67\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}},$ $\quad (ii)\ \cos 67\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}},$

$(iii)\ \tan 67\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} = \sqrt{2}+1,$ $\quad (iv)\ \cot 67\frac{1}{2}^0 = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{2}-1.$

20. For $A, B \in R$ we have

(i) $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$
(ii) $\sin(A+B)-\sin(A-B) = 2\cos A\sin B$
(iii) $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$
(iv) $\cos(A+B)-\cos(A-B) = -2\sin A\sin B$

21. For any $C, D \in R$ we have

(i) $\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C-\sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C-\cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Answers
Exercise 6(a)

(1) $(i)\ \frac{-\sqrt{3}}{2}$ (ii) 2 (iii) 0 (iv) -1 (v) 2 (vi) 1

(3) $(i)\ \frac{1}{2}$ (ii) 1 (iii) 0 (iv) 2

(6) (i) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ $(ii)\ xy=ab$ $(iii)\ \left(x^2y\right)^{2/3}-\left(xy^2\right)^{2/3}=1$

(7) (i) $-2\sqrt{2}, \frac{2\sqrt{2}}{3}$ (ii) $-\frac{3}{5}$

Exercise 6(b)

(1) $(i)\ \frac{3-\sqrt{3}}{4\sqrt{2}}$ $(ii)\ \frac{3+\sqrt{3}}{4\sqrt{2}}$ $(iii)1$ $(iv)0$ $(v)\ \frac{1}{2}$

(3) Ist Quadrant

(4) (i) $-\frac{3}{4}$ and $\frac{3}{5}$ (ii) $\frac{1-\tan\alpha}{1+\tan\alpha}$

7. TRIGONOMETRIC EQUATIONS

Introduction:

In earlier classes, we have solved the simple equations involving a single variable. Here we solve the equations involving trigonometric functions as variables.

7.1 General solution of trigonometric equations:

In this Section, we shall find the general solution of simple trigonometric equations like $\sin x = k, \cos x = k, \tan x = k$ etc.

7.1.1 Definition:

An equation consisting of the trigonometric functions of a variable angle $x \in R$ is called *trigonometric equation.*

7.1.2 Example:

The following are examples of *simple trigonometric equations.*

$$(i) \sin x = \frac{1}{\sqrt{2}}$$

$$(ii) \tan x = \sqrt{3}$$

7.1.3 Definition:

The values of the variable angle $x \in R$ satisfying the given trigonometric equation is called a *solution* of the trigonometric equation. The set of all solutions of a trigonometric equation is called a *solution set* of the trigonometric equation. The *general solution* is a functional form of the solution set.

7.1.4 Example:

The equation $\sin x = \frac{\sqrt{3}}{2}$ has a solution $x = \frac{\pi}{3}$. But $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \ldots.$ are solutions of this equation. If x is a solution of the equation then $2n\pi + x, x \in Z$ is also a solution.

Now we define the concept of the principle solution and formula for finding the general solution of *trigonometric equations*.

7.1.5 Definition:

The function $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$ by $f(x)=\sin x$ is a bijection. For each $k\in[-1,1]$ there exists a unique $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ such that $\sin x=k$. This $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called the *principal solution* of the equation $\sin x=k$.

7.1.6 Definition:

The function $f:[0,\pi]\to[-1,1]$ by $f(x)=\cos x$ is a bijection. For each $k\in[-1,1]$ there exists a unique $x\in[0,\pi]$ such that $\cos x=k$. This $x\in[0,\pi]$ is called the *principal solution* of the equation $\cos x=k$.

7.1.7 Definition:

The function $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ by $f(x)=\tan x$ is a bijection. For each $k\in R$ there exists a unique $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that $\tan x=k$. This $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is called the *principal solution* of the equation $\tan x=k$.

7.1.8 Definition:

The function $f:(0,\pi)\to R$ by $f(x)=\cot x$ is a bijection. For each $k\in R$ there exists a unique $x\in(0,\pi)$ such that $\cot x=k$. This $x\in(0,\pi)$ is called the *principal solution* of the equation $\cot x=k$.

7.1.9 Definition:

The function $f:[0,\pi]-\left\{\frac{\pi}{2}\right\}\to(-\infty,-1]\cup[1,\infty)$ by $f(x)=\sec x$ is a bijection. For each $k\in(-\infty,-1]\cup[1,\infty)$ there exists a unique $x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$ such that $\sec x=k$. This $x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$ is called the *principal solution* of the equation $\sec x=k$.

7.1.10 Definition:

The function $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}\rightarrow(-\infty,-1]\cup[1,\infty)$ by $f(x)=\operatorname{cosec} x$ is a bijection. For each $k\in(-\infty,-1]\cup[1,\infty)$ there exists a unique $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ such that $\operatorname{cosec} x=k$. This $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ is called the *principal solution* of the equation $\operatorname{cosec} x=k$.

7.1.11 General solution of the equations $\sin x=0, \cos x=0$ and $\tan x=0$:

(i) If $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ then $\sin x=0$ if and only if $x=0.$ Hence the principal solution of $\sin x=0$ is $x=0.$ Then there exists an integer k such that

$$2k\pi\leq x<2(k+1)\pi$$

That is, $0\leq x-2k\pi<2\pi$

$$\therefore x=2k\pi \ or \ x=(2k+1)\pi$$

The general solution of $\sin x=0$ is $x=n\pi, n\in Z.$

(ii) If $x\in[0,\pi]$ then $\cos x=0$ if and only if $x=\frac{\pi}{2}.$ Hence the principal solution of $\cos x=0$ is $x=\frac{\pi}{2}.$

$$\cos x=0\Leftrightarrow \sin\left(x-\frac{\pi}{2}\right)=0 \Leftrightarrow x-\frac{\pi}{2}=n\pi, n\in Z \Leftrightarrow x=n\pi+\frac{\pi}{2}, n\in Z$$

$$\therefore x=(2n+1)\frac{\pi}{2}, n\in Z$$

The general solution of $\cos x=0$ is $x=(2n+1)\frac{\pi}{2}, n\in Z$

(iii) If $\tan x=0$ then $\sin x=0$ if and only if $x=0.$ Hence the principal solution of $\tan x=0$ is $x=0.$

$$\tan x=0\Leftrightarrow \sin x=0 \Leftrightarrow x=n\pi, n\in Z$$

$\therefore x = n\pi, n \in Z$

The general solution of $\tan x = 0$ is $x = n\pi, n \in Z$

7.1.12 General solution of the equation $\sin x = k \, (-1 \le k \le 1)$**:**

Let $k \in [-1,1]$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the principal solution of $\sin x = k$.

That is $\sin x = k = \sin \alpha \quad \Leftrightarrow \sin x - \sin \alpha = 0$

$$\Leftrightarrow 2\cos\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$\Leftrightarrow \cos\left(\frac{x+\alpha}{2}\right) \quad or \quad \sin\left(\frac{x-\alpha}{2}\right) = 0$$

If $\cos\left(\frac{x+\alpha}{2}\right) = 0 \Leftrightarrow \frac{x+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in Z \quad \Leftrightarrow x + \alpha = (2n+1)\pi, n \in Z$

$$\Leftrightarrow x = (2n+1)\pi - \alpha, n \in Z$$

If $\sin\left(\frac{x-\alpha}{2}\right) = 0 \Leftrightarrow \frac{x-\alpha}{2} = n\pi, n \in Z \quad \Leftrightarrow x - \alpha = 2n\pi, n \in Z$

$$\Leftrightarrow x = 2n\pi + \alpha, n \in Z$$

Thus $x = n\pi + (-1)^n \alpha, n \in Z$ is the general solution of $\sin x = k$.

7.1.13 General solution of the equation $\cos x = k \, (-1 \le k \le 1)$**:**

Let $k \in [-1,1]$ and $\alpha \in [0, \pi]$ be the principal solution of $\cos x = k$.

That is $\cos x = k = \cos \alpha \quad \Leftrightarrow \cos x - \cos \alpha = 0$

$$\Leftrightarrow -2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right) = 0$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$\Leftrightarrow \sin\left(\frac{x+\alpha}{2}\right) \quad or \quad \sin\left(\frac{x-\alpha}{2}\right) = 0$$

If $\sin\left(\frac{x+\alpha}{2}\right)=0 \Leftrightarrow \frac{x+\alpha}{2}=n\pi, n\in Z \quad \Leftrightarrow x+\alpha=2n\pi, n\in Z$

$$\Leftrightarrow x=2n\pi-\alpha, n\in Z$$

If $\sin\left(\frac{x-\alpha}{2}\right)=0 \Leftrightarrow \frac{x-\alpha}{2}=n\pi, n\in Z \quad \Leftrightarrow x-\alpha=2n\pi, n\in Z$

$$\Leftrightarrow x=2n\pi+\alpha, n\in Z$$

Thus $x=2n\pi\pm\alpha, n\in Z$ is the general solution of $\cos x=k$.

7.1.14 General solution of the equation $\tan x=k\,(k\in R)$**:**

Let $k\in R$ and $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be the principal solution of $\tan x=k$.

That is $\tan x=k=\tan\alpha \quad \Leftrightarrow \tan x-\tan\alpha=0 \Leftrightarrow \frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}=0$

$$\Leftrightarrow \frac{\sin x\cos\alpha-\cos x\sin\alpha}{\cos x\cos\alpha}=0 \Leftrightarrow \sin x\cos\alpha-\cos x\sin\alpha=0$$

$$\Leftrightarrow \sin(x-\alpha)=0$$

If $\sin(x-\alpha)=0 \Leftrightarrow x-\alpha=n\pi, n\in Z$

$$\Leftrightarrow x=n\pi+\alpha, n\in Z$$

Thus $x=n\pi+\alpha, n\in Z$ is the general solution of $\tan x=k$.

7.1.15 General solution of the equation $\sec x=k\left(k\in(-\infty,-1]\cup[1,\infty)\right)$**:**

Let $|k|\geq 1$ and α be the principal solution of $\cos x=\frac{1}{k}$.

That is $\sec x=k \quad \Leftrightarrow \cos x=\frac{1}{k} \Leftrightarrow x=2n\pi\pm\alpha, n\in Z \quad (by\ 7.1.13)$

Thus $x=2n\pi\pm\alpha, n\in Z$ is the general solution of $\sec x=k$ where α is the principal solution of $\cos x=\frac{1}{k}$.

7.1.16 General solution of the equation $\operatorname{cosec} x=k\left(k\in(-\infty,-1]\cup[1,\infty)\right)$**:**

Let $|k| \geq 1$ and α be the principal solution of $\sin x = \frac{1}{k}$.

That is $\operatorname{cosec} x = k \quad \Leftrightarrow \sin x = \frac{1}{k} \Leftrightarrow x = n\pi + (-1)^n \alpha, n \in Z \quad (by\ 7.1.12)$

Thus $x = n\pi + (-1)^n \alpha, n \in Z$ is the general solution of $\operatorname{cosec} x = k$ where α is the principal solution of $\sin x = \frac{1}{k}$.

7.1.17 General solution of the equation $\cot x = k\,(k \in R)$**:**

We know that $\cot x = k$ has a solution for all $k \in R$.

Case (i): Let $k \in R - \{0\}$ and α be the principal solution of $\tan x = \frac{1}{k}$.

That is $\cot x = k \quad \Leftrightarrow \tan x = \frac{1}{k} \Leftrightarrow x = n\pi + \alpha, n \in Z \quad (by\ 7.1.14)$

Thus $x = n\pi + \alpha, n \in Z$ is the general solution of $\cot x = k$ where α is the principal solution of $\tan x = \frac{1}{k}$.

Case (ii): Let $k = 0$ and α be the principal solution of $\cot x = 0$

That is $\cot x = 0 \quad \Leftrightarrow \cos x = 0 \Leftrightarrow x = (2n+1)\frac{\pi}{2}, n \in Z \;\Leftrightarrow x = n\pi + \frac{\pi}{2}, n \in Z$

$$\Leftrightarrow x = n\pi + \alpha, n \in Z$$

Thus in either case $x = n\pi + \alpha, n \in Z$ is the general solution of $\cot x = k$ where α is the principal solution of $\cot x = k$

7.1.18 Solved Problems:

1. Problem: Solve $\operatorname{Sin} x = \frac{1}{\sqrt{2}}$.

Solution: Given $\operatorname{Sin} x = \frac{1}{\sqrt{2}}$.

$\Rightarrow \operatorname{Sin} x = \frac{1}{\sqrt{2}} = \operatorname{Sin} \frac{\pi}{4}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore x = \frac{\pi}{4}$ is the principal solution.

General solution is $x = n\pi + (-1)^n \frac{\pi}{4}, n \in Z$

2. Problem: Solve $\operatorname{Sin} 2x = \frac{\sqrt{5}-1}{4}$.

Solution: Given $\operatorname{Sin} 2x = \frac{\sqrt{5}-1}{4}$.

$\Rightarrow \operatorname{Sin} 2x = \frac{\sqrt{5}-1}{4} = \operatorname{Sin}\frac{\pi}{10}$ and $\frac{\pi}{10} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow 2x = \frac{\pi}{10}$

$\therefore x = \frac{\pi}{20}$ is the principal solution.

General solution is $2x = n\pi + (-1)^n \frac{\pi}{10}, n \in Z$

$\Rightarrow x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{20}, n \in Z$

3. Problem: Solve $3\operatorname{cosec} x = 4\sin x$.

Solution: Given $3\operatorname{cosec} x = 4\sin x$

$\Rightarrow \frac{3}{\sin x} = 4\sin x \quad \Rightarrow 4\sin^2 x = 3 \Rightarrow \sin^2 x = \frac{3}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow x = \pm\frac{\pi}{3}$

$\therefore$ Principal solutions are $x = \pm\frac{\pi}{3}$

General solution is $x = n\pi \pm \frac{\pi}{3}, n \in Z$

4. Problem: Solve $\tan^2 x = 3$.

Solution: Given $\tan^2 x = 3$

$\Rightarrow \tan x = \pm\sqrt{3}$

$\Rightarrow x = \pm\dfrac{\pi}{3}$

$\therefore$ Principal solutions are $x = \pm\dfrac{\pi}{3}$

General solution is $x = n\pi \pm \dfrac{\pi}{3}, n \in Z$

5. Problem: Solve $\cos 3x = \sin 2x$.

Solution: Given $\cos 3x = \sin 2x$

$\Rightarrow \cos 3x = \cos\left(\dfrac{\pi}{2} - 2x\right) \qquad \left[\because \sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)\right]$

$\Rightarrow 3x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right) \qquad [\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$

$\Rightarrow 5x = 2n\pi + \dfrac{\pi}{2}$ *or* $x = 2n\pi - \dfrac{\pi}{2}$

$\Rightarrow x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}$ *or* $x = 2n\pi - \dfrac{\pi}{2}$

General solution is $x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}$ *or* $x = 2n\pi - \dfrac{\pi}{2}, n \in Z$

6. Problem: Solve $7\sin^2 x + 3\cos^2 x = 4$.

Solution: Given $7\sin^2 x + 3\cos^2 x = 4$

$\Rightarrow 7\sin^2 x + 3(1 - \sin^2 x) = 4$

$\Rightarrow 4\sin^2 x = 1$

$\Rightarrow \sin x = \pm\dfrac{1}{2}$

$\Rightarrow x = \pm\dfrac{\pi}{6}$

$\therefore$ Principal solutions are $x = \pm\dfrac{\pi}{6}$

General solution is $x = n\pi \pm \frac{\pi}{6}, n \in Z$

7. Problem: Solve $2\cos^2 x - \sqrt{3}\sin x + 1 = 0$.

Solution: Given $2\cos^2 x - \sqrt{3}\sin x + 1 = 0$

$\Rightarrow 2(1 - \sin^2 x) - \sqrt{3}\sin x + 1 = 0$

$\Rightarrow 2\sin^2 x + \sqrt{3}\sin x - 3 = 0$

$\Rightarrow (\sin x + \sqrt{3})(2\sin x - \sqrt{3}) = 0$

$\Rightarrow \sin x + \sqrt{3} = 0 \quad or \quad 2\sin x - \sqrt{3} = 0$

$\Rightarrow \sin x = -\sqrt{3} \quad or \quad \sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2} \left[\because \sin x \neq -\sqrt{3}\right]$

$\Rightarrow \sin x = \sin\frac{\pi}{3}$

$\therefore$ Principal solutions is $x = \frac{\pi}{3}$

General solution is $x = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$

8. Problem: Solve $4\cos^2 x + \sqrt{3} = 2(\sqrt{3} + 1)\cos x$.

Solution: Given $4\cos^2 x + \sqrt{3} = 2(\sqrt{3} + 1)\cos x$

$\Rightarrow 4\cos^2 x - 2\sqrt{3}\cos x - 2\cos x + \sqrt{3} = 0$

$\Rightarrow (2\cos x - \sqrt{3})(2\cos x - 1) = 0$

$\Rightarrow 2\cos x - 1 = 0 \quad or \quad 2\cos x - \sqrt{3} = 0$

$\Rightarrow \cos x = \frac{1}{2} \quad or \quad \cos x = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos x = \cos\frac{\pi}{3} \quad or \quad \cos x = \cos\frac{\pi}{6}$

$\therefore$ Principal solutions is $x = \frac{\pi}{6} or \frac{\pi}{3}$

General solution is $x = 2n\pi \pm \frac{\pi}{6}$ or $x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

9. Problem: Solve $\tan x + 3\cot x = 5\sec x$.

Solution: Given $\tan x + 3\cot x = 5\sec x$

$$\Rightarrow \frac{\sin x}{\cos x} + 3\frac{\cos x}{\sin x} = \frac{5}{\cos x}$$

$$\Rightarrow \sin^2 x + 3\cos^2 x = 5\sin x \Rightarrow \sin^2 x + 3(1-\sin^2 x) = 5\sin x$$

$$\Rightarrow 2\sin^2 x + 5\sin x - 3 = 0 \Rightarrow (\sin x + 3)(2\sin x - 1) = 0$$

$$\Rightarrow \sin x + 3 = 0 \quad or \quad 2\sin x - 1 = 0 \Rightarrow \sin x = -3 \quad or \quad \sin x = \frac{1}{2}$$

$$\Rightarrow \sin x = \frac{1}{2} \; [\because \sin x \neq -3 \;] \Rightarrow \sin x = \sin\frac{\pi}{6}$$

$\therefore$ Principal solutions is $x = \frac{\pi}{6}$

General solution is $x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

10. Problem: Solve $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0$.

Solution: Given $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0$

$$\Rightarrow \cot^2 x - \sqrt{3}\cot x - \cot x + \sqrt{3} = 0$$

$$\Rightarrow (\cot x - \sqrt{3})(\cot x - 1) = 0 \Rightarrow \cot x - 1 = 0 \quad or \quad \cot x - \sqrt{3} = 0$$

$$\Rightarrow \cot x = 1 \quad or \quad \cot x = \sqrt{3} \Rightarrow \tan x = 1 \quad or \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan x = \tan\frac{\pi}{4} \quad or \quad \tan x = \tan\frac{\pi}{6}$$

$\therefore$ Principal solutions is $x = \frac{\pi}{6} or \frac{\pi}{4}$

General solution is $x = n\pi + \frac{\pi}{6}$ or $x = n\pi + \frac{\pi}{4}, n \in Z$

11. Problem: Solve $1 + \sin^2 x = 3\sin x \cos x.$

Solution: Given $1 + \sin^2 x = 3\sin x \cos x$

Dividing on both sides with $\cos^2 x$ we get

$$\Rightarrow \frac{1 + \sin^2 x}{\cos^2 x} = \frac{3\sin x \cos x}{\cos^2 x} \Rightarrow \sec^2 x + \tan^2 x = 3\tan x$$

$$\Rightarrow 2\tan^2 x - 3\tan x + 1 = 0 \Rightarrow (2\tan x - 1)(\tan x - 1) = 0$$

$$\Rightarrow 2\tan x - 1 = 0 \quad or \quad \tan x - 1 = 0 \Rightarrow \tan x = 1$$

$\therefore$ Principal solution is $x = \frac{\pi}{4}$

General solution is $x = n\pi + \frac{\pi}{4}, n \in Z$

Let α be the principal solution of $\tan x = \frac{1}{2}$

General solution is $x = n\pi + \alpha, n \in Z$

12. Problem: Solve $\sin 5x + \sin x = \sin 3x.$

Solution: Given $\sin 5x + \sin x = \sin 3x$

$$\Rightarrow 2\sin\left(\frac{5x + x}{2}\right)\cos\left(\frac{5x - x}{2}\right) = \sin 3x \quad \left[\because \sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)\right]$$

$$\Rightarrow 2\sin 3x \cos 2x = \sin 3x \quad \Rightarrow \sin 3x\left(2\cos 2x - 1\right) = 0$$

$$\Rightarrow \sin 3x = 0 \quad or \quad 2\cos 2x - 1 = 0$$

$$\Rightarrow \sin 3x = 0 \quad or \quad \cos 2x = \frac{1}{2}$$

$$\Rightarrow \sin 3x = \sin 0 \quad or \quad \cos 2x = \cos\frac{\pi}{3}$$

General solution is $3x = n\pi \ or \ 2x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

$$\Rightarrow x = \frac{n\pi}{3} \ or \ x = n\pi \pm \frac{\pi}{6}, n \in Z$$

13. Problem: Solve $\cos 8x + \cos 2x = \cos 5x.$

Solution: Given $\cos 8x + \cos 2x = \cos 5x$

$$\Rightarrow 2\cos\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right) = \cos 5x$$

$$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$\Rightarrow 2\cos 5x \cos 3x = \cos 5x \quad \Rightarrow \cos 5x\left(2\cos 3x - 1\right) = 0$$

$$\Rightarrow \cos 5x = 0 \quad or \quad 2\cos 3x - 1 = 0 \Rightarrow \cos 5x = 0 \quad or \quad \cos 3x = \frac{1}{2}$$

$$\Rightarrow \cos 5x = \cos\frac{\pi}{2} \quad or \quad \cos 3x = \cos\frac{\pi}{3}$$

General solution is $5x = 2n\pi \pm \frac{\pi}{2} \ or \ 3x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

$$\Rightarrow x = \frac{2n\pi}{5} \pm \frac{\pi}{10} \ or \ x = \frac{2n\pi}{3} \pm \frac{\pi}{9}, n \in Z$$

14. Problem: Solve $\cos x \cos 2x \cos 3x = \frac{1}{4}.$

Solution: Given $\cos x \cos 2x \cos 3x = \frac{1}{4}$

$$\Rightarrow 4\cos x \cos 2x \cos 3x = 1$$

$$\Rightarrow 2\left(2\cos 3x \cos x\right)\cos 2x = 1$$

$$\Rightarrow 2\left(\cos\left(3x + x\right) + \cos\left(3x - x\right)\right)\cos 2x = 1$$

$$\left[\because 2\cos A \cos B = \cos\left(A+B\right) + \cos\left(A-B\right)\right]$$

$$\Rightarrow 2\cos 4x \cos 2x + 2\cos^2 2x - 1 = 0$$

$$\Rightarrow 2\cos 4x \cos 2x + \cos 4x = 0\left[\because 2\cos^2 A - 1 = \cos 2A\right]$$

$$\Rightarrow \cos 4x(2\cos 2x+1)=0$$

$$\Rightarrow \cos 4x=0 \quad or \quad 2\cos 2x+1=0 \Rightarrow \cos 4x=0 \quad or \quad \cos 2x=-\frac{1}{2}$$

$$\Rightarrow \cos 4x=\cos\frac{\pi}{2} \quad or \quad \cos 2x=\cos\frac{2\pi}{3}$$

General solution is $4x=2n\pi\pm\frac{\pi}{2}$ or $2x=2n\pi\pm\frac{2\pi}{3}, n\in Z$

$$\Rightarrow x=\frac{n\pi}{2}\pm\frac{\pi}{8} \; or \; x=n\pi\pm\frac{\pi}{3}, n\in Z$$

15. Problem: Solve $\sqrt{3}\cos x+\sin x=\sqrt{2}$.

Solution: Given $\sqrt{3}\cos x+\sin x=\sqrt{2}$

Dividing on both sides with $\sqrt{(\sqrt{3})^2+1^2}=\sqrt{3+1}=\sqrt{4}=2$ we get

$$\Rightarrow \frac{\sqrt{3}\cos x+\sin x}{2}=\frac{\sqrt{2}}{2}$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos x\cos\frac{\pi}{6}+\sin x\sin\frac{\pi}{6}=\frac{1}{\sqrt{2}} \Rightarrow \cos\left(x-\frac{\pi}{6}\right)=\cos\frac{\pi}{4}$$

General solution is $x-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}, n\in Z$

$$\Rightarrow x=2n\pi+\frac{5\pi}{12}, x=2n\pi-\frac{\pi}{12}, n\in Z$$

16. Problem: Solve $8^{1+\cos x+\cos^2 x+\cos^3 x+...\infty}=4^3$ for all $x\in(-\pi,\pi)$

Solution: Given $8^{1+\cos x+\cos^2 x+\cos^3 x+...\infty}=4^3$

For $x=0$ the given equation has no solution.

For $x\neq 0$ we have $|\cos x|<1$

Then $1+\cos x+\cos^2 x+\cos^3 x+...\infty=\frac{1}{1-\cos x}$

Now $8^{1+\cos x+\cos^2 x+\cos^3 x+\ldots\infty} = 4^3 \quad \Rightarrow 2^{3(1+\cos x+\cos^2 x+\cos^3 x+\ldots\infty)} = (2^2)^3$

$$\Rightarrow 2^{3(\frac{1}{1-\cos x})} = 2^6 \Rightarrow \frac{3}{1-\cos x} = 6 \Rightarrow 1-\cos x = \frac{1}{2} \Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{3}, -\frac{\pi}{3}$$

17. Problem: If $a\cos 2\theta + b\sin 2\theta = c$ then prove that

$$\tan\theta_1 + \tan\theta_2 = \frac{2b}{c+a}, \tan\theta_1 . \tan\theta_2 = \frac{c-a}{c+a}.$$

Solution: Given $a\cos 2\theta + b\sin 2\theta = c$

We have $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}, \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Now $a\cos 2\theta + b\sin 2\theta = c \quad \Rightarrow a\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = c$

$$\Rightarrow a\left(1-\tan^2\theta\right) + b\left(2\tan\theta\right) = c\left(1+\tan^2\theta\right)$$

$$\Rightarrow \left(c+a\right)\tan^2\theta - 2b\tan\theta + \left(c-a\right) = 0$$

This is a quadratic equation in $\tan\theta$ and $\tan\theta_1, \tan\theta_2$ are solutions then we get

$$\tan\theta_1 + \tan\theta_2 = \frac{2b}{c+a}, \tan\theta_1 . \tan\theta_2 = \frac{c-a}{c+a}.$$

Exercise 7

I 1. Find the principle solution of the following equations:

$(i)\, 2\cos^2 x = 1 \qquad (ii) 3\cot^2 x = 1 \qquad (iii) \sqrt{3}\sec x + 2 = 0 \qquad (iv) \cos 2x = \dfrac{\sqrt{5}+1}{4}$

$(v) \sin 3x = \dfrac{\sqrt{3}}{2} \qquad (vi) \cos^2 x = \dfrac{3}{4}$

2. Find the general solution of the following equations:

$(i)\, 2\sin^2 x = 3\cos x \qquad (ii) \sin^2 x - \cos x = \dfrac{1}{4} \qquad (iii) 7\sin^2 x + 5\cos^2 x = 6$

$(iv)\, 2\sin^2 x - 4 = 5\cos x \qquad (v)\, 2\sin^2 x + \sin^2 2x = 2$

II 1. Solve the following equations:

$(i)\ \sqrt{3}\sin x - \cos x = \sqrt{2}$ $\quad (ii)\ \cot x + \operatorname{cosec} x = \sqrt{3}$ $\quad (iii)\ \tan x + \sec x = \sqrt{3}$

2. Solve the following equations:

$(i)\ \sin 2x - \cos 2x = \sin x - \cos x$ $\quad (ii)\ 4\sin x \sin 2x \sin 4x = \sin 3x$

$(iii)\ \sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

$(iv)\ \cos 3x - \cos 4x = \cos 5x - \cos 6x$

Key Concepts

1. An equation consisting of the trigonometric functions of a variable angle $x \in R$ is called *trigonometric equation.* The values of the variable angle $x \in R$ satisfying the given trigonometric equation is called a *solution* of the trigonometric equation. The set of all solutions of a trigonometric equation is called a *solution set* of the trigonometric equation. The *general solution* is a functional form of the solution set.

2. The function $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ by $f(x) = \sin x$ is a bijection then $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal solution* of the equation $\sin x = k$.

3. The function $f: [0,\pi] \to [-1,1]$ by $f(x) = \cos x$ is a bijection then $x \in [0,\pi]$ is called the *principal solution* of the equation $\cos x = k$.

4. The function $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ by $f(x) = \tan x$ is a bijection then $x \in [0,\pi]$ is called the *principal solution* of the equation $\tan x = k$.

5. The function $f: (0,\pi) \to R$ by $f(x) = \cot x$ is a bijection then $x \in (0,\pi)$ is called the *principal solution* of the equation $\cot x = k$.

6. The function $f: [0,\pi] - \left\{\frac{\pi}{2}\right\} \to (-\infty,-1] \cup [1,\infty)$ by $f(x) = \sec x$ is a bijection then $x \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$ is called the *principal solution* of the equation $\sec x = k$.

7. The function $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to (-\infty,-1] \cup [1,\infty)$ by $f(x) = \operatorname{cosec} x$ is a bijection then $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal solution* of the equation $\operatorname{cosec} x = k$.

8. The general solution of $\sin x = 0$ is $x = n\pi, n \in Z$.

9. The general solution of $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}, n \in Z$

10. The general solution of $\tan x = 0$ is $x = n\pi, n \in Z$

11. The general solution of $\sin x = k$ is $x = n\pi + (-1)^n \alpha, n \in Z$

12. The general solution of $\cos x = k$ is $x = 2n\pi \pm \alpha, n \in Z$

13. The general solution of $\tan x = k$ is $x = n\pi + \alpha, n \in Z$

14. The general solution of $\sec x = k$ is $x = 2n\pi \pm \alpha, n \in Z$ the where α is the principal solution of $\cos x = \frac{1}{k}$.

15. The general solution of $\operatorname{cosec} x = k$ is $x = n\pi + (-1)^n \alpha, n \in Z$ the where α is the principal solution of $\sin x = \frac{1}{k}$.

16. The general solution of $\cot x = k$ is $x = n\pi + \alpha, n \in Z$ the where α is the principal solution of $\tan x = \frac{1}{k}$.

Answers

Exercise 7

I

(1) $(i) 45^0, 135^0$ $(ii) \pm 60^0$ $(iii) 150^0$ $(iv) 18^0$ $(v) 20^0, 40^0, 140^0, 160^0$ $(vi) 30^0, 150^0$

(2) $(i)\, x = 2n\pi \pm \frac{\pi}{3}, n \in Z$ $(ii) x = 2n\pi \pm \frac{\pi}{3}, n \in Z$ $(iii) x = n\pi \pm \frac{\pi}{4}, n \in Z$

$(iv) x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$ $(v) x = (2n+1)\frac{\pi}{2}, n\pi \pm \frac{\pi}{4}, n \in Z$

II

(1) $(i)\, x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{6}, n \in Z$ $(ii) x = 2n\pi + \frac{\pi}{3}, n \in Z$ $(iii) x = \frac{\pi}{6}$

(2) $(i)\, x = 2n\pi, \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$ $(ii) x = \frac{n\pi}{3} \pm \frac{\pi}{9}, n \in Z$ $(iii) x = 2n\pi \pm \frac{2\pi}{3}, \frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$

$(iv) x = (2n+1)\frac{\pi}{2}, n\pi - \frac{\pi}{4}, 2n\pi + \frac{\pi}{2}, 2n\pi,\ n \in Z$

8. HYPERBOLIC FUNCTIONS

In this chapter, we discuss definitions of hyperbolic functions, definitions of inverse hyperbolic functions and addition and subtraction formulas of hyperbolic functions.

8.1 Definitions of hyperbolic functions:

8.1.1 Definition: The function $f : R \to R$ is defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$ is called *hyperbolic sine function.* It is denoted by $\sinh x$. *i.e* $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

8.1.2 Definition: The function $f : R \to R$ is defined by $f(x) = \dfrac{e^x + e^{-x}}{2}$ is called *hyperbolic cosine function.* It is denoted by $\cosh x$. *i.e* $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

8.1.3 Definition: The function $f : R \to R$ is defined by $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is called *hyperbolic tangent function.* It is denoted by $\tanh x$. *i.e* $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

8.1.4 Definition: The function $f : R - \{0\} \to R$ is defined by $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ is called *hyperbolic cotangent function.* It is denoted by $\coth x$. *i.e* $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

8.1.5 Definition: The function $f : R \to R$ is defined by $f(x) = \dfrac{2}{e^x + e^{-x}}$ is called *hyperbolic secant function.* It is denoted by $\operatorname{sech} x$. *i.e* $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$.

8.1.6 Definition: The function $f : R - \{0\} \to R$ is defined by $f(x) = \dfrac{2}{e^x - e^{-x}}$ is called *hyperbolic cosecant function.* It is denoted by $\operatorname{cosech} x$. *i.e* $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$.

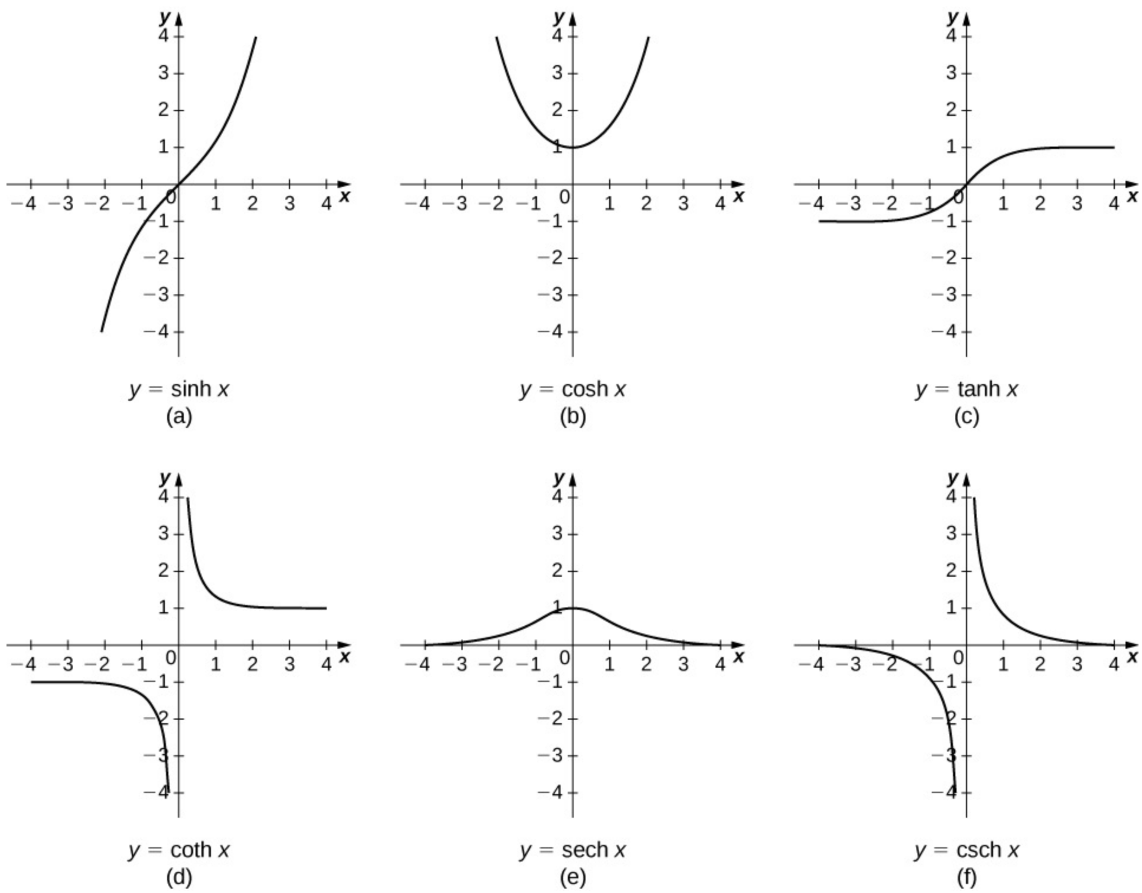
8.1.7 Domain and Range of hyperbolic functions:

The domain and ranges of the hyperbolic trigonometric functions are as follows.

Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1,\infty)$
$\tanh x$	R	$(-1,1)$
$\coth x$	$R-\{0\}$	$(-\infty,-1)\cup(1,\infty)$
$\text{sech}\, x$	R	$(0,1]$
$\text{cosech}\, x$	$R-\{0\}$	$R-\{0\}$

8.1.8 Graphs of hyperbolic functions:

The graphs of the hyperbolic trigonometric functions is as follows.



$y = \sinh x$ (a) $y = \cosh x$ (b) $y = \tanh x$ (c)

$y = \coth x$ (d) $y = \text{sech}\, x$ (e) $y = \text{csch}\, x$ (f)

8.1.9 Identities of hyperbolic trigonometric functions:

$i)\cosh^2 x-\sinh^2 x=1 \;\; for\, all\, x\in R$

$ii)1-\tanh^2 x=\text{sech}^2 x \; for\, all\, x\in R$

$iii)\coth^2 x-1=\text{cosech}^2 x \; for\, all\, x\in R-\{0\}.$

8.1.10 Note :

1. For any $x \in R$

   $i)\sinh(-x) = -\sinh x,\quad ii)\cosh(-x) = \cosh x,\ iii)\tanh(-x) = -\tanh x$
   $iv)\coth(-x) = -\coth x,\ \ v)\operatorname{sech}(-x) = \operatorname{sech} x,\ vi)\operatorname{cosech}(-x) = -\operatorname{cosech} x.$

2. $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0,$

3. $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1,$

4. $\tanh 0 = \dfrac{e^0 - e^{-0}}{e^0 + e^{-0}} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0.$

8.1.11 Formulas of hyperbolic trigonometric functions:

In the following we give formulae to evaluate $\sinh(x \pm y), \cosh(x \pm y), \tanh(x \pm y),$ $\coth(x \pm y), \sinh 2x, \cosh 2x, \tanh 2x$ and $\coth 2x$ as in hyperbolic trigonometric functions.

8.1.11(a) Theorem:

For any $x, y \in R$

$i)\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,\ ii)\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y,$
$iii)\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, iv)\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y.$

8.1.11(b) Theorem:

For any $x, y \in R$

$$i)\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y},\ ii)\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y},$$
$$iii)\coth(x+y) = \frac{\coth x \coth y + 1}{\coth y + \coth x}, iv)\coth(x-y) = \frac{\coth x \coth y - 1}{\coth y - \coth x}.$$

8.1.11(c) Theorem:

For any $x \in R$

$$i)\sinh 2x = 2\sinh x \cosh x,\ ii)\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$iii)\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}, iv)\coth 2x = \frac{\coth^2 x + 1}{2\coth x}.$$

8.2 Definitions of Inverse hyperbolic functions:

8.2.1 Definition: Let the function $f: R \to R$ is defined by $f(x) = \sinh x$ be a bijective function. The inverse function of f i.e $f^{-1}: R \to R$ is also a bijective function and it is called *inverse hyperbolic sine function.* It is denoted by $\sinh^{-1} x$.

8.2.2 Definition: Let the function $f: [0,\infty) \to [1,\infty)$ is defined by $f(x) = \cosh x$ be a bijective function. The inverse function of f i.e $f^{-1}: [1,\infty) \to [0,\infty)$ is also a bijective function and it is called *inverse hyperbolic cosine function.* It is denoted by $\cosh^{-1} x$.

8.2.3 Definition: Let the function $f: R \to (-1,1)$ is defined by $f(x) = \tanh x$ be a bijective function. The inverse function of f i.e $f^{-1}: (-1,1) \to R$ is also a bijective function and it is called *inverse hyperbolic tangent function.* It is denoted by $\tanh^{-1} x$.

8.2.4 Definition: Let the function $f: R - \{0\} \to (-\infty,-1) \cup (1,\infty)$ is defined by $f(x) = \coth x$ be a bijective function. The inverse function of f i.e $f^{-1}: (-\infty,-1) \cup (1,\infty) \to R - \{0\}$ is also a bijective function and it is called *inverse hyperbolic cotangent function.* It is denoted by $\coth^{-1} x$.

8.2.5 Definition: Let the function $f: [0,\infty) \to (0,1]$ is defined by $f(x) = \text{sech}\, x$ be a bijective function. The inverse function of f i.e $f^{-1}: (0,1] \cup (1,\infty) \to [0,\infty)$ is also a bijective function and it is called *inverse hyperbolic secant function.* It is denoted by $\text{sech}^{-1} x$.

8.2.6 Definition: Let the function $f: R - \{0\} \to R - \{0\}$ is defined by $f(x) = \text{cosech}\, x$ be a bijective function. The inverse function of f i.e $f^{-1}: R - \{0\} \to R - \{0\}$ is also a bijective function and it is called *inverse hyperbolic cosecant function.* It is denoted by $\text{cosech}^{-1} x$.

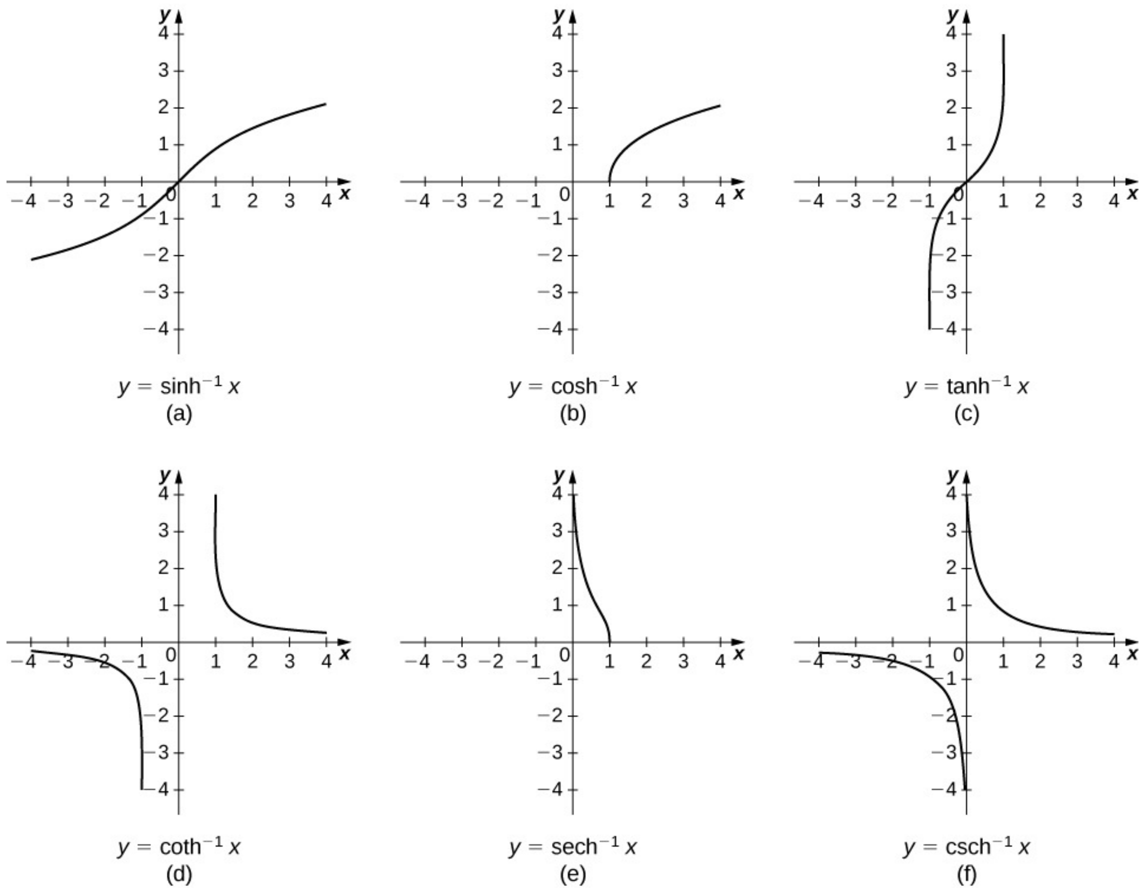
8.2.7 Domain and Range of Inverse hyperbolic functions:

The domain and ranges of the inverse hyperbolic trigonometric functions is as follows.

Function	Domain	Range
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\tanh^{-1} x$	$(-1,1)$	R
$\coth^{-1} x$	$(-\infty,-1) \cup (1,\infty)$	$R - \{0\}$
$\text{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\text{cosech}^{-1} x$	$R - \{0\}$	$R - \{0\}$

8.2.8 Graphs of Inverse hyperbolic functions:

The graphs of the inverse hyperbolic trigonometric functions is as follows.



8.2.9 Formulas of inverse hyperbolic trigonometric functions:

In the following we give formulae to evaluate $\sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x,\ \coth^{-1} x,$ $\text{sech}^{-1} x$ and $\cos\text{ech}^{-1} x$.

8.2.9(a) Theorem:

For any $x \in R$

$i) \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}),$

$ii) \cosh x = \log(x + \sqrt{x^2 - 1})$ for $x \geq 1,$

$iii) \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ for $x \in (-1,1),$

$iv)\ \coth^{-1} x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) \text{for} |x| > 1,$

$v)\ \text{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) \text{for } x \in (0,1],$

$$vi)\ \text{cosec h}^{-1} x = \begin{cases} \log\left(\frac{1+\sqrt{1+x^2}}{x}\right) \text{for } x > 0 \\ \log\left(\frac{1-\sqrt{1+x^2}}{x}\right) \text{for } x < 0. \end{cases}$$

8.2.10 Solved Problems:

1. Problem: If $\sinh x = 3$, Prove that $x = \log(3+\sqrt{10})$.

Solution: Given that $\sinh x = 3$

To prove that $x = \log(3+\sqrt{10})$

since $\sinh x = 3 \Rightarrow x = \sinh^{-1} 3$

we have $\sinh^{-1} x = \log(x+\sqrt{x^2+1})$

Put $x = 3$

$\therefore \sinh^{-1} 3 = \log(3+\sqrt{3^2+1})$
$\Rightarrow \sinh^{-1} 3 = \log(3+\sqrt{10})$

$\therefore x = \log(3+\sqrt{10})$

2. Problem: If $\sinh x = \frac{3}{4}$, Prove that $\cosh 2x = \frac{17}{8}$ and $\sinh 2x = \frac{15}{8}$.

Solution: Given that $\sinh x = \frac{3}{4}$

To prove that $\cosh 2x = \frac{17}{8}$ and $\sinh 2x = \frac{15}{8}$.

we have $\cosh 2x = 1 + 2\sinh^2 x$

$$\Rightarrow \cosh 2x = 1 + 2\left(\frac{3}{4}\right)^2 \Rightarrow \cosh 2x = 1 + 2\left(\frac{9}{16}\right) \Rightarrow \cosh 2x = 1 + \frac{18}{16} \Rightarrow \cosh 2x = 1 + \frac{9}{8}$$

$\therefore \cosh 2x = \frac{17}{8}$

we have $\cosh^2 2x - \sinh^2 2x = 1$

$\Rightarrow \sinh 2x = \sqrt{\cosh^2 2x - 1} \Rightarrow \sinh 2x = \sqrt{\left(\frac{17}{8}\right)^2 - 1} \Rightarrow \sinh 2x = \sqrt{\left(\frac{289}{64}\right) - 1}$

$\Rightarrow \sinh 2x = \sqrt{\frac{289-64}{64}} \Rightarrow \sinh 2x = \sqrt{\frac{225}{64}} \Rightarrow \sinh 2x = \frac{15}{8}$

$\therefore \sinh 2x = \frac{15}{8}$

3. Problem: Prove that $\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log 3.$

Solution: We have $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ for $x \in (-1,1)$

Put $x = \frac{1}{2}$

$$\tanh^{-1}\frac{1}{2} = \frac{1}{2}\log\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$\Rightarrow \tanh^{-1}\frac{1}{2} = \frac{1}{2}\log\left(\frac{3/2}{1/2}\right) \Rightarrow \tanh^{-1}\frac{1}{2} = \frac{1}{2}\log 3$$

$\therefore \tanh^{-1}\frac{1}{2} = \frac{1}{2}\log 3$

4. Problem: Find the value of $\cosh 2 + \sinh 2$

Solution: We have $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \Rightarrow \cosh x + \sinh x = \frac{e^x + e^{-x} + e^x - e^{-x}}{2}$$

$\Rightarrow \cosh x + \sinh x = e^x$

Put $x = 2$ we get $\cosh 2 + \sinh 2 = e^2$

5. Problem: If $\cosh^{-1} x = \log(2+\sqrt{3})$ then find the value of x

Solution: We have $\cosh^{-1} x = \log(x+\sqrt{x^2-1})$

$$\log(2+\sqrt{3}) = \log(2+\sqrt{2^2-1}) = \cosh^{-1} 2$$

$$\therefore x = 2$$

6. Problem: If $x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ then prove that $\cosh x = \sec\theta$.

Solution: Given $x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

To prove $\cosh x = \sec\theta$

Since $x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \Rightarrow e^x = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

and $e^{-x} = \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$

Now $e^x + e^{-x} = \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$

$$\Rightarrow e^x + e^{-x} = \frac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{\left(1+\tan\frac{\theta}{2}\right)\left(1-\tan\frac{\theta}{2}\right)}$$

$$\Rightarrow e^x + e^{-x} = \frac{2\left(1+\tan^2\frac{\theta}{2}\right)}{\left(1-\tan^2\frac{\theta}{2}\right)} \Rightarrow \frac{e^x+e^{-x}}{2} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \Rightarrow \cosh x = \sec\theta$$

$$\therefore \cosh x = \sec\theta$$

7. Problem: For any $x, y \in R$ prove that

$i)\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y,\ ii)\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y,$
$iii)\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y, iv)\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y.$

Solution: (i) We have $\cosh x=\dfrac{e^x+e^{-x}}{2}$ $,\sinh x=\dfrac{e^x-e^{-x}}{2}.$

$\text{R.H.S}=\sinh x\cosh y+\cosh x\sinh y$

$$=\frac{e^x-e^{-x}}{2}.\frac{e^y+e^{-y}}{2}+\frac{e^x+e^{-x}}{2}.\frac{e^y-e^{-y}}{2}$$

$$=\frac{1}{4}\left(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}+e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}\right)$$

$$=\frac{2\left(e^{x+y}-e^{-x-y}\right)}{4}$$

$$=\frac{\left(e^{x+y}-e^{-x-y}\right)}{2}=\sinh\left(x+y\right)=\text{L.H.S}$$

$\therefore\sinh\left(x+y\right)=\sinh x\cosh y+\cosh x\sinh y$

(ii) We have $\cosh x=\dfrac{e^x+e^{-x}}{2}$ $,\sinh x=\dfrac{e^x-e^{-x}}{2}.$

$\text{R.H.S}=\sinh x\cosh y-\cosh x\sinh y$

$$=\frac{e^x-e^{-x}}{2}.\frac{e^y+e^{-y}}{2}-\frac{e^x+e^{-x}}{2}.\frac{e^y-e^{-y}}{2}$$

$$=\frac{1}{4}\left(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}-e^{x+y}+e^{x-y}-e^{-x+y}+e^{-x-y}\right)$$

$$=\frac{2\left(e^{x-y}-e^{-x+y}\right)}{4}$$

$$=\frac{\left(e^{x-y}+e^{-x+y}\right)}{2}=\sinh\left(x-y\right)=\text{L.H.S}$$

$\therefore\sinh\left(x-y\right)=\sinh x\cosh y-\cosh x\sinh y$

(iii) We have $\cosh x=\dfrac{e^x+e^{-x}}{2}$ $,\sinh x=\dfrac{e^x-e^{-x}}{2}.$

$\text{R.H.S} = \cosh x \cosh y + \sinh x \sinh y$

$$= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right)$$

$$= \frac{2}{4}\left(e^{x+y} + e^{-x-y}\right)$$

$$= \frac{\left(e^{x+y} + e^{-x-y}\right)}{2} = \cosh\left(x+y\right) = \text{L.H.S}$$

$\therefore \cosh\left(x+y\right) = \cosh x \cosh y + \sinh x \sinh y$

(iv) We have $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

$\text{R.H.S} = \cosh x \cosh y - \sinh x \sinh y$

$$= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} - \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} - e^{x+y} + e^{x-y} + e^{-x+y} - e^{-x-y}\right)$$

$$= \frac{2}{4}\left(e^{x-y} + e^{-x+y}\right)$$

$$= \frac{\left(e^{x-y} + e^{-x+y}\right)}{2} = \cosh\left(x-y\right) = \text{L.H.S}$$

$\therefore \cosh\left(x-y\right) = \cosh x \cosh y - \sinh x \sinh y$

8. Problem: For any $x, y \in R$ prove that

$$i)\ \tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y},\quad ii)\ \tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y},$$

$$iii)\ \coth(x+y) = \frac{\coth x \coth y + 1}{\coth y + \coth x},\quad iv)\ \coth(x-y) = \frac{\coth x \coth y - 1}{\coth y - \coth x}.$$

Solution: (i) We have $\tanh x = \dfrac{\sinh x}{\cosh x}$

$$\text{L.H.S} = \tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)}$$

$$= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$$

Dividing the numerator and denominator with $\cosh x \cosh y$ we get

$$= \frac{\dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y + \sinh x \sinh y}{\cosh x \cosh y}}$$

$$= \frac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$$

$$= \frac{\dfrac{\sinh x}{\cosh x} + \dfrac{\sinh y}{\cosh y}}{1 + \dfrac{\sinh x}{\cosh x} . \dfrac{\sinh y}{\cosh y}} = \frac{\tanh x + \tanh y}{1 + \tanh x . \tanh y} = \text{R.H.S}$$

$$\therefore \tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

(ii) We have $\tanh x = \dfrac{\sinh x}{\cosh x}$

$$\text{L.H.S} = \tanh(x-y) = \frac{\sinh(x-y)}{\cosh(x-y)}$$

$$= \frac{\sinh x \cosh y - \cosh x \sinh y}{\cosh x \cosh y - \sinh x \sinh y}$$

Dividing the numerator and denominator with $\cosh x \cosh y$ we get

$$= \frac{\dfrac{\sinh x \cosh y - \cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y - \sinh x \sinh y}{\cosh x \cosh y}}$$

$$= \frac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} - \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} - \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$$

$$=\frac{\frac{\sinh x}{\cosh x}-\frac{\sinh y}{\cosh y}}{1-\frac{\sinh x}{\cosh x}.\frac{\sinh y}{\cosh y}}=\frac{\tanh x-\tanh y}{1-\tanh x.\tanh y}=\text{R.H.S}$$

$$\therefore \tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}$$

(iii) We have $\coth x=\frac{\cosh x}{\sinh x}$

$$\text{L.H.S}=\coth(x+y)=\frac{\cosh(x+y)}{\sinh(x+y)}$$

$$=\frac{\cosh x\cosh y+\sinh x\sinh y}{\sinh x\cosh y+\cosh x\sinh y}$$

Dividing the numerator and denominator with $\sinh x\sinh y$ we get

$$=\frac{\frac{\cosh x\cosh y+\sinh x\sinh y}{\sinh x\sinh y}}{\frac{\sinh x\cosh y+\cosh x\sinh y}{\sinh x\sinh y}}$$

$$=\frac{\frac{\cosh x\cosh y}{\sinh x\sinh y}+\frac{\sinh x\sinh y}{\sinh x\sinh y}}{\frac{\sinh x\cosh y}{\sinh x\sinh y}+\frac{\cosh x\sinh y}{\sinh x\sinh y}}$$

$$=\frac{\frac{\cosh x}{\sinh x}\frac{\cosh y}{\sinh y}+1}{\frac{\cosh y}{\sinh y}+\frac{\cosh x}{\sinh x}}=\frac{\coth x\coth y+1}{\coth y+\coth x}=\text{R.H.S}$$

$$\therefore \coth(x+y)=\frac{\coth x\coth y+1}{\coth y+\coth x}$$

(iv) We have $\coth x=\frac{\cosh x}{\sinh x}$

$$\text{L.H.S}=\coth(x-y)=\frac{\cosh(x-y)}{\sinh(x-y)}$$

$$=\frac{\cosh x\cosh y-\sinh x\sinh y}{\sinh x\cosh y-\cosh x\sinh y}$$

Dividing the numerator and denominator with $\sinh x \sinh y$ we get

$$= \frac{\dfrac{\cosh x \cosh y - \sinh x \sinh y}{\sinh x \sinh y}}{\dfrac{\sinh x \cosh y - \cosh x \sinh y}{\sinh x \sinh y}}$$

$$= \frac{\dfrac{\cosh x \cosh y}{\sinh x \sinh y} - \dfrac{\sinh x \sinh y}{\sinh x \sinh y}}{\dfrac{\sinh x \cosh y}{\sinh x \sinh y} - \dfrac{\cosh x \sinh y}{\sinh x \sinh y}}$$

$$= \frac{\dfrac{\cosh x}{\sinh x}\dfrac{\cosh y}{\sinh y} - 1}{\dfrac{\cosh y}{\sinh y} - \dfrac{\cosh x}{\sinh x}} = \frac{\coth x \coth y - 1}{\coth y - \coth x} = \text{R.H.S}$$

$$\therefore \coth(x+y) = \frac{\coth x \coth y - 1}{\coth y - \coth x}$$

9. Problem: For any $x \in R$ prove that

$i) \sinh 2x = 2\sinh x \cosh x,\ ii) \cosh 2x = \cosh^2 x + \sinh^2 x,$

$iii) \tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}, iv) \coth 2x = \dfrac{\coth^2 x + 1}{2\coth x}.$

Solution: (i) We have $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Replace y with x we get $\sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x$

$\therefore \sinh 2x = 2\sinh x \cosh x$

(ii) We have $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Replace y with x we get $\cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x$

$\therefore \cosh 2x = \cosh^2 x + \sinh^2 x$

(iii) We have $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1+\tanh x \tanh y}$

Replace y with x we get $\tanh(x+x) = \dfrac{\tanh x + \tanh x}{1+\tanh x \tanh x}$

$\therefore \tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(iv) We have $\coth(x+y)=\dfrac{\coth x\coth y+1}{\coth y+\coth x}$

Replace y with x we get $\coth(x+x)=\dfrac{\coth x\coth x+1}{\coth x+\coth x}$

$\therefore \coth 2x=\dfrac{\coth^2 x+1}{2\coth x}$

Exercise 8

1. Find the values of the following $i)\cosh^{-1}1$ $ii)\sinh^{-1}2^{3/2}$ $iii)\text{sech}^{-1}\dfrac{1}{2}$
2. If $\sinh^{-1}x=\log(5+2\sqrt{6})$ then find the value of x
3. If $\cosh x=\dfrac{5}{2}$ then prove that $\cosh 2x=\dfrac{23}{2}$ and $\sinh 2x=\dfrac{5\sqrt{21}}{2}$.
4. If $x=\log\tan\left(\dfrac{\pi}{4}+\theta\right)$ then prove that $\cosh x=\sec 2\theta$.
5. If $x=\log\cot\left(\dfrac{\pi}{4}+\theta\right)$ then prove that $\sinh x=-\tan 2\theta$.
6. If $e^{\sinh^{-1}(\tan\theta)}=k$ then prove that $\sec\theta+\tan\theta=k$.
7. Prove that $(\cosh x+\sinh x)^n=\cosh nx+\sinh nx$.
8. Prove that $(\cosh x-\sinh x)^n=\cosh nx-\sinh nx$.
9. If $\tanh^2 x=\tan^2\theta$ then prove that $\cosh 2x=\sec 2\theta$.
10. For any $x\in R$ prove that

    $i)\sinh 3x=3\sinh x+4\sinh^3 x$, $ii)\cosh 3x=4\cosh^3 x-3\cosh x$,

    $iii)\tanh 3x=\dfrac{3\tanh x+\tanh^3 x}{1+3\tanh^2 x}$.

Key Concepts

1. $\sinh x=\dfrac{e^x-e^{-x}}{2}$ 2. $\cosh x=\dfrac{e^x+e^{-x}}{2}$ 3. $\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ 4. $\coth x=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$.

5. $\text{sech}\, x=\dfrac{2}{e^x+e^{-x}}$ 6. $\text{cosech}\, x=\dfrac{2}{e^x-e^{-x}}$.

7. The domain and ranges of the hyperbolic trigonometric functions are as follows.

Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1,\infty)$
$\tanh x$	R	$(-1,1)$
$\coth x$	$R-\{0\}$	$(-\infty,-1)\cup(1,\infty)$
$\text{sech}\, x$	R	$(0,1]$
$\text{cosech}\, x$	$R-\{0\}$	$R-\{0\}$

8. $i)\cosh^2 x-\sinh^2 x=1 \; for\, all\, x\in R$

$ii)1-\tanh^2 x=\text{sech}^2 x \; for\, all\, x\in R$

$iii)\coth^2 x-1=\text{cosech}^2 x \; for\, all\, x\in R-\{0\}.$

9. For any $x\in R$

$i)\sinh(-x)=-\sinh x,\quad ii)\cosh(-x)=\cosh x,\; iii)\tanh(-x)=-\tanh x$

$iv)\coth(-x)=-\coth x,\quad v)\text{sech}(-x)=\text{sech}\, x,\; vi)\text{cosech}(-x)=-\text{cosech}\, x.$

10. $\sinh 0=0$ 11. $\cosh 0=1$ 12. $\tanh 0=0$

13. For any $x\in R$

$i)\sinh(-x)=-\sinh x,\quad ii)\cosh(-x)=\cosh x,\; iii)\tanh(-x)=-\tanh x$

$iv)\coth(-x)=-\coth x,\quad v)\text{sech}(-x)=\text{sech}\, x,\; vi)\text{cosech}(-x)=-\text{cosech}\, x.$

14. For any $x,y\in R$

$i)\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y,\; ii)\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y,$

$iii)\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y, iv)\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y.$

15. For any $x,y\in R$

$$i)\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y},\; ii)\tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y},$$

$$iii)\coth(x+y)=\frac{\coth x\coth y+1}{\coth y+\coth x}, iv)\coth(x-y)=\frac{\coth x\coth y-1}{\coth y-\coth x}.$$

16. For any $x\in R$

$i)\sinh 2x=2\sinh x\cosh x,\; ii)\cosh 2x=\cosh^2 x+\sinh^2 x,$

$$iii)\tanh 2x=\frac{2\tanh x}{1+\tanh^2 x}, iv)\coth 2x=\frac{\coth^2 x+1}{2\coth x}.$$

17. For any $x,y\in R$ prove that

$$i)\ \tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y},\ ii)\ \tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y},$$
$$iii)\ \coth(x+y)=\frac{\coth x\coth y+1}{\coth y+\coth x},\ iv)\ \coth(x-y)=\frac{\coth x\coth y-1}{\coth y-\coth x}.$$

18. For any $x\in R$

$$i)\ \sinh^{-1}x=\log(x+\sqrt{x^2+1}),\quad ii)\ \cosh x=\log(x+\sqrt{x^2-1})\ \text{for } x\geq 1,$$
$$iii)\ \tanh^{-1}x=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\text{for } x\in(-1,1),\ iv)\ \coth^{-1}x=\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)\text{for } |x|>1,$$

$$v)\ \text{sech}^{-1}x=\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\text{for } x\in(0,1],$$
$$vi)\ \text{cosech}^{-1}x=\begin{cases}\log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right) & \text{for } x>0\\ \log\left(\dfrac{1-\sqrt{1+x^2}}{x}\right) & \text{for } x<0.\end{cases}$$

19. The domain and ranges of the inverse hyperbolic trigonometric functions is as follows.

Function	Domain	Range
$\sinh^{-1}x$	R	R
$\cosh^{-1}x$	$[1,\infty)$	$[0,\infty)$
$\tanh^{-1}x$	$(-1,1)$	R
$\coth^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$R-\{0\}$
$\text{sech}^{-1}x$	$(0,1]$	$[0,\infty)$
$\text{cosech}^{-1}x$	$R-\{0\}$	$R-\{0\}$

Answers
Exercise 8

(1) $(i)0\,(ii)\log(3+2\sqrt{2})\ (iii)\log(2+\sqrt{3})$ (2) $x=5$

9. LIMITS AND CONTINUITY

Introduction:

Calculus can be considered as the subject that studies the problems of change. This mathematical discipline stems from the 17^{th} century investigations of *Isaac Newton* (1642-1727) and *Gottfried Leibnitz* (1646-1716) and today it stands as the quantitative language of science and technology

The very basic notion of a limit was conceived in 1680's by Newton and Leibnitz simultaneously, while they were struggling with the creation of calculus. They gave a loose verbal definition of limit which led to many problems. There were other mathematicians of the same era who proposed other definitions of the intuitive concept of limit. But none of these were adequate to provide a basis for rigorous proofs. Of course there are evidences that the idea of limit was first known to *Archimedes* (281-212B.C).

It is *Augustin-Louis Cauchy* (1789-1857) who formulated the definition and presented the arguments with greater care than his predecessors in his monumental work 'Cours d Analyse'. But the concept of a limit still remained elusive.

The precise definition of limit that as we use today, was given by *Karl Weierstrass* (1815-1897).

9.1 Intervals and neighbourhoods:

First we look into the concepts of intervals and neighbourhoods, which are very much useful in studying limits and continuity.

9.1.1 Intervals:

Let $a, b \in R$ such that $a \le b$ .Then the set

1. $\{x \in R : a \le x \le b\}$ Denoted by $[a,b]$ is called a *Closed Interval*.
2. $\{x \in R : a < x < b\}$ Denoted by (a,b) is called an *Open Interval*.

In a similar way, some more intervals are given below.

3. $(a,b] = \{x \in R : a < x \le b\}$
4. $[a,b) = \{x \in R : a \le x < b\}$
5. $[a,\infty) = \{x \in R : x \ge a\}$
6. $(a,\infty) = \{x \in R : x > a\}$
7. $(-\infty,a] = \{x \in R : x \le a\}$
8. $(-\infty,a) = \{x \in R : x < a\}$

The intervals in 1, 2, 3 and 4 are said to be intervals of finite length $b-a$. Others are intervals of infinite length.

9.1.2 Neighbourhoods:

Let $a, b \in R$. If $\delta > 0$, then the open interval $(a-\delta, a+\delta)$ is called the δ-*neighbourhood of* a. That is $\{x \in R : a-\delta < x < a+\delta\}$.Figure 9.1 shows the location of $(a-\delta, a+\delta)$ on the number line.

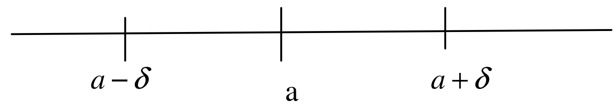


Figure.9.1

The set obtained by deleting the point a from this neighbourhood is called the *deleted neighbourhood of* a. That is, the deleted $\delta-$neighbourhood of a is $(a-\delta, a) \cup (a, a+\delta)$ or $(a-\delta, a+\delta) - \{a\}$.

9.1.3 Note:

1. Any interval (c,d) is a neighbourhood of some $a \in (c,d)$ .In fact, take $a = \dfrac{c+d}{2}$ and $\delta = \dfrac{d-c}{2} > 0.$ Then $(a-\delta, a+\delta) = \left(\dfrac{c+d}{2} - \dfrac{d-c}{2}, \dfrac{c+d}{2} + \dfrac{d-c}{2}\right) = (c,d)$

Therefore (c,d) is the δ neighbourhood of a.

2. The set $\{x \in R : 0 < |x-a| < \delta\}$ is the deleted $\delta-$neighbourhood of a, because

$$0 < |x-a| < \delta \Leftrightarrow |x-a| < \delta \;\&\; x \neq a$$

$$\Leftrightarrow -\delta < x-a < \delta \;\&\; x \neq a$$

$$\Leftrightarrow a-\delta < x < a+\delta \;\&\; x \neq a$$

$$\Leftrightarrow x \in (a-\delta, a+\delta) - \{a\}$$

3. If $\delta_1, \delta_2 .... \delta_n$ are positive real numbers and $a \in R$ then

$\bigcap_{k=1}^{n} (a-\delta_k, a+\delta_k)$ is also a neighbourhood of a.

For, take $\delta = \min\{\delta_1, \delta_2 .... \delta_n\}$.Then clearly $\delta > 0$ and

$$(a-\delta, a+\delta) = \bigcap_{k=1}^{n} (a-\delta_k, a+\delta_k)$$
$$\Rightarrow x \in (a-\delta, a+\delta)$$

$$\Rightarrow a-\delta_k < x < a+\delta_k \quad (k = 1, 2, ...n)$$

$$\Rightarrow x \in (a-\delta_k, a+\delta_k) \quad \text{for all } k$$

$$\Rightarrow x \in \bigcap_{k=1}^{n} (a-\delta_k, a+\delta_k)$$

Conversely if $\bigcap_{k=1}^{n} (a-\delta_k, a+\delta_k)$ then $a-\delta_k < x < a+\delta_k$ for all k.

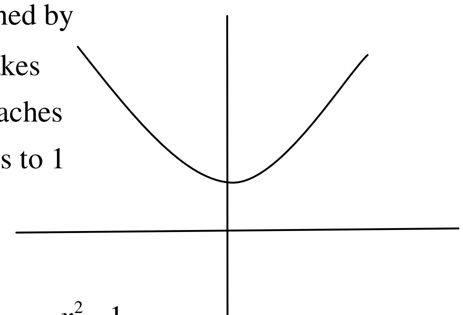
But $\delta = \min\{\delta_1, \delta_2 ....\delta_n\}$ implies that $\delta = \delta_i$ for some i among $1, 2, 3, ..., n$

Hence $x \in (a-\delta_i, a+\delta_i) = (a-\delta, a+\delta)$.

9.1.4 Limits: We illustrate some examples to get familiarity on the concept of limits.

1. **Example:** Let $f : R \to R$ be the function defined by $f(x) = x^2 + 1, x \in R$ .Here we observe that x takes values very close to 0, the value of $f(x)$ approaches to 1(see fig).In this case we say that $f(x)$ tends to 1 as x tends to 0 and we write it as $\lim_{x \to 0} f(x) = 1$

   That is limit of $f(x)$ is 1 as x tends to 0.



2. **Example:** Let us define $f : R - \{1\} \to R$ by $f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$

In the following table we compute the values of $f(x)$ for certain values

on either side of $x = 1$.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1

From the table we observe that these values are very near to 2.This can be illustrated by considering the graph of the function $f(x)$ given in the fig. Here we note that the limit of $f(x)$ at 1 exists even through $f(x)$ is not defined at 1.

3. **Example:** Let $f : R - \{-2\} \to R$ be defined by $f(x) = \frac{x-1}{x+2}$ for each $R - \{-2\}$

   Consider the following table of values of x close to $x = 3$ on either side or the corresponding values of $f(x)$.

x	2.9	2.99	2.999	2.9999	2.99999	3.00001	3.0001	3.001	3.01	3.1
f(x)	0.38776	0.39880	0.39988	0.39999	0.399999	0.400001	0.40001	0.40012	0.40120	0.41176

The above table shows that as x gets close to 3, the function $f(x)$ is approaching to 0.4

4. **Example:** Let $f: R-\{2\} \to R$ be defined by $f(x) = \dfrac{x^2+3x-10}{x-2}$

Here is a table of values of x near 2 and corresponding $f(x)$.

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
f(x)	6.9	6.99	6.999	6.9999	7.001	7.001	7.01	7.1

Though f is not defined at 2, but $f(x)$ is approaching to 7 as x is nearing to 2. The same can be seen from table and the fig.

5. **Example:** Let us look at the value of x near 0 and corresponding $f(x)$.

x	0.01	0.001	0.0001	0.00001
f(x)	0.1	0.0316	0.01	0.0031

From the table we observe that $f(x)$ approaches zero as x approaches zero. In each of the above examples it is clear that $f(x)$ approaches value l when x is nearing to a particular point c. This leads to an important concept called the limit of a function. In the third century (B.C) Archimedes of Greece (287-212 B.C) was the first person who formulated this concept. But a precise definition of the limit of a function, that we use today, is due to the German mathematician Karl Weierstrass (1815-1897), We introduce this concept in the present section.

9.1.5 Definition of the limit: Let $E \subseteq R$ and $f: E \to R$.let $a \in R$ be such that $((a-r, a+r)-\{a\}) \cap E$ is non empty for every $r > 0$. If there exists a real number l satisfying the condition below then l is said to be a *limit* of f at a:

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x)-l| < \varepsilon$ whenever $x \in E$ and $0 < |x-a| < \delta$.In this case, we say that limit of the function $f(x)$ as x tends to a exists and it is 'l' and we write it as $\lim\limits_{x \to a} f(x) = l$ or $f(x) \to l$ as $x \to a$

If such an l does not exist we say that $\lim\limits_{x \to a} f(x)$ does not exist.

9.1.6 Note: Let f, E, a, l be as given in definition 9.1.1. Also let $m \in R$ be such that $\lim\limits_{x \to a} f(x) = m.$ Then it can shown that $l = m.$ In other words, the limit of a function at a given point if exists is unique. This is proved as follows.

Given that $\lim\limits_{x \to a} f(x) = m$ and $\lim\limits_{x \to a} f(x) = l$

In order to show that l = m it is sufficient to show that $|l - m| < \varepsilon$ for every $\varepsilon > 0$

Let $\varepsilon > 0$

Since $\lim\limits_{x \to a} f(x) = l$ for $\dfrac{\varepsilon}{2} > 0, \exists \delta_1 > 0$ suh that

$x \in R$ and $0 < |x-a| < \delta_1 \Rightarrow |f(x)-l| < \dfrac{\varepsilon}{2}$.........(I)

Since $\lim_{x\to a} f(x) = m$ for $\frac{\varepsilon}{2} > 0, \exists \delta_2 > 0$ such that

$x \in E$ and $0 < |x-a| < \delta_2 \Rightarrow |f(x) - m| < \frac{\varepsilon}{2}$.........(II)

Let $\delta = \min\{\delta_1, \delta_2\}$.Then if $x \in E$ and $0 < |x-a| < \delta$ we have by (I) and (II)

$|l-m| = |l - f(x) + f(x) - m| \le |l - f(x)| + |f(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$i.e., |l-m| < \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we get $l = m$.

9.1.7 Examples: In the following we illustrate the definition of limit through examples with δ, ε notation

1. Suppose $f : (0, \infty) \to R$ is defined by $f(x) = \sqrt{x}$.Then $\lim_{x\to 0} f(x) = 0$

   Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$.Then $\delta > 0$ and for all x with $x \in (0, \infty), 0 < |x| < \delta$

   i.e., $0 < x < \delta$ we have $|f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon$

   Hence $\lim_{x\to 0} f(x) = 0$

2. Suppose $f : (R \setminus \{0\}) \to R$ is given by $f(x) = \frac{1}{x}, x \ne 0$.Then $\lim_{x\to 0} = \frac{1}{x}$ does not exist.

   If possible suppose that $\lim_{x\to 0} = \frac{1}{x}$ exists and is equal to say l

   Then for $\varepsilon = 1$ there is a $\delta > 0$ such that

   $$0 < |x| < \delta \Rightarrow \left|\frac{1}{x} - l\right| < 1$$

   $$i.e.,\ 0 < |x| < \delta \Rightarrow \left|\frac{1}{x}\right| = \left|\frac{1}{x} - l + l\right| \le \left|\frac{1}{x} - l\right| + |l| < 1 + |l|$$

   That is $|x| > \frac{1}{1+|l|}; 0 < x < \delta$

   But if we choose a y_0 such that $0 < y_0 < \min\{\delta, \frac{1}{1+|l|}\}$, then $0 < |y_0| = y_0 < \delta$

   and $|y_0| = y_0 < \frac{1}{1+|l|}$, contradicting. Therefore $\lim_{x\to 0} = \frac{1}{x}$ does not exist.

9.1.8Note: $\lim_{x\to a} x = a$ (Try to give a proof)

We state the following theorem (without) proof which is helpful in finding the limits.

9.1.9 Theorem: Let $f: E \to R, g: E \to R$ and let $a \in R$ be such that $E \cap ((a-r, a+r) - \{a\})$ is non empty for every $r > 0$.

Let $k \in R$ .Suppose that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ .Then the following are true.

(i) $\lim_{x\to a}(f+g)(x) = l+m, \lim_{x\to a}(f-g)(x) = l-m, \lim_{x\to a}(fg)(x) = lm$
$\lim_{x\to a}(kf)(x) = kl$

(ii) If $h: E \to R$ and $\lim_{x\to a} h(x) = n \neq 0$ then h is never zero in

$E \cap ((a-r, a+r) - \{a\})$ for some $r > 0$, $\lim_{x\to a}\left(\frac{1}{h}\right)(x) = \frac{1}{n}$ & $\lim_{x\to a}\left(\frac{f}{h}\right)(x) = \frac{1}{n}$.

As an illustration we prove the following.

9.1.10 Theorem: If p is a polynomial function (*i.e.*, a function $p(x)$ of the form $a_0 + a_1 x + ... + a_k x^k, k \geq 1$) then $\lim_{x\to a} p(x) = p(a)$

9.1.11 Remark: During the course of proof of theorem 9.1.10, we proved that

$$\lim_{x\to a} x^n = a^n, a \in R, n \in N$$

9.1.12 Theorem: Let $E \subseteq R$, let $f, g: E \to R$ be two functions. Let $a \in R$ be such that $((a-r, a+r) - \{a\}) \cap E \neq \varphi$ for every $r > 0$. Assume that $f(x) \leq g(x)$ for all x in E with x not equal to a. If both $\lim_{x\to a} f(x) = l$ & $\lim_{x\to a} g(x) = m$ then $l \leq m$ .That is $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

9.1.13 Theorem (Sandwich theorem): Let $E \subseteq R$, $f, g, h: E \to R$ and let $a \in R$ be such that $\left((a-r, a+r) - \{a\}\right) \cap E \neq \varphi$ for every $r > 0$. If $f(x) \leq g(x) \leq h(x)$ for all $x \in R, x \neq a$ and if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x)$ exists and is equal to *l.*

9.1.14 Theorem: If F and G are polynomials such that $F(x) = (x-a)^k f(x), G(x) = (x-a)^k g(x)$ for some $k \in N$ and for some polynomials $f(x)$ and $g(x)$ with $g(a) \neq 0$ then $\lim_{x\to a}\left(\frac{F}{G}\right)(x) = \frac{f(a)}{g(a)}$

We shall now make use of theorems to compute some limits. Hereafter if the domain of a function f is not explicitly given, then by convention, the domain of f is to be taken as the set of all those real x for which $f(x)$ is real.

9.1.15 Solved problems:

1. Problem: Find $\lim\limits_{x\to 3}\dfrac{x^3-6x^2+9x}{x^2-9}$

Solution: Write $F(x)=x^3-6x^2+9x=x(x-3)^2=(x-3)f(x)$ where $f(x)=x(x-3)$

Write $G(x)=x^2-9=(x+3)(x-3)=(x-3)g(x)$ where $g(x)=x+3$

Therefore $\dfrac{F(x)}{G(x)}=\dfrac{(x-3)f(x)}{(x-3)g(x)}=\dfrac{f(x)}{g(x)}$ & $g(3)=6\neq 0$

Now by applying theorem 9.1.14, we get

$$\lim_{x\to 3}\frac{x^3-6x^2+9x}{x^2-9}=\lim_{x\to 3}\frac{F(x)}{G(x)}=\frac{f(3)}{g(3)}=\frac{3(3-3)}{3+3}=0$$

2. Problem: Find $\lim\limits_{x\to 3}\dfrac{x^3-3x^2}{x^2-5x+6}$

Solution: We write $F(x)=x^3-3x^2=x^2(x-3)=(x-3)=(x-3)f(x); f(x)=x^2$

$G(x)=x^2-5x+6=(x-2)(x-3)=(x-3)g(x)$ where $g(x)=x-2$ with $g(3)=3-2=1\neq 0$

$\therefore$ By applying theorem 9.1.14, we get

$$\lim_{x\to 3}\frac{x^3-3x^2}{x^2-5x+6}=\lim_{x\to 3}\frac{F(x)}{G(x)}=\frac{f(3)}{g(3)}=\frac{3^2}{3-2}=9$$

Exercise 9(a)

1. $\lim\limits_{x\to 0}x^2\cos\dfrac{2}{x}$ 2. $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$ 3. $\lim\limits_{x\to 3}\dfrac{x^2-8x+15}{x^2-9}$

4. If $f(x)=-\sqrt{25-x^2}$ then find $\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}$

9.1.16 Right and left hand limits:

We studied limit of function $f(x)$ at a given point $x=a$ as the approaching value of $f(x)$ when x tends to 'a'. Hence we note that there are two ways x could approach 'a' either from the left of 'a' or front the right of 'a'. This naturally leads to two limits namely the' right hand limit' and 'left hand limit'. Right hand limit of a function f at $x=a$ is the limit of the values of $f(x)$ as x tends to 'a' when x takes values greater than 'a'. We denote the right hand limit of f at 'a' by $\lim_{x \to a+} f(x)$ .Similarly we describe the left hand limit of $f(x)$ at 'a' and we denote it by $\lim_{x \to a-} f(x)$.

1. Example: Define $f : R \to R$ by f(x)= $\begin{cases} 1 & if \ \ x \leq 0 \\ -1 & if \ \ x > 0 \end{cases}$

It is clear that the $\lim_{x \to 0-} f(x) = 1$ and $\lim_{x \to 0+} f(x) = -1$.

Hence the right and left hand limits of $f(x)$ at 0 are different. We observe that the limit of $f(x)$ as x tends to 0 does not exist.

To formulate these concepts analogous to the definition of a limit in the following.

9.1.17 Definition (Right and left hand limits):

Let $E \subseteq R$ and $f : E \to R$

(i). Suppose $a \in R$ is such that $E \cap (a, a+r)$ is non-empty for every $e > 0$. We say that $l \in R$ is a *right hand limit* of at a, and we write $\lim_{x \to a+} f(x) = l$, if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < x - a < \delta$ and $x \in E$.

(ii) Suppose $a \in R$ is such that $E \cap (a-r, a)$ is non empty for every $r > 0$. We say that $m \in R$ is a *left hand limit* of f at a and we write $\lim_{x \to a-} f(x) = m$ if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - m| < \varepsilon$ whenever $0 < a - x < \delta$ and $x \in E$.

The limits $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ are called one –sided limits.

These limits (if exists) are unique.

9.1.18 Note:

(i) If $E = (a, b)$ then it is clear from definitions 9.1.17 and 9.1.5 that $f : E \to R$ has limit at a if and only if has right hand limit at a.
In this case $\lim_{x \to a} f(x) = \lim_{x \to a+} f(x)$

(ii) Also f has limit at b if and only if it has left hand limit b.
In this case $\lim_{x \to b} f(x) = \lim_{x \to b-} f(x)$.

(iii) If $a < c < b$, f has limit at c if and only if the left hand limit and the right hand limit both exist at c and are equal.
In this case $\lim_{x \to c} f(x) = \lim_{x \to c-} f(x) = \lim_{x \to c+} f(x)$

The following theorem relates limit of a function to one sided limits.

9.1.19 Theorem: Let $E=(a-r,a+r)-\{a\}$ for all $r>0$ and $f:E\to R$.

Then $\lim_{x\to a} f(x)=l \Leftrightarrow \lim_{x\to a+} f(x)=l=\lim_{x\to a-} f(x)$.

9.1.20 Note:

If $\lim_{x\to a+} f(x)$ & $\lim_{x\to a-} f(x)$ exist then $\lim_{x\to a+} f(x)$ & $\lim_{\substack{h\to 0\\ h>0}} f(a+h)$ and $\lim_{x\to a-} f(x)$ & $\lim_{\substack{h\to 0\\ h>0}} f(a-h)$

9.1.21 Solved Problems:

1. Problem: Show that $\lim_{x\to 0+}\frac{|x|}{x}=1$ and $\lim_{x\to 0-}\frac{|x|}{x}=-1$

Solution: Here $\frac{|x|}{x}=\begin{cases}1 & if\ \ x>0\\ -1 & if\ \ x<0\end{cases}$

Therefore $\lim_{x\to 0+}\frac{|x|}{x}=1=\lim_{x\to 0+}1=1$ and $\lim_{x\to 0-}\frac{|x|}{x}=-1$

2. **Problem:** Let $f:R\to R$ be defined by $f(x)=\begin{cases}2x-1 & if\ \ x<3\\ 5 & if\ \ x\geq 3\end{cases}$.

   Show that $\lim_{x\to 3} f(x)=5$.

Solution: $\lim_{x\to 3+} f(x)=5$, since $f(x)=5$ for $x>3$ and $\lim_{x\to 3-} f(x)=\lim_{x\to 3-}(2x-1)=5$

Hence $\lim_{x\to 3} f(x)=5$.

3. **Problem:** Show that $\lim_{x\to 2-}\sqrt{x^2-4}=0=\lim_{x\to 2}\sqrt{x^2-4}$

Solution: Observe that $\sqrt{x^2-4}$ is not defined over $(-2,2)$

But $\lim_{x\to 2+}\sqrt{x^2-4}=0$ and $\lim_{x\to 2-}\sqrt{x^2-4}=0$

Therefore $\lim_{x\to 2}\sqrt{x^2-4}=0=\lim_{x\to -2}\sqrt{x^2-4}$

4. **Problem:** If $f(x)=\begin{cases}x^2 & if\ \ x\leq 1\\ 2x-1 & if\ \ x>1\end{cases}$, then find $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$.

   Does $\lim_{x\to 1} f(x)$ exist?

Solution: $\lim_{x\to 1+} f(x)=\lim_{\substack{h\to 0\\ h>0}} f(1+h)=\lim_{h\to 0}(2(1+h)-1)=1$

And $$\lim_{x \to 1-} f(x) = \lim_{\substack{h \to 0 \\ h>0}} f(1-h) = \lim_{h \to 0}(1-h)^2 = 1$$

Since $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = 1$, we get $\lim_{x \to 1} f(x)$ exists and $\lim_{x \to 1} f(x) = 1$.

Exercise 9(b)

Find the right and left hand limits of the functions at a point a mentioned against them .Hence check whether the functions have limits at those $a's$.

1. $f(x) = \begin{cases} 1-x & if \ x \le 1 \\ 1+x & if \ x > 1 \end{cases}$ $\quad a = 1.$
2. $f(x) = \begin{cases} x+2 & if \ -1 < x \le 3 \\ x^2 & if \ 3 < x < 5 \end{cases}$ $\quad a = 3.$
3. $f(x) = \begin{cases} \dfrac{x}{2} & if \ x < 2 \\ \dfrac{x^2}{3} & if \ x \ge 2 \end{cases}$ $\quad a = 2.$
4. Show that $\lim_{x \to 2-} \dfrac{|x-2|}{x-2} = -1$
5. Show that $\lim_{x \to 0+} \left(\dfrac{2|x|}{x} + x + 1 \right) = 3$
6. Compute $\lim_{x \to 2+} \left([x] + x\right)$ & $\lim_{x \to 2-} \left([x] + x\right)$

9.1.22 Standard limits:

We shall now obtain the limits of some standard functions in the following theorems. Using these we can find the limits of some functions easily.

9.1.23 Theorem: If $a > 0, n \in R$ then $\lim_{x \to a} x^n = a^n$

9.1.24 Theorem: Let n be a rational number and a be a positive real number.

Then $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

9.1.25 Theorem: $\lim_{x \to 0} \cos x = 1$ & $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

9.1.26 Theorem: $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

9.1.27 Theorem: $\lim_{x \to 0} \left(\dfrac{a^x - 1}{x} \right) = \log_e a$

9.1.27 Corollary: $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1$

9.1.28 Theorem: $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$

9.1.29 Solved problems:

1. Problem: Compute $\lim_{x\to 0}\left(\frac{e^x-1}{\sqrt{1+x}-1}\right)$

Solution: For $o<|x|<1$

$$\text{Let } L=\lim_{x\to 0}\frac{e^x-1}{\sqrt{1+x}-1} \quad =\frac{e^0-1}{\sqrt{1+0}-1} \quad =\frac{1-1}{\sqrt{1}-1} \quad =\frac{1-1}{1-1}=\frac{0}{0}\ form$$

$$\text{Now } L=\lim_{x\to 0}\frac{e^x-1}{\sqrt{1+x}-1}=\left(\lim_{x\to 0}\frac{\frac{e^x-1}{x}}{\frac{\sqrt{1+x}-1}{x}}\right)=\frac{\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)}{\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)}$$

$$=\frac{1}{1/2} \quad \left[\because \lim_{y\to 0}\frac{e^y-1}{y}=1\right]$$

$$=1\times\frac{2}{1}=2$$

2. Problem: Compute $\lim_{x\to 0}\frac{a^x-1}{b^x-1},(a>0,b>0,b\neq 1)$

Solution: Let $L=\lim_{x\to 0}\frac{a^x-1}{b^x-1} \quad =\frac{a^0-1}{b^0-1} \quad =\frac{1-1}{1-1}=\frac{0}{0}\ form$

$$\text{Now } L=\lim_{x\to 0}\frac{a^x-1}{b^x-1}=\lim_{x\to 0}\left[\frac{\frac{a^x-1}{x}}{\frac{b^x-1}{x}}\right]=\left[\frac{\lim_{x\to 0}\frac{a^x-1}{x}}{\lim_{x\to 0}\frac{b^x-1}{x}}\right]$$

$$=\frac{\log_e a}{\log_e b} \quad \left[\because \lim_{y\to 0}\frac{a^y-1}{y}=\log_e a\right]$$

$$=\log_b a \quad \left[\because \frac{\log_k a}{\log_k b}=\log_b a\right]$$

3. Problem: Compute $\lim_{x\to 0}\frac{e^x-\sin x-1}{x}$

Solution: We have

$$L=\lim_{x\to 0}\frac{e^x-\sin x-1}{x}=\lim_{x\to 0}\left[\frac{e^x-1}{x}-\frac{\sin x}{x}\right]=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)-\lim_{x\to 0}\frac{\sin x}{x}=1-1=0$$

5. **Problem:** Evaluate $\lim_{x\to 1}\frac{\log_e x}{x-1}$

Solution: Put $y=x-1$. Then y implies 0

Now $\lim_{x\to 1}\frac{\log_e x}{x-1}=\lim_{y\to 0}\frac{\log(1+y)}{y}=1$

Exercise 9(c)

1. $\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\left(x-\frac{\pi}{2}\right)}$ 2. $\lim_{x\to a}\frac{\tan(x-a)}{x^2-a^2}(a\neq 0)$ 3. $\lim_{x\to 0}\frac{\sin(a+bx)-\sin(a-bx)}{x}$

4. $\lim_{x\to 0}\frac{e^{3+x}-e^3}{x}$ 5. $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$ 6. $\lim_{x\to 1}\frac{(2x-1)(\sqrt{x}-1)}{(2x^2+x-3)}$

9.1.30 Infinite limits and limits at infinity: Consider $f(x)=x^{-4}$ for x not equal to zero. At the points very close to zero the values of $f(x)$ would be increasing rapidly. Thus we can't have the concept o limit at zero for this function. We shall try to describe this nature of the function in the present section.

(i) Let $E\subseteq R, f:E\to R \,\&\, a\in R$ be such that $E\cap((a-r,a+r)-\{a\})$ is non empty for every $r>0$. We say that $f(x)$ tends to infinity as $x\to a$ and write $\lim_{x\to a}f(x)=\infty$ is given $\alpha\in R$ there exists a $\delta>0$ such that $f(x)>\alpha$ for all $x\in R$ with $0<|x-a|<\delta$.

(ii) $f(x)$ is said that to tend to $-\infty$ as $x\to a$ and write $\lim_{x\to a}f(x)=-\infty$, if given $\beta\in R$ there exists a $\delta>0$ such that $f(x)<\beta$,for all $x\in E$ with $o<x-a<\delta$.

(iii) Let $E\subseteq R, f:E\to R$ .Suppose $(a,\infty)\subseteq E$ for some $a\in R$.Then we say that l belongs R is a limit of $f(x)$ as x tends to infinity and write $\lim_{x\to\infty}f(x)=l$, if given $\varepsilon>0$ there exists a $K>a$ such that $f(x)-l<\varepsilon$ or all x>K Such an l if exists, is unique.

(iv) Let $E \subseteq R, f: E \to R$. Suppose $(-\infty, a) \subseteq E$ for some a belongs to R. Then we say that l belongs to R is a limit of $f(x)$ as x tends to $-\infty$ and write $\lim_{x \to -\infty} f(x) = l$, if given $\varepsilon > 0$ there exists a K< a such that $|f(x) - l| < \varepsilon$ for all x<K. Such that an l, if exists is unique.

(v) Let $E \subseteq R, f: E \to R$. Suppose $(a, \infty) \subseteq E$ for some $a \in R$. Then we say f(x) tends to ∞ as x tends to ∞ and write $\lim_{x \to -\infty} f(x) = \infty$, if given $\alpha \in R$ there exists a K> a such that $f(x) > \alpha$ for all x>K.

(vi) Let $E \subseteq R, f: E \to R$. Suppose $(a, \infty) \subseteq E$ for some $a \in R$. Then we say $f(x)$ tends to -∞ as x tends to ∞ and write $\lim_{x \to -\infty} f(x) = -\infty$, if given $\alpha \in R$ there exists a K> a such that $f(x) < \alpha$ for all x>K.

(vii) Let $E \subseteq R, f: E \to R$. Suppose $(-\infty, a) \subseteq E$ for some $a \in R$. Then we say $f(x)$ tends to ∞ as x tends to -∞ and write $\lim_{x \to -\infty} f(x) = \infty$, if given $\alpha \in R$ there exists a K< a such that $f(x) > \alpha$ for all $x <$ K.

(viii) Let $E \subseteq R, f: E \to R$. Suppose $(-\infty, a) \subseteq E$ for some $a \in R$. Then we say $f(x)$ tends to $-\infty$ as x tends to $-\infty$ and write $\lim_{x \to -\infty} f(x) = -\infty$, if given $\alpha \in R$ there exists a K $< a$ such that $f(x) < \alpha$ for all $x <$ K.

In order to compute the limits defined in (i) through (vii) the following theorem is of great use. We state the theorem without proof as the proof is beyond the scope of this book.

9.1.31 Theorem: Let $E \subseteq R, f: E \to R \,\&\, a \in R$ be such that $E \cap ((a - r, a + r) - \{a\})$ is non empty for every $r > 0$.

(i) Suppose $\lim_{x \to a} f(x) = \infty$. Then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(ii) Suppose $\lim_{x \to a} f(x) = -\infty$. Then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(iii) If $\lim_{x \to a} \frac{1}{f(x)} = 0$ and f is positive in a deleted neighbourhood of a, then $\lim_{x \to a} \frac{1}{f(x)} = \infty$.

(iv) If $\lim_{x \to a} \frac{1}{f(x)} = 0$ and f is negative in a deleted neighbourhood of a, then $\lim_{x \to a} \frac{1}{f(x)} = -\infty$

9.1.32 Theorem:

Let $E \subseteq R, f: E \to R, g: E \to R, h: E \to R$ and let $(a, \infty) \subset E$ for some $a \in R$. If $\lim_{x \to \infty} g(x) = l = \lim_{x \to \infty} h(x) \,\&\, g(x) \le f(x) \le h(x)$ for all $x \in R$ then $\lim_{x \to \infty} f(x) = 1$.

9.1.33 Solved Problems:

1. **Problem:** Show that $\lim_{x\to\infty}\frac{1}{x^2}=0$.

Solution: Given $\varepsilon>0$, choose $\alpha=\frac{1}{\sqrt{\varepsilon}}>0$. Then

$$x>\alpha \Rightarrow x>\frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2>\frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2}<\varepsilon \Rightarrow \left|\frac{1}{x^2}-0\right|<\varepsilon$$

Hence $\lim_{x\to\infty}\frac{1}{x^2}=0$

2. **Problem:** Show that $\lim_{x\to\infty}e^x=\infty$

Solution: Given $K>0$, let $\alpha=\log K$. Then $x>\alpha \Rightarrow e^x>e^\alpha=K$

Hence $\lim_{x\to\infty}e^x=\infty$

3. **Problem:** Compute $\lim_{x\to 2}\frac{x^2+2x-1}{x^2-4x+4}$

Solution: Write $f(x)=\frac{x^2-4x+4}{x^2+2x-1}=\frac{(x-2)^2}{x^2+2x-1}$

Clearly $f(x)>0$ in a deleted neighbourhood of 2.

Hence $\lim_{x\to 2}\frac{x^2+2x-1}{x^2-4x+4}=\infty$.

4. **Problem:** If $f(x)=\frac{a_nx^n+.........+a_1x+a_0}{b_mx^m+.........+b_1x+b_0}$ with $a_n>0, b_m>0$ then show that $\lim_{x\to\infty}f(x)=\infty$ if $n>m$.

Solution: $$f(x)=\frac{x^n\left(a_n+\frac{a_{n-1}}{x}+.....+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)}{x^n\left(b_m+\frac{b_{m-1}}{x}+.....+\frac{b_1}{x^{m-1}}+\frac{b_0}{x^m}\right)}$$

As $x\to\infty$, all the quotients approach to zero. Therefore the quantity in the big bracket above approaches $\frac{a_n}{b_m}(>0)$. But $\lim_{x\to\infty}x^{n-m}=\infty$ (since $n>m$). Hence $\lim_{x\to\infty}f(x)=\infty$.

Exercise 9(d)

1. $\lim_{x\to 3}\frac{x^2+3x+2}{x^2-6x+9}$ 2. $\lim_{x\to 1-}\frac{1+5x^3}{1-x^2}$ 3. $\lim_{x\to\infty}\frac{6x^2-x+7}{x+3}$ 4. $\lim_{x\to\infty}e^{-x^2}$ 5. $\lim_{x\to\infty}\frac{8|x|+3x}{3|x|-2x}$

6. $\lim_{x\to\infty}\frac{x^2+5x+2}{2x^2-5x+1}$ 7. $\lim_{x\to\infty}\frac{2x^2-x+3}{x^2-2x+5}$ 8. $\lim_{x\to\infty}\frac{11x^3-3x43}{13x^3-5x^2-7}$ 9. $\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

10. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$ 11. $\lim_{x\to-\infty}\left(\frac{2x+3}{\sqrt{x^2-1}}\right)$ 12. $\lim_{x\to-\infty}\frac{2+\sin x}{x^2+3}$ 13. $\lim_{x\to-\infty}\frac{2+\cos^2 x}{x+2007}$

14. $\lim_{x\to\infty}\frac{\cos x+\sin^2 x}{x+1}$

9.2 Definition of Continuity and simple problems:

In this section, we shall define one of the most important concepts of mathematical analysis, namely, the continuity of a function at a point and on a set. We shall also discuss the relation between limits and continuity.

We start this section with some examples.

1. **Example**: Define $f:R\to R$ by $f(x)=2x+1,\ x\in R$. Note that this function is defines at every point of R. The graph of this function is given in the figure.

   We observe that the left hand limit of f at $x=0$ is 1 and also the right hand limit of f at $x=0$ is 1.

   Thus the $\lim_{x\to 0+} f(x)=\lim_{x\to 0-} f(x)=1$ and this value equal to $f(0)=1$.

Here, it is worth mentioning that it is possible to draw the graph of the function around the point $x=0$ without lifting the pen from the plane of the paper. Since the same is true for every point in R, graphically the function $f(x)=2x+1$, defines a line without any breaks.

2. **Example:** Define $f:R\to R$ by $f(x)=\begin{cases}1 & if\ x\le 0\\ 2 & if\ x>0\end{cases}$.

This function is defined at every point of R. The graph of this function is given in the figure. It is east to see that $\lim_{x\to 0+} f(x)=2\ne f(0)\ \&\ \lim_{x\to 0-} f(x)=1$.

Here we note that it is not possible to draw the graph of the function on the plane of the paper without lifting the pen at $x=0$. The graph of the function and has a break at $x=0$.

3. **Example:** Define $f:R\to R$ by $f(x)=x^2, x\in R$ we observe that $\lim_{x\to 0+} f(x)=0$ with $f(0)=0$ so that $\lim_{x\to 0} f(x)=\lim_{x\to 0+} f(x)=\lim_{x\to 0-} f(x)=0=f(0)$

9.2.1 Definition: Let $E \subseteq R, f : E \to R \,\&\, a \in E$, Then we say that f is continuous at 'a' if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in E$ and $0 < |x - a| < \delta$.

If f is continuous at every point of E then we say that f is continuous on E.

If f is not continuous at a, we say that f is discontinuous at a.

Observe that, we talk of continuity or discontinuity of f at a point 'a' only when 'a' is in the domain of f.

9.2.2 Remark: Let $E \subseteq R, f : E \to R \,\&\, a \in E$, Suppose that there exists a positive real number r such that $(a - r, a + r) \cap E = \{a\}$.Then f is continuous at a, a being arbitrary, f is continuous on N.

For let $\varepsilon > 0$ be given. Then

$$x \in E \text{ And } |x - a| < r \Rightarrow (x = a) \text{(since } (a - r, a + r) \cap E = \{a\})$$

$$\Rightarrow f(x) = f(a) \Rightarrow |f(x) - f(a)| = 0 < \varepsilon$$

As a particular case, any function $f : N \to R$ is continuous on N. In fact for any given $a \in N$, $\left(a - \frac{1}{2}, a + \frac{1}{2}\right) \cap N = \{a\}$ .Hence f is continuous at a. As being arbitrary, f is continuous on N.

9.2.3 Geometric interpretation of continuity at a point:

If $f : R \to R$ is a function recall that the set $\{(x, f(x)) \in R \times R; x \in R\}$ is called the graph of f.

If $[a,b]$ and $[c,d]$ are intervals in R then the Cartesian product $[a,b] * [c,d]$ is called a rectangle R in the plane. Infact $R = [a,b] \times [c,d] = [(x, y) : a \le x \le b, c \le y \le d$ (see the fig).Note that $b - a$ is the width and $d - c$ is the height of the rectangle.

Now recall that f is continuous at a point x_0 in R if and only if to each $\varepsilon > 0$,there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.That is f is continuous at x_0 if and only if to each $\varepsilon > 0$ there is a $\delta > 0$ such that $(x, f(x)) \in (x_0 - \delta, x_0 + \delta) \times (f(x_0) - \varepsilon, f(x_0) + \varepsilon) = R_{\varepsilon\delta}$ (say).

Thus is continuous at $x_0 \in R$ if and only if to each $\varepsilon > 0$ there is a $\delta > 0$ such that the part of the graph of f is given by

$\{(x, f(x) : x \in (x_0 - \delta, x_0 + \delta)\} \subseteq R_{\varepsilon\delta}$ (see fig)

That is, as the height 2ε of the rectangle $R_{\varepsilon\delta}$ is sufficiently small, a part of the graph of f is contained in $R_{\varepsilon\delta}$.

9.2.4 Remark: Let $E \subseteq R, f : E \to R \,\&\, a \in E$ be such that $((a-r, a+r) - \{a\}) \cap E$ is non-empty for every $r > 0$. Then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$. The following conditions should be valid.

(i) f should be defined at a,

(ii) $\lim_{x \to a} f(x)$ must exist

(iii) $f(a) = \lim_{x \to a} f(x)$

Theorem analogous to theorem 9.2.5 can also be had for continuous functions.

9.2.5 Theorem: Let $E \subseteq R$, f, g be functions from E into R and c belongs R. Suppose a belongs to E and f, g are continuous at a, then

(i) $f+g, f-g, fg, cf$ are all continuous at a.

(ii) If, in addition $g(a)$ is not equal to 0 then the quotient $\frac{f}{g}$ is continuous at a

9.2.6 Observation:

When f and g are continuous at a we have $\lim_{x \to a} f(x) = f(a) \,\&\, \lim_{x \to a} g(x) = g(a)$ .Now by theorem 9.2.5 we get

(i) $\lim_{x \to a} (f+g)x = f(a) + g(a)$

(ii) $\lim_{x \to a} (f-g)x = f(a) - g(a)$

(iii) $\lim_{x \to a} (fg)x = f(a)g(a)$

(iv) $\lim_{x \to a} (cf)x = cf(a)$ for any c belongs to R

(v) Also if g(a) not equal to 0 then $\lim_{x \to a} \left(\frac{f}{g}\right)x = \frac{f(a)}{g(a)}$

9.2.7 Theorem:

Let $*E \subseteq R$ f, g be real valued continuous functions on E and c belongs to R. Then

(i) $f+g, f-g, fg, cf$ are all continuous on E

(ii) If, in addition $g(x)$ not equal to zero for all x belongs to E then $\frac{f}{g}$ is continuous on E.

Proofs of these two theorems are not given, as they are beyond the scope of this book.

9.2.8 Definition (Right and left continuities):

Let $E \subseteq R$, $f : E \to R$ be a function. Let $a \in E$ & $((a-r, a+r)-\{a\}) \cap E \neq 0$ for every $r > 0$ We say that a function f is right continuous at 'a' if $\lim_{x \to a+} f(x)$ exists and is equal to $f(a)$. Similarly we say that f is left continuous at 'a' if $\lim_{x \to a-} f(x)$ exists and is equal to $f(a)$.

9.2.9 Theorem: Let $E \subseteq R$, $f : E \to R$ be a function. Let a belongs to E. Then f is continuous at a if and only if $\lim_{x \to a+} f(x)$ & $\lim_{x \to a-} f(x)$ both exist and

$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$.

The proof of this theorem is beyond the scope of the book.

9.2.10 Note:

1. f is continuous on the closed interval $[a, b]$ if
   (i) f is continuous in (a, b)
   (ii) $\lim_{x \to a+} f(x) = f(a)$
   (iii) $\lim_{x \to a-} f(x) = f(b)$
2. Let $E \subseteq R, a \in E$, f is discontinuous at a point $x = a$ in any one of the following cases
   (i) $\lim_{x \to a+} f(x)$ & $\lim_{x \to a-} f(x)$ exist, but are not equal.
   (ii) $\lim_{x \to a+} f(x)$ & $\lim_{x \to a-} f(x)$ exist and are equal, but not equal to $f(a)$.
   (iii) One or both of the two limits $\lim_{x \to a+} f(x)$ & $\lim_{x \to a-} f(x)$ fail to exist.

9.2.11 Theorem: Let $A \subseteq R, f : A \to R$ and let $f(x) \geq 0$ for all x in A. Let $\sqrt{f}$ be defined for x belongs to A by $\left(\sqrt{f}\right)(x) = \sqrt{f(x)}$ .then the following conclusions hold

(i) If f is continuous at a point c in A, then $\sqrt{f}$ is continuous at c

(ii) If f is continuous on A, then $\sqrt{f}$ is continuous on A.

The continuity behaviour of the composition of two continuous functions is given in the follow theorems.

9.2.12 Theorem:

Let $A, B \subseteq R, f : A \to R$ and $g : B \to R$ be two functions such that $f(A) \subseteq B$ continuous at a point c in A and g is continuous at $b = f(c)$ in B then the composition $g \circ f$: is continuous at c.

9.2.13 Theorem: Let $A, B \subseteq R, f : A \to R$ be continuous on A and let be continuous of $f(A) \subseteq B$ then the composite function $g \circ f$: A tends to R is continuous on A.

9.2.14 Solved problems:

1. **Problem:** Show that $f(x)=[x](x \in R)$ is continuous at only those real numbers that are integers.

Solution:

Case 1: If $a \in Z$ then $f(a)=[a]=a$.

Now $\lim_{x \to a-} f(x) = \lim_{h \to 0}[a-h] = a-1, \lim_{x \to a+} f(x) = \lim_{h \to 0}[a+h] = a$

Hence, $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$ so that $\lim_{x \to a} f(x)$ does not exist.

Case 2: If $a \notin Z$, then there exists $n \in Z$ such that $n < a < n+1$ and $f(a)=[a]=n$

Now, $\lim_{x \to a-} f(x) = \lim_{h \to 0}[a-h] = 0, \lim_{x \to a+} f(x) = \lim_{h \to 0}[a+h] = n$

So $\lim_{x \to a} f(x) = n = f(a)$

Hence f is continuous at $x = a$.

2. **Problem:** Show that the function 'f' defined on R by $f(x) = \cos x^2, x \in R$ is continuous function.

Solution: We define $h : R \to R$ by $h(x) = x^2$ and $g : R \to R$ by $g(x) = \cos x$. Now for x belongs to R we have $(goh)(x) = g(h(x)) = g(x^2) = \cos x^2 = f(x)$

Since g and h are continuous on their respective domains, by theorem 9.2.13, it follows that 'f' is a continuous function on R.

3. **Problem:** If $f : R \to R$ is such that $f(x+y) = f(x) + f(y) \forall x, y \in R$, the '$f$' is continuous on R if it continuous at a single point in R.

Solution: Let 'f' be continuous at $x_0 \in R$

Then $\lim_{t \to x_o} f(t) = f(x_o)$ or $\lim_{h \to 0} f(x_o + h) = f(x_o)$

Let $x \in R$. Now since $f(x+h) - f(x) = f(h) = f(x_o + h) - f(x_o)$, we have

$$\lim_{h \to 0}(f(x+h) - f(x)) = \lim_{h \to 0}(f(x_o + h) - f(x_o)) = 0$$

Therefore 'f' is continuous at 'x'. Since $x \in R$ is arbitrary, 'x' is continuous on R.

4. **Problem:** Show that the function 'f' is defined on R by $f(x) = \left|1 + 2x + |x|\right|, x \in R$ is a continuous function.

Solution: We define $g: R \to R$ by $g(x) = 1 + 2x + |x|, x \in R$ and $h: R \to R$ by $h(x) = |x|, x \in R$. Then $(hog)(x) = h(g(x)) = h(1 + 2x + |x|) = f(x)$

By example we have 'h' is a continuous function. Since 'g' is the sum of the polynomial function $1 + 2x$ and the modulus function $|x|$ and since both are continuous functions, by theorem 9.2.7 (i), 'g' is continuous.

Since 'f' is the composition of two continuous functions 'h' and 'g' by theorem 9.2.13, it follows that 'f' is continuous.

Exercise 9(e)

I. 1. Is the function f , defined by $f(x) = \begin{cases} x^2 & if \quad x \le 1 \\ x & if \quad x > 1 \end{cases}$,continuous on R

2. Is f defined by $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & if \quad x \ne 0 \\ 1 & if \quad x = 0 \end{cases}$ continuous at 0

3. Show that the function $f(x) = [\cos(x^{10} + 1)]^{1/3}, x \in R$ is a continuous function.

II. 1. Check the continuity of the following function at 2.

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & if \quad 0 < x < 2 \\ 0 & if \quad x = 2 \\ 2 - 8x^{-3} & if \quad x > 2 \end{cases}$$

3. Check the continuity of f given by $f(x) = \begin{cases} \dfrac{(x^2 - 9)}{x^2 - 2x - 3} & if \quad 0 < x < 5 \\ 1.5 & if \quad x = 3 \end{cases}$ & $x \ne 3$ the point 3.

3. Show that f , given by $f(x) = \dfrac{x - |x|}{x} (x \ne 0)$ is continuous on $R - \{0\}$.

4. If f is a function defined by $f(x) = \begin{cases} \dfrac{x - 1}{\sqrt{x} - 1} & if \quad x > 1 \\ 5 - 3x & if \quad -2 \le x \le 1 \\ \dfrac{6}{x - 10} & if \quad x < -2 \end{cases}$ then discuss the continuity of f .

5. If f is given by $f(x)=\begin{cases} k^2x-k & if \quad x\geq 1 \\ 2 & if \quad x<1 \end{cases}$ is a continuous function on R then find the values of k.

6. Prove that the functions $\sin x$ and $\cos x$ are continuous on R.

III 1. Check the continuity of f given by $f(x)=\begin{cases} 4-x^2 & if \quad x\leq 0 \\ x-5 & if \quad 0<x\leq 1 \\ 4x^2-9 & if \quad 1<x<2 \\ 3x+4 & if \quad x\geq 2 \end{cases}$ at the points 0, 1, 2.

2. Find real consonants a,b so that the unction f given by

$$f(x)=\begin{cases} \sin x & if \quad x\leq 0 \\ x^2+a & if \quad 0<x<1 \\ bx+3 & if \quad 1\leq x\leq 2 \\ -3 & if \quad x>3 \end{cases}$$

is continuous on R.

3. Show that $f(x)=\begin{cases} \dfrac{\cos ax-\cos bx}{x^2} & if \quad x\neq 0 \\ \dfrac{1}{2}\left(b^2-a^2\right) & if \quad x=0 \end{cases}$ where a and b are real constants, is continuous at 0.

Key Concepts

1. If $\lim\limits_{x\to a} f(x)=l$ and $\lim\limits_{x\to a} g(x)=m$. And k ∈ R, then

(i) $\lim\limits_{x\to a}(f+g)(x)=l+m, \lim\limits_{x\to a}(f-g)(x)=l-m, \lim\limits_{x\to a}(fg)(x)=lm$

$\lim\limits_{x\to a}(kf)(x)=kl$

(ii) If $h:E\to R$ and $\lim\limits_{x\to a} h(x)=n\neq 0$ then h is never zero in

$E\cap((a-r,a+r)\setminus\{a\})$ for some r>0 , $\lim\limits_{x\to a}\left(\dfrac{1}{h}\right)(x)=\dfrac{1}{n}$ & $\lim\limits_{x\to a}\left(\dfrac{f}{h}\right)(x)=\dfrac{1}{n}$.

2. If p is a polynomial function (i.e. a function p(x) of the form $a_0+a_1x+.......+a_kx^k, k\geq 1$) then $\lim\limits_{x\to a} p(x)=p(a)$

3. $\lim\limits_{x\to a} x^n=a^n, a\in R, n\in N$

4. (Sandwich theorem)

Let $E \subseteq R$, $f, g, h: E \to R$ and let $a \in R$ be such that $((a-r, a+r)\setminus\{a\}) \cap E \neq \varphi$ for every r>0.If $f(x) \leq g(x) \leq h(x)$ for all $x \in R, x \neq a$ and if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x),$ then $\lim_{x\to a} g(x)$ exists and is equal to *l*.

5. If F and G are polynomials such that $F(x) = (x-a)^k f(x), G(x) = (x-a)^k g(x)$ for some $k \in N$ and for some polynomials f(x) and g(x) with $g(a) \neq 0$ then $\lim_{x\to a}\left(\frac{F}{G}\right)(x) = \frac{f(a)}{g(a)}$

6. If a>0, $n \in R$ then $\lim_{x\to a} x^n = a^n$

7. Let n be a rational number and a be a positive real number. Then $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

8. $\lim_{x\to 0} \cos x = 1 \,\&\, \lim_{x\to 0} \frac{\sin x}{x} = 1$

9. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

10. $\lim_{x\to 0}\left(\frac{a^x - 1}{x}\right) = \log_e a$

11. $\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = 1$

12. $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

13. Let $E \subseteq R, f: E \to R \,\&\, a \in R$ be such that $E \bigcap ((a-r, a+r)\setminus\{a\})$ is non empty for every r>0.

(i) Suppose $\lim_{x\to a} f(x) = \infty$ .Then $\lim_{x\to a} \frac{1}{f(x)} = 0$

(ii) Suppose $\lim_{x\to a} f(x) = -\infty$ .Then $\lim_{x\to a} \frac{1}{f(x)} = 0$

(iii) If $\lim_{x\to a} \frac{1}{f(x)} = 0$ and f is positive in a deleted neighbourhood of a,

then $\lim_{x\to a} \frac{1}{f(x)} = \infty$.

(iv) If $\lim_{x\to a}\frac{1}{f(x)}=0$ and f is negative in a deleted neighbourhood of a ,then

$\lim_{x\to a}\frac{1}{f(x)}=-\infty$

14. Let $E\subseteq R, f:E\to R, g:E\to R, h:E\to R$ and let $(a,\infty)\subset E$ for some $a\in R$. If $\lim_{x\to\infty} g(x)=l=\lim_{x\to\infty} h(x)$ & $g(x)\le f(x)\le h(x)$ for all $x\in R$ then $\lim_{x\to\infty} f(x)=1$.

Answers

Exercise 9(a)

1. 0 2. $\frac{108}{7}$ 3. $\frac{-1}{3}$ 4. $\frac{1}{\sqrt{24}}$

Exercise 9(b)

1. $Rf(1)=2,\ Lf(1)=0$. 2. $Rf(3)=9,\ Lf(3)=5$. 3. $Rf(2)=\frac{4}{3},\ Lf(2)=1$. 6. 4, 3.

Exercise 9(c)

1. -1 2. $\frac{1}{2a}$ 3. $2b\cos a$ 4. e^3 5. 1 6. $\frac{1}{10}$

Exercise 9(d)

1. ∞ 2. ∞ 3. ∞ 4. 0 5. 11 6. $\frac{1}{2}$ 7. 2 8. $\frac{11}{13}$ 9. 0

10. $\frac{1}{2}$ 11. -2 12. 0 13. 0 14. 0

Exercise 9(e)

I. 1. yes 2. discontinuous at 0

II. 1. discontinuous at 2 2. continuous at 3 4. continuous at 1, discontinuous at -2

5. $k=-1,2$

III 1. discontinuous at 0, discontinuous at 1, discontinuous at 2. . .

3. $a=0, b=-2$

10. DIFFERENTIATION

Introduction:

We shall discuss in this chapter, the derivative which forms a basis for the fundamental concepts like velocity, acceleration and the slope of a tangent to a curve and so on. The credit goes to the great English mathematician Sir Isaac Newton (1642-1727) and the noted German mathematician Gottfried Wilhelm Leibnitz (1646-1716) who independently conceived this idea simultaneously. Sir Isaac Newton was the most distinguished student of his distinguished teacher Isaac Barrow.

Suppose $f: I \to R$ is a function, I being an interval. We usually denote it by the equation $y = f(x)$, where x is the independent variable and y is the dependent variable. Let c be a point in I. Let $c+h$ also be a point in I lying either to the left side of c or to the right side of c ($c+h < c$ if $h < 0$; $c+h > c$ if $h > 0$).Then $f(c+h) - f(c)$ denotes the change in $f(x)$ corresponding to a change h in x at c. The ratio $\dfrac{f(c+h)-(c)}{h}$ is called the average change in $f(x)$ corresponding to a change of h in x at c. If this ratio tends to a finite limit as h approaches zero, then the limit the derivative (or the rate of change) of f at c.

10.1 Derivative of a function:

We begin with the definition of the derivative of a function and later prove certain important elementary properties of the derivatives.

10.1.1 Definition: Let I be an interval in R, $f: I \to R, a \in I$ and let $|h| \neq 0$ be sufficiently small such that $a+h \in I$. If $\lim_{h \to 0} \dfrac{f(c+h)-(c)}{h}$ exists, then f is said to be differentiable at a and the limit is called the derivative of f at a (or the differential coefficient of f at a). The derivative of f at a is denoted by any one of the forms.

$$f'(a) \text{ or } \left(\frac{dy}{dx}\right)_{x=a} \text{ or } y'(a) \text{ where } y = f(x)$$

The definition of derivative is also called the 'first principle of derivative'

Observe that if a is not an end point of the interval I, then $f^{'}(a)$ exists if and only if $\lim_{h \to 0+} \dfrac{f(a+h)-(a)}{h}$ & $\lim_{h \to 0-} \dfrac{f(a+h)-(a)}{h}$ both exist and are equal. The limits $\lim_{h \to 0+} \dfrac{f(a+h)-(a)}{h}$ & $\lim_{h \to 0-} \dfrac{f(a+h)-(a)}{h}$, if exists are denoted by $f'(a+)$ & $f'(a-)$ respectively and are called the right and left hand derivative of f at a.

If $f:[c,d]\to R$, then f is said to be differentiable.

(i) At c if $f'(c+)$ exists

(ii) At d if $f'(d-)$ exists

Suppose $A\subseteq I$ and f is differentiable at every $x\in A$.The function that assigns $f'(x)$ to each x belongs to A is called the derived function or the derivate of f and is denoted by f'. The process of finding the derivative of a function is called differentiation.

10.1.2 Note: If we denote the change h in x by Δx and the change in y by Δy, then $\Delta y = f(a+\Delta x)-f(a)$. Moreover $f'(a)=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-(a)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

We also note that $f'(a)=\lim_{x\to a}\frac{f(x)-(a)}{x-a}$

10.1.3 Solved problems:

1. **Problem:** If $f(x)=x^2(x\in R)$, prove that f is differentiable on R and find its derivative.

Solution: Given that $f(x)=x^2(x\in R)$

For $x,h\in R$ we have $f(x+h)-f(x)=(x+h)^2-x^2=2xh+h^2=h(2x+h)$

Hence for $h\neq 0, \frac{f(x+h)-f(x)}{h}=2x+h$

Therefore $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2x$

Therefore f is differentiable on R and $f'(x)=2x$ for each $x\in R$.

2. **Problem:** Suppose $f(x)=\sqrt{x}(x>0)$. Prove that f is differentiable on $(0,\infty)$ and find $f'(x)$.

Solution: Let $x\in(0,\infty), h\neq 0\,\&\,|h|<x$

Then

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\left(\sqrt{x+h}-\sqrt{x}\right)(\sqrt{x+h}+\sqrt{x})}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Therefore $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{1}{2\sqrt{x}}$

Hence f is differentiable at 'x' and $f'(x)=\dfrac{1}{2\sqrt{x}}$ for each $x\in(0,\infty)$

3. **Problem:** If $f(x)=\dfrac{1}{x^2+1}(x\in R)$, prove that f is differentiable on R and find $f'(x)$.

Solution: Let $x\in R$. Then for $h\neq 0$, we have

$$\frac{f(x+h)-f(x)}{h}=\frac{1}{h}\left[\frac{1}{(x+h)^2+1}-\frac{1}{x^2+1}\right]=\frac{-h(2x+h)}{h(x^2+1)\left[(x+h)^2+1\right]}=\frac{-(2x+h)}{(x^2+1)\left[(x+h)^2+1\right]}$$

Therefore $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\dfrac{-2x}{(x^2+1)^2}$

Hence f is differentiable at 'x' and $f'(x)=\dfrac{-2x}{(x^2+1)^2}$ for each x belongs to R.

4. **Problem:** If $f(x)=\sin x(x\in R)$, then show that f is differentiable on R and $f'(x)=\cos x$.

Solution: Let $x\in R$. Then for $h\neq 0$, we have

$$\frac{f(x+h)-f(x)}{h}=\frac{\sin(x+h)-\sin x}{h}=\frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h}$$

Therefore, $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left\{\cos\left(x+\dfrac{h}{2}\right).\dfrac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}=\cos x$

Hence f is differentiable at x on R and $f'(x)=\cos x$ for each x belongs to R.

5. **Problem:** Show that $f(x)=|x|(x\in R)$ is not differentiable at zero and is differentiable at any $x\neq 0$.

Solution: We have to show that $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist.

Given that $f(x)=|x|, f(h)=\begin{cases} h & if\ h\geq 0 \\ -h & if\ h<0\end{cases}$

Thus, for $h\neq 0$, $\dfrac{f(0+h)-f(0)}{h}=\dfrac{f(h)}{h}=\begin{cases} 1 & if\ h>0 \\ -1 & if\ h<0\end{cases}$

Therefore $f'(0+)=1\ \&\ f'(0-)=-1$

Hence f is differentiable at zero. It can be easily proved that f is differentiable at any $x \neq 0$ and that $f^{'}(x) = \begin{cases} 1 & if\ x > 0 \\ -1 & if\ x < 0 \end{cases}$

6. **Problem:** Check whether the following function is differentiable at zero $f(x) = \begin{cases} 3 + x\ \ if\ \ x \geq 0 \\ 3 - x\ if\ x < 0 \end{cases}$

Solution: We show that f has the left and the right hand derivative at zero and find them.

First we observe that, for $h \neq 0$, $f(x) = \begin{cases} 3 + x\ if\ x \geq 0 \\ 3 - x\ if\ x < 0 \end{cases}$ and $f(0) = 3$

Therefore, for $h > 0$, we have $\dfrac{f(0+h) - f(0)}{h} = \dfrac{f(h) - 3}{h} = \dfrac{3 + h - 3}{h} = 1$

Hence, $\lim\limits_{h \to 0+} \dfrac{f(0+h) - f(0)}{h} = 1$. Thus f has right hand derivative at zero and $f'(0+) = 1$

Similarly, $\lim\limits_{h \to 0-} \dfrac{f(0+h) - f(0)}{h} = -1$, so that f has left hand derivative at zero and $f'(0-) = -1$.

Therefore $f'(0-) \neq f'(0+)$.

Hence f is not differentiable at zero.

Note that the function in the problem can be rewritten as $f(x) = 3 + |x|$, which is not differentiable at zero. (see problem 5 above)

7. **Problem:** Show that the derivative of a constant on an interval is zero.

Solution: Let f be a constant function on an interval I, Then x for all $f(x) = \sin x (x \in R)$ belongs to I for some constant c. Let $a \in I$ .Then for $h \neq 0$, $\dfrac{f(a+h) - f(a)}{h} = \dfrac{c - c}{h} = 0$ for sufficiently small $|h|$.

Hence $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = 0$

Hence f is differentiable at a and $f'(a) = 0$.

Thus the derivative of a constant function is zero.

10.1.3 Elementary properties:

Now we shall prove certain important properties of derivatives of functions.

10.1.4 Theorem: Let I be an interval in R, $f: I \rightarrow R$ & $a \in I$. If f is differentiable at a, then f is continuous at a.

Proof: Suppose that f is differentiable at a. Then we have $\lim_{x \to a} \frac{f(x)-(a)}{x-a} = f'(a)$

Now $f(x) - f(a) = \frac{f(x)-f(a)}{x-a}.(x-a)$ $(x \neq a)$ and

Therefore $\lim_{x \to a} [f(x) - f(a)] = f'(a).0 = 0$

That is $\lim_{x \to a} f(x) = f(a)$ proving f is continuous at a.

If f is differentiable at a, then f is continuous at a.

10.1.4 Note: The converse of the above theorem is not true. That is If f is continuous at a, then f need not be differentiable at a.

For example, $f(x) = |x| (x \in R)$ is continuous at zero but not differentiable at zero.

10.1.5 Theorem (the derivate of the sum and difference of two functions):

Let I be an interval in R, u and v be real valued functions on I and x belongs to I, Suppose that u and v are differentiable at x. Then $u+v$ is also differentiable at x and $(u+v)'(x) = u'(x) + v'(x)$.

Proof: Let $f = u + v$

Then for sufficiently small non –zero values of $|h|$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{u(x+h)+v(x+h)-u(x)-v(x)}{h} = \left[\frac{u(x+h)-u(x)}{h}\right] + \left[\frac{v(x+h)-v(x)}{h}\right]$$

Which tends to $u'(x) + v'(x)$ as h tends to 0. Hence f is differentiable at x and $f'(x) = u'(x) + v'(x) \Rightarrow (u+v)'(x) = u'(x) + v'(x)$

We may similarly prove that $(u-v)'(x) = u'(x) - v'(x)$

10.1.6 Corollary: If $u_1, u_2, u_3 .... u_n$ are real valued functions on an interval I and are differentiable at x belongs to I then $v = u_1, u_2, u_3 .... u_n$ is also differentiable at x and $v'(x) = u_1'(x), u_2'(x), u_3'(x) .... u_n'(x)$. (Proof is easy).

10.1.7 Theorem (The derivative of the product of two functions):

Let I be an interval, u and v be real valued functions on I and x belongs to I. Suppose that u and v are differentiable at x. Then $u.v$ is differentiable at x and $(uv)'(x) = u(x)v'(x) + u'(x)v(x)$.

Proof: Let $f = u.v$. Then for sufficiently small non-zero of $|h|$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{u(x+h)v(x+h)-u(x)v(x)}{h} = u(x+h)\left[\frac{v(x+h)-v(x)}{h}\right] + v(x)\left[\frac{u(x+h)-u(x)}{h}\right]$$

Since u is differentiable at x, it is continuous at x so that $\lim_{h\to 0} u(x+h) = u(x)$ .Hence

$\frac{f(x+h)-f(x)}{h} \to u(x)v'(x) + v(x)u'(x)$ as h tends to 0. Thus $f = u.v$ is differential at x and $(uv)'(x) = u(x)v'(x) + v(x)u'(x)$.

As a consequence of mathematical induction and theorem 9.2.5 the following result follows.

10.1.8 Corollary: If $u_1, u_2, u_3 .... u_n$ are real valued functions on an interval I and are differentiable at x belongs to I, then $u_1, u_2, u_3 .... u_n$ is also differentiable at x and

$$u'(x) = \sum_{j=1}^{n} (u_1, u_2 .... u_{j-1}, u_{j+1} .... u_n)(x) u_j'(x).$$

10.1.9 Corollary: If u, v are real valued functions on an interval I and are differentiable at x belongs I and α, β are any constants, then $\alpha u + \beta v$ is also differentiable at x and

$$(\alpha u + \beta v)' = \alpha u' + \beta v'$$

10.1.10 Note: If u is a real valued function on an interval I and is differentiable at x belongs to I then $v = u'(n \in N)$ is differentiable at x and $v'(x) = nu^{n-1}(x).u'(x)$.

For take $u_1 = u_2 = ... = u_n = u$ in corollary 9.2.6.

10.1.11 Theorem (The derivative of the reciprocal of a function):

Let f be a function defined on an interval I such that $f(t) \neq 0$ for any t belongs to I and f is differential at x belongs to I. Then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$.

Proof: Since f is differentiable at x, it is continuous at x. Given that $f(x)$ is not equal to 0.

Now write $g = \frac{1}{f}$. Then for sufficiently small non- zero values of $|h|$, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x)-f(x+h)}{hf(x)f(x+h)}$$

$$\Rightarrow \frac{g(x+h)-g(x)}{h} = \left[\frac{f(x+h)-f(x)}{h}\right].\frac{1}{f(x)f(x+h)} \text{....(I)}$$

From the hypothesis, we have $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ & $\lim_{h\to 0} f(x+h) = f(x)$

Hence from (I) it follows that $\frac{g(x+h)-g(x)}{h} \to \frac{f'(x)}{(f(x))^2}$ as h tends to 0.

Therefore g is differentiable at x and $g'(x) = -\frac{f'(x)}{[f(x)]^2}$

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}.$$

10.1.12 Theorem (The derivative of the quotient of two functions):

Let u and v be real valued functions on an interval I such that v is never zero on I and let u and v be differentiable at x belongs to I. Then $\frac{u}{v}$ is differentiable at x and $\left(\frac{u}{v}\right)'(x) = -\frac{1}{[v(x)]^2}\left[v(x)u'(x)-u(x)v'(x)\right]$.

Proof: From the theorem 9.2.9 it follows that $\frac{1}{v}$ is differentiable at x and $\left(\frac{1}{v}\right)'(x) = -\frac{1}{[v(x)]^2}$. From theorem 9.2.5 it follows that $u.\frac{1}{v}$ is differentiable at x and

$$\left(\frac{u}{v}\right)'(x) = \left(u.\frac{1}{v}\right)'(x) = u(x)\left(\frac{1}{v}\right)'(x) + u'(x)\frac{1}{v}(x)$$

$$= u(x)\left(\frac{-v'(x)}{(v(x))^2}\right) + \frac{u'(x)}{v(x)} = \frac{1}{(v(x))^2}\left[v(x)u'(x)-u(x)v'(x)\right]$$

$$\left(\frac{u}{v}\right)'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{\left[v(x)\right]^2}$$

10.1.13 Theorem (The derivative of a composite function):

Let I be an interval, $g : I \to R$ and f be a real valued function on an interval containing g(I). Suppose that g is differentiable at x and f is differentiable at $g(x)$. Let $F = f \circ g$ (so that $F(x) = f(g(x))$).Then F is differentiable at x and $F'(x) = f'(g(x))g'(x)$.(This is also known as *chain rule* for differentiation)

Proof: Write $y = g(x)$

Let us define a function φ in a neighbourhood of zero as follows.

$$\varphi(\kappa) = \begin{cases} \dfrac{f(y+k)-f(y)}{k} - f'(y) & if \;\; k \neq 0 \\ 0 & if \;\; k = 0 \end{cases}$$

Since f is differentiable at $y = g(x)$ we have $\lim\limits_{k \to 0} \dfrac{f(y+k)-f(y)}{k} = f'(y)$

Hence $\lim\limits_{k \to 0} \varphi(k) = 0$

Moreover $f(y+k) - f(y) = kf'(y) + k\varphi(k)$ for k not equal to zero

Write $\psi(h) = g(x+h) - g(x)$ for h not equal to zero.

Then $\dfrac{F(x+h)-F(x)}{h} = \dfrac{f(g(x+h)) - f(g(x))}{h}$

$$= \frac{f(y+\psi(h)) - f(y)}{h} = \frac{\psi(h)}{h} f'(y) + \frac{\psi(h)}{h}\theta(\psi(h)) \quad \text{(by 1)}$$

$$= \left[\frac{g(x+h)-g(x)}{h}\right] f'(y) + \left[\frac{g(x+h)-g(x)}{h}\right]\varphi(\psi(h))$$

$$\to g'(x) f'(y) + g'(x).0; h \to 0$$

Since $g(x)$ is differentiable at x, $\psi(h) \to 0 \;\; as \;\; h \to 0, \;\; \varphi(k) \to 0 \;\; as \;\; k \to 0$

Hence $\lim\limits_{h \to 0} \dfrac{F(x+h)-F(x)}{h}$ exists and is equal to $f'(y)g'(x)$

i.e.., F is differentiable at x and $F'(x) = f'(g(x))g'(x)$.

Thus $(fog)'(x) = f'(g(x)).g'(x)$.

10.1.14 Note: If we write $z = f(y), y = g(x)$ in the above theorem, We get $\frac{dz}{dx}=\frac{dz}{dy}.\frac{dy}{dx}$.

The derivative of the inverse of a function is given in the following theorem.

10.1.15 Theorem (The derivative of the inverse of a function):

Let $f:[a,b]\to[c,d]$ be a bijection and g denote the inverse of f. Suppose that f is differentiable at $x\in(a,b), f'(x)\neq 0$ and g is continuous at $f(x)$. Then g is differentiable at $f(x)$ and $g'(f(x))=\frac{1}{f'(x)}$.

Proof: Let $y=f(x)$ Let k be a non zero real number such that $y+k\in[c,d]$

Let $g\left(y+k\right)-g\left(y\right)=h.$ Since g is one-to-one, h not equal to zero.

We have $g\left(y+k\right)=g\left(y\right)+h=x+h$

Hence, $f\left(x+h\right)=y+k$.

Hence $k=(y+k)-y=f(x+h)-f(x)$, since $g(x)$ is continuous at y $g(y+k)\to g(y)\;\; as\;\; k\to 0$ Hence $h\to 0\;\; as\;\; k\to 0$ .Since f is differentiable at x.

$$\frac{f(x+h)-f(x)}{h}\to f'(x)\;\; as\;\; h\to 0$$

Since $f'(x)\neq 0, \frac{h}{f(x+h)-f(x)}\to\frac{1}{f'(x)}\;\; as\;\; h\to 0$

We have $\frac{g(y+k)-g(y)}{k}=\frac{h}{f(x+h)-f(x)}$

Hence $\frac{g(y+k)-g(y)}{k}\to\frac{1}{f'(x)}\;\; as\;\; h\to 0$

Therefore g is differentiable at $f(x)$ and $g'(f(x))=\frac{1}{f'(x)}$

10.1.16 Note: If $y=f(x)$ then $x=f^{-1}(y)\,\&\,\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$

We shall now find the derivative of some standard functions.

10.1.17 Example: If $f(x)=e^{x}(x\in R)$, then show that $f'(x)=e^{x}$ by first principle.

Solution: From $f(x)=e^x$, we have for $h \neq 0$

$$\frac{f(x+h)-f(x)}{h}=\frac{e^{x+h}-e^x}{h}=e^x\,\frac{e^h-1}{h}$$

Therefore $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=e^x.\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}=e^x.1=e^x$

Therefore $f'(x)=e^x$ for each x belongs to R.

10.1.18 Example: If $f(x)=\log x (x>0)$, then $f'(x)=\frac{1}{x}$ by first principle.

Solution: Now for $h \neq 0$

$$\frac{f(x+h)-f(x)}{h}=\frac{\log(x+h)-\log x}{h}=\frac{1}{h}\log\left(1+\frac{h}{x}\right)=\frac{1}{h}.\frac{h}{x}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=\frac{1}{x}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

Now, putting $\frac{h}{x}=z$, we get $z\to 0$ as $h\to 0$

Therefore $\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=\log(1+z)^{\frac{1}{z}}\to \log e=1$ as $z\to 0$

Hence $\frac{f(x+h)-f(x)}{h}\to\frac{1}{x}$ as $h\to 0$.

Thus $f'(x)=\frac{1}{x}$ for each $x>0$.

$$\frac{d}{dx}(\log x)=\frac{1}{x}$$

10.1.19 Example: If $f(x)=a^x (x\in R)(a>0)$, then show that $f'(x)=a^x\log_e a$ by first principle.

Solution: For $h\neq 0$, $\frac{f(x+h)-f(x)}{h}=\frac{a^{x+h}-a^x}{h}=a^x\left(\frac{a^h-1}{h}\right)$

We know that $\frac{a^h-1}{h}\to\log a$ as $h\to 0$

Hence $f'(x)=a^x\log_e a$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

10.1.20 Solved problems:

1. **Problem:** If $f(x) = (ax+b)^n \left(x > \frac{-b}{a} \right)$, then find $f^{'}(x)$.

Solution: Write $u = ax + b$ so that $f(x) = u^n$. Then

$$f'(x) = \frac{d}{dx}(u^n).\frac{du}{dx}, \text{ by Note 9.2.12}$$

$$= nu^{n-1}.a = an(ax+b)^{n-1}.$$

2. **Problem:** Find the derivative of $f(x) = e^x(x^2+1)$.

Solution: Write $u = e^x, v = x^2 + 1$, so that $f(x) = u(x)v(x)$ and

$$f'(x) = u(x)v'(x) + u'(x)v(x), \text{ by theorem 9.2.5}$$

Now $u'(x) = e^x$ and $v'(x) = 2x$ imply that $f'(x) = e^x(2x) + (x^2+1)e^x = (x+1)^2 e^x$.

3. **Problem:** If $y = \frac{a-x}{a+x}(x \neq -a)$, find $\frac{dy}{dx}$

Solution: Write $u(X) = a - x, v(x) = a + x$, so that $y = \frac{u}{v}$

$$u'(x) = -1 \,\&\, v'(x) = 1$$

Therefore

$$\frac{dy}{dx} = \frac{1}{[v(x)]^2}[v(x)u'(x) - v'(x)u(x)] = \frac{1}{(a+x)^2}[(a+x)(-1) - (a-x)(1)] = \frac{-2a}{(a+x)^2}$$

4. **Problem:** If $f(x) = e^{2x}.\log x(x > 0)$, then $f'(x)$

Solution: Write $u(x) = e^{2x}, v(x) = \log x$, so that

$$f(x) = u(x)v(x), u'(x) = 2e^{2x}, v'(x) = \frac{1}{x}$$

Therefore $f'(x) = u(x)v'(x) + u'(x)v(x)$

$$= e^{2x}\frac{1}{x} + 2e^{2x}\log x = e^{2x}\left(\frac{1}{x} + 2\log x\right)$$

5. **Problem:** If $f(x)=\sqrt{\dfrac{1+x^2}{1-x^2}}(|x|<1)$, then find $f'(x)$.

Solution: Write $u(x)=\dfrac{1+x^2}{1-x^2}$ and $y=f(x)$. Then $y=f(x)=u^{1/2}$

Now by the chain rule, we get $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$ where in

$$\frac{dy}{du}=\frac{1}{2}u^{\frac{1}{2}-1}=\frac{1}{2}u^{\frac{-1}{2}}=\frac{1}{2\sqrt{u}}$$

And $$\frac{du}{dx}=\frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2}=\frac{4x}{(1-x^2)^2}$$

Therefore $$f'(x)=\frac{dy}{dx}=\frac{1}{2\sqrt{u}}.\frac{4x}{(1-x^2)^2}=\frac{2x}{(1-x^2)\sqrt{1-x^4}}$$

6. **Problem:** If $f(x)=x^2 2^x\log x(x>0)$, find $f'(x)$

Solution: Write $u(x)=x^2, v(x)=2^x$ & $w(x)=\log x$ so that $f(x)=(uvw)(x)$.

Then $$f'(x)=u'(x)v(x)w(x)+u(x)v'(x)w(x)+u(x)v(x)w'(x)$$

$$= 2x(2^x\log x)+(2^x\log 2)(x^2\log x)+x^2 2^x.\frac{1}{x}$$

$$= x2^x[\log x^2+x\log x\log 2+1]$$

7. **Problem:** If $f(x)=7^{x^3+3x}(x>0)$, then find $f'(x)$

Solution: Write $u(x)=x^3+3x$, so that $\dfrac{du}{dx}=3x^2+3$ and $f(x)=7^u$

Therefore, by the chain rule, we get $f'(x)=\dfrac{df}{du}.\dfrac{du}{dx}$

$$=(7^u\log 7)(3x^2+3)=3(x^2+1)7^{x^3+3x}\log 7$$

Exercise 10(a)

1. Find the derivatives of the following functions $f(x)$.

   (i) $\sqrt{x}+2x^{\frac{3}{4}}+3x^{\frac{5}{6}}(x>0)$ (ii) $\sqrt{2x-3}+\sqrt{7-3x}$ (iii) $(x^2-3)(4x^3+1)$

$(iv)\ (\sqrt{x}-3x)(x+\frac{1}{x})$ $(v)\ (\sqrt{x}+1)(x^2-4x+2)(x>0)$ $(vi)\ (ax+b)^n(cx+d)^m$

$(vii)\ 5\sin x+e^x\log x$ $(viii)\ 5^x+\log x+x^3e^x$ $(ix)\ e^x+\sin x\cos x$

$(x)\ \frac{px^2+qx+r}{ax+b}(|a|+|b|\neq 0)$ $(xi)\ \log_7(\log x)(x>0)$

$(xii)\ \frac{1}{ax^2+bx+c}(|a|+|b|+|c|)\neq 0$ $(xiii)\ e^{2x}\log(3x+4)\left(x>\frac{-4}{3}\right)$

$(xiv)\ (4+x^2)e^{2x}$ $(xv)\ \frac{ax+b}{cx+d}(|c|+|d|\neq 0)$

2. If $f(x)=1+x+x^2+..............+x^{100}$ then find $f'(1)$
3. If $f(x)=2x^2+3x-5$ then prove that $f'(0)+3f'(-1)=0$

4. Find the derivatives of the following functions from the first principles.

   $(i)\ x^3$ $(ii)\ x^4+4$ $(iii)\ ax^2+bx+c$ $(iv)\ \sqrt{x+1}$ $(v)\ \sin 2x$ $(vi)\ \cos ax$ $(vii)\ \tan 2x$ $(viii)\cot x$
5. Find the derivatives of the following functions

   $(i)\ \frac{1-x\sqrt{x}}{1+x\sqrt{x}}(x>0)$ $(ii)\ x^n n^x\log(nx)(x>0, n\in N)$ $(iii)\ ax^{2n}\log x+bx^n e^{-x}$

   $(iv)\ \left(\frac{1}{x}-x\right)^3 e^x$
6. Show that the function $f(x)=|x|+|x-1|, x\in R$ is differentiable for all real numbers except for 0 and 1.

7. Verify whether the following function is differentiable at 1 and 3.

$$f(x)=\begin{cases} x & if\ x<1 \\ 3-x & if\ 1\le x\le 3 \\ x^2-4x+3 & if\ x>3 \end{cases}$$

8. Is the following function f derivable at 2? Justify

$$f(x)=\begin{cases} x\ if\ 0\le x\le 2 \\ 2\ if\ x\ge 2 \end{cases}$$

10.2 Trigonometric, Inverse Trigonometric, Hyperbolic, Inverse Hyperbolic functions-Derivatives:

In this section we find the derivatives of trigonometric and hyperbolic function and also of their inverses.

10.2.1 Derivatives of trigonometric functions:

1. $\frac{d}{dx}(\sin x) = \cos x$

We have already proved this result.

2. $\frac{d}{dx}(\cos x) = -\sin x$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \cos\left(\frac{\pi}{2} - x\right).\frac{d}{dx}\left(\frac{\pi}{2} - x\right) = -\sin x$$

This result can be obtained from the first principle also.

$$\frac{d}{dx}(\cos x) = -\sin x$$

3. If $y = \tan x, x \in R - \left\{(2n+1)\frac{\pi}{2}; n \in Z\right\}$, then $\frac{dy}{dx} = \sec^2 x$

Now $y = \tan x = \frac{\sin x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x}\left[\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)\right]$

$$= \frac{1}{\cos^2 x}\left[\cos^2 x - \sin^2 x\right] = \sec^2 x$$

Similarly $\frac{d}{dx}(\tan x) = \sec^2 x$.

4. If $y = \cot x, x \in R - \{n\pi : n \in Z\}$, then $\frac{dy}{dx} = -\cos ec^2 x$

$$\frac{d}{dx}(\cot x) = -\cos ec^2 x$$

5. If $y = \sec x, x \in R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$, then

$$y = \frac{1}{\cos x} \;\&\; \frac{dy}{dx} = \frac{-1}{\cos^2 x}\frac{d}{dx}(\cos x) = \frac{\sin x}{\cos^2 x} = \tan x.\sec x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

6. If $y = \operatorname{cosec} x, x \in R - \{n\pi : n \in Z\}$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

10.2.2 Derivatives of inverse trigonometric functions:

Let us recall that if f and g are functions that $f(g(x)) = x$ and $g(f(y)) = y$ for any x any y and $f'(y) \neq 0$, then $g'(y) = \frac{1}{f'(y)}$, where $y = g(x)$.

Hence we have $y = g(x) \Leftrightarrow x = f(y)$ & $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$.

1. If $y = \sin^{-1} x, x \in [-1,1]$ then its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$y = \sin^{-1} x \Leftrightarrow x = \sin y \ \& \ \frac{dx}{dy} = \cos y .$$

If $-1 < x < 1$ then $\frac{-\pi}{2} < y < \frac{\pi}{2}$

Hence $\frac{dx}{dy} = \cos y > 0$. This implies

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} .$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

2. If $y = \cos^{-1} x, x \in [-1,1]$, then we have $y \in [0, \pi]$

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

$$x = \cos y \Rightarrow \frac{dx}{dy} = -\sin y$$

Hence $\frac{dy}{dx} = \frac{1}{-\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} .$$

3. If $y=\tan^{-1}x, x\in R$, then we know that $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$x=\tan y\Rightarrow\frac{dy}{dx}=\sec^2 y=1+\tan^2 y=1+x^2>0$$

Therefore $\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1}=\frac{1}{1+x^2}$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}.$$

4. $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}(x\in R,\cot^{-1}x\in(0,\pi))$

$$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}.$$

5. If $y=\sec^{-1}x, x\in R-[-1,1]$ then $y\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$.

$$x=\sec y\Rightarrow\frac{dx}{dy}=\sec y\tan y$$

$$|x|>1\Rightarrow y\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},0\right) \qquad (1)$$

$$\Rightarrow\frac{dx}{dy}=\sec y\tan y=\sec^2 y\sin y>0\left(y\neq\frac{\pi}{2}\right)$$

Now $x<-1\Rightarrow\sec y<-1\Rightarrow\tan y<0$ (from 1)

And $\tan^2 y=\sec^2 y-1, \tan y=-\sqrt{\sec^2 y-1}=-\sqrt{x^2-1}$ (since tany<0)

Therefore $\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1}=\frac{1}{\sec y\tan y}=\begin{cases}\frac{1}{x\sqrt{x^2-1}}; x>1\\ \frac{1}{-x\sqrt{x^2-1}}; x<-1\end{cases}$

$$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$$

6. If $y=\cos ec^{-1}x$, then $\frac{dy}{dx}=\frac{-1}{|x|\sqrt{x^2-1}}\left[x\in R-[-1,1], y\in\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\right]$

$$\frac{d}{dx}(co\sec^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}.$$

10.2.3 Derivatives of hyperbolic functions:

1. If $y=\sinh x(x\in R)$ then $\frac{dy}{dx}=\cosh x$.

For $y=\sinh x=\frac{e^x-e^{-x}}{2} \Rightarrow \frac{dy}{dx}=\frac{e^x+e^{-x}}{2}=\cosh x$

$$\frac{d}{dx}(\sinh x)=\cosh x$$

2. If $y=\cosh x(x\in R)$ then $\frac{dy}{dx}=\sinh x$.

For $y=\frac{e^x+e^{-x}}{2} \Rightarrow \frac{dy}{dx}=\frac{e^x-e^{-x}}{2}=\sinh x$.

$$\frac{d}{dx}(\cosh x)=\sinh x.$$

3. If $y=\tan hx\ (x\in R)$ then $\frac{dy}{dx}=\sec h^2 x$.

For $y=\tanh x=\frac{\sinh x}{\cosh x} \Rightarrow \frac{dy}{dx}=\frac{1}{\cosh^2 x}\left[(\cosh x)\frac{d}{dx}(\sinh x)-\sinh x\frac{d}{dx}(\cosh x)\right]$

$$=\frac{1}{\cosh^2 x}(\cosh^2 x-\sinh^2 x)=\frac{1}{\cosh^2 x}=\sec h^2 x$$

$$\frac{d}{dx}(\tanh x)=\sec h^2 x.$$

4. If $y=\sec hx(x\in R)$ then $\frac{dy}{dx}=-\sec hx\tanh x$.

For $y=\sec hx=\frac{1}{\cosh x} \Rightarrow \frac{dy}{dx}=\frac{-1}{\cosh^2 x}.\frac{d}{dx}(\cosh x)=-\frac{\sinh x}{\cosh^2 x}=-\sec hx\tanh x$.

$$\frac{d}{dx}(\sec hx)=-\sec hx\tanh x$$

5. If $y=\cos echx\ (x\in R-\{0\})$ then $\frac{dy}{dx}=-\cos echx\coth x$.

For $y=\cos echx=\frac{1}{\sinh x} \Rightarrow \frac{dy}{dx}=\frac{-1}{\sinh x}.\frac{d}{dx}(\sinh x)=\frac{-\cosh x}{\sinh^2 x}=-\cos echx\coth x$

$$\frac{d}{dx}(co\sec hx)=-\cos echx\coth x.$$

6. If $y=\coth x(x\in R-\{0\})$ then $\frac{dy}{dx}=-\cos ech^2 x$.

For $y = \coth x = \frac{1}{\tanh x} \Rightarrow \frac{dy}{dx} = \frac{-1}{\tanh^2 x}\frac{d}{dx}(\tanh x) = \frac{-\sec h^2 x}{\tanh^2 x} = -\cos ech^2 x$.

$$\frac{d}{dx}(\coth x) = -\cos ec^2 x$$

10.2.4 Derivatives of inverse hyperbolic functions:

1. If $y = \sinh^{-1} x (x \in R)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

$y = \sinh^{-1} x \Rightarrow x = \sinh y$

Hence $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

2. If $y = \cosh^{-1} x (x \in (1,\infty))$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

For $x > 1, x = \cosh y \Rightarrow \frac{dx}{dy} = \sinh y > 0$.

Therefore for $x > 1$, we have

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}.$$

3. If $y = \tanh^{-1} x (x \in (-1,1))$, then $\frac{dy}{dx} = \frac{1}{1-x^2}$.

For $x = \tanh y, \frac{dx}{dy} = \sec h^2 y = 1 = 1 - \tanh^2 y = 1 - x^2 > 0$.

Therefore $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{1-x^2}$.

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

4. If $y = \text{sech}^{-1} x (x \in (0,1))$, then $\dfrac{dy}{dx} = \dfrac{-1}{x\sqrt{1-x^2}}$.

For $x \in (0,1), y = \sec h^{-1} x = \cosh^{-1}\left(\dfrac{1}{x}\right)$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{\left(\dfrac{1}{x}\right)^2 - 1}} \times \dfrac{-1}{x^2} = \dfrac{-1}{x\sqrt{1-x^2}}$.

$$\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$$

5. If $y = \cos ec\text{h}^{-1} x \ (x \in R - \{0\})$ then $\dfrac{dy}{dx} = \dfrac{-1}{|x|\sqrt{1-x^2}}$

$$y = \cos ech^{-1} x = \sinh^{-1}\left(\frac{1}{x}\right)$$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+\left(\dfrac{1}{x}\right)^2}} \times \dfrac{-1}{x^2} = \dfrac{-1}{|x|\sqrt{1+x^2}}$.

$$\frac{d}{dx}(\cos ec\text{h}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}.$$

6. If $y = \coth^{-1} x (x \in (-\infty, -1) \cup (1, \infty))$ then $\dfrac{dy}{dx} = \dfrac{1}{1-x^2}$.

$$y = \coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{1-\dfrac{1}{x^2}} \times \left(\frac{-1}{x^2}\right) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

Observe that through the formulae for the derivatives of $\tanh^{-1} x, \coth^{-1} x$ are the same, their domains are disjoint.

10.2.5 Note: The formulae mentioned under 1, 2, 3 above can also be obtained by using the following identities.

$$\sinh^{-1} x = \log(x + \sqrt{1+x^2})$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})(x \geq 1)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)(1-x^2>0).$$

10.2.6 Note: Hereafter, in order to find the derivative of a function, even though its domain is not explicitly mentioned, we mean that in its appropriate domain the derivative exists and we have to find the same.

10.2.7 Solved Problems:

1. Problem: Find the derivative of $f(x) = \dfrac{x\cos x}{\sqrt{1+x^2}}$

Solution: Write $u(x) = x\cos x, v(x) = \sqrt{1+x^2}$ so that

$$f(x) = \left(\frac{u}{v}\right)(x) \text{ and } f'(x) = \frac{1}{[v(x)]^2}[v(x)u'(x) - v'(x)u(x)]$$

Here $\qquad u'(x) = \dfrac{d}{dx}(x\cos x) = \cos x - x\sin x$

And $\qquad v'(x) = \dfrac{d}{dx}\left(\sqrt{1+x^2}\right) = \dfrac{2x}{2\sqrt{1+x^2}} = \dfrac{x}{\sqrt{1+x^2}}$

Therefore the derivative of $f(x)$ is

$$f'(x) = \frac{1}{1+x^2}\left[\sqrt{1+x^2}(\cos x - x\sin x) - \frac{x^2\cos x}{\sqrt{1+x^2}}\right]$$

$$= (1+x^2)^{-3/2}[\cos x - x(1+x^2)\sin x]$$

2. **Problem:** If $f(x) = \log(\sec x + \tan x)$, then find $f'(x)$.

Solution: Write $u(x) = \sec x + \tan x$, $y = f(x)$ so that

$$y = \log u \;\&\; \frac{dy}{dx} = f'(x) = \frac{dy}{du}\times\frac{du}{dx}$$

Now $\quad \dfrac{dy}{du} = \dfrac{1}{u}, \dfrac{du}{dx} = \sec x\tan x + \sec^2 x$

Therefore $f'(x) = \dfrac{1}{u}(\sec x\tan x + \sec^2 x) = \sec x$

3. **Problem:** If $y = \sin^{-1}\sqrt{x}$, then find $\dfrac{dy}{dx}$

Solution: Write $u(x)=\sqrt{x}$, then $y=\sin^{-1}u$ and $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Hence $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\times\frac{1}{\sqrt{1-u^2}}=\frac{1}{2\sqrt{x-x^2}}$

4. **Problem:** If $y=\sec(\sqrt{\tan x})$, then find $\frac{dy}{dx}$.

Solution: Write $u=\sqrt{\tan x}$ then $y=\sec u, u=\sqrt{v}, v=\tan x$

Imply that $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$

Now $\frac{dy}{du}=\sec u\tan u, \frac{du}{dv}=\frac{1}{2\sqrt{v}}\ \&\ \frac{dv}{dx}=\sec^2 x$

Therefore $\frac{dy}{dx}=\frac{\sec^2 x}{2\sqrt{\tan x}}.\sec\left(\sqrt{\tan x}\right).\tan\left(\sqrt{\tan x}\right)$

5. **Problem:** If $y=\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$, then find $\frac{dy}{dx}$

Solution: Write $u=x\sin^{-1}x, v=\sqrt{1-x^2}$ so that $y=\frac{u}{v}$

Now $\frac{du}{dx}=\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x$ and $\frac{dv}{dx}=\frac{-2x}{2\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}$

$$\frac{dy}{dx}=\frac{1}{v^2}[vu'-v'u]$$

$$=\frac{1}{(1-x^2)}\left[\sqrt{1-x^2}\left(\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x\right)+\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}\right]$$

$$=\frac{1}{(1-x^2)^{3/2}}\left[x\sqrt{1-x^2}+\sin^{-1}x\right]$$

6. **Problem:** If $y=\log(\cosh 2x)$, then find $\frac{dy}{dx}$

Solution: Let $u=\cosh 2x$, so that $y=\log u$

Then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Here $\frac{dy}{du}=\frac{1}{u}\ \&\ \frac{du}{dx}=2\sinh 2x$

Hence $\frac{dy}{dx}=\frac{2}{u}\sinh 2x=\frac{2\sinh 2x}{\cosh 2x}=2\tanh 2x$

7. Problem: If $y=\log(\sin(\log x))$, then find $\frac{dy}{dx}$.

Solution: Write $v=\log x, u=\sin v$ so that $y=\log u$

Therefore $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$

$$=\frac{1}{u}\times\cos v\times\frac{1}{x}=\frac{\cos(\log x)}{x\sin x(\log x)}=\frac{1}{x}.\cot(\log x)$$

8. Problem: If $y=\left(\cot^{-1}x^3\right)^2$, then find $\frac{dy}{dx}$.

Solution: Put $u=\cot^{-1}x^3$ so that $y=u^2$

Then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=2u\times\frac{-1}{(1+x^6)}.3x^2=\frac{-6x^2\cot^{-1}(x^3)}{1+x^6}$.

Exercise 10(b)

1. Find the derivatives of the following functions

   $(i)\ \cot^n x$ $\quad(ii)\ \cos ec^4 x$ $\quad(iii)\ \tan(e^x)$ $\quad(iv)\ \frac{1-\cos 2x}{1+\cos 2x}$ $\quad(v)\ \sin^m x\cos^n x$

   $(vi)\ \sin mx.\cos nx$ $\ (vii) x\tan^{-1}x$ $\ (viii)\sin^{-1}(\cos x)$ $\ (ix)\log(\tan 5x)$ $\ (x)\sin^{-1}\left(\frac{3x}{4}\right)$

   $(xi)\tan^{-1}(\log x)$ $\ (xii)\ \log\left(\frac{x^2+x+2}{x^2-x+2}\right)$ $\ (xiii)\ \log(\sin^{-1}(e^x))$ $\ (xiv)\ (\sin x)^2(\sin^{-1}x)^2$

   $(xv)\ \frac{\cos x}{\sin x+\cos x}$ $\ (xvi)\ \frac{x(1+x^2)}{\sqrt{1+x^2}}$ $\ (xvii) e^{\sin^{-1}x}$ $\ (xvii)\cos(\log x+e^x)$ $\ (xix)\frac{\sin(x+a)}{\cos x}$

   $(xx)\cot^{-1}(\cos ec3x)$
2. Find the derivatives of the following functions

   $(i)\ x=\sinh^2 y$ $\quad(ii)\ x=\tanh^2 y$ $\quad(iii)\ x=e^{\sinh y}$

   $(iv)\ x=\tan(e^{-y})$ $\quad(v)\ x=\log(1+\sin^2 y)$ $\quad(vi)\ x=\log(1+\sqrt{y})$
3. Find the derivatives of the following functions

$(i)\ \cos(\log(\cot x))\ \ (ii)\ \sinh^{-1}\left(\dfrac{1-x}{1+x}\right)\ \ (iii)\ \log(\cot(1-x^2))\ \ (iv)\ \sin(\cos(x^2))$

$(v)\ \sin(\tan^{-1}(e^x))\ \ (vi)\ \dfrac{\sin(ax+b)}{\cos(cx+d)}\ (vii)\ \sin x.(\tan^{-1} x)^2$

4. Find the derivatives of the following functions

$(i)\ \sin^{-1}\left(\dfrac{b+a\sin x}{a+b\sin x}\right)(a>0, b>0)\qquad (ii)\ \cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)(a>0, b>0)$

$(iii)\ \tan^{-1}\left(\dfrac{\cos x}{1+\cos x}\right)$

10.3 Methods of differentiation:

Some times the formulae obtained so far, may prove to be difficult in finding the derivative of some typical functions. There are some special methods of differentiation to deal with such situations. Our main aim in this section is to discuss such methods.

This method is well illustrated in the following examples.

1. If $y=\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}(|x|<1)$, we shall find $\dfrac{dy}{dx}$

Substituting $x=\cos u (u\in(0,\pi))$ in y, we get

$$\frac{1-x}{1+x}=\frac{1-\cos u}{1+\cos u}=\frac{2\sin^2(u/2)}{2\cos^2(u/2)}=\tan^2(u/2)$$

So that $\qquad \sqrt{\dfrac{1-x}{1+x}}=\tan\left(\dfrac{u}{2}\right)$

And $\qquad y=\tan^{-1}\left(\tan\left(\dfrac{u}{2}\right)\right)=\dfrac{u}{2}$

Therefore, $\qquad \dfrac{dy}{du}=\dfrac{dy}{dx}.\dfrac{dx}{du}\Rightarrow\dfrac{1}{2}=\dfrac{dy}{dx}(-\sin u)$

Hence $\qquad \dfrac{dy}{dx}=\dfrac{-1}{2\sin u}=\dfrac{-1}{2\sqrt{1-x^2}}$

Observe that $\tan^{-1} x, \sqrt{\dfrac{1-x}{1+x}}$ and $\cos u$ are the functions that stand for $f(x), g(x)$ and $h(u)$ respectively, mentioned in the method.

2. If $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)(|x|<1)$ then we shall find $\dfrac{dy}{dx}$.

Substituting $x=\tan u$

We get $\dfrac{2x}{1-x^2} = \dfrac{2\tan u}{1-\tan^2 u} = \tan 2u$

And $y = \tan^{-1}(\tan 2u) = 2u$

Therefore from $\dfrac{dy}{du} = \dfrac{dy}{dx}.\dfrac{dx}{du}$

We get that $2 = \dfrac{dy}{dx}.\sec^2 u$

Therefore $\dfrac{dy}{dx} = 2\cos^2 u = \dfrac{2}{1+\tan^2 u} = \dfrac{2}{1+x^2}$

10.3.1 Substitution methods:

Let $y = f \circ g$ If we are able to find a function h such that $g \circ h = f^{-1}$, then the substitution $x = h(u)$ may give the derivative of y with respect to x easily (Here f is a bijection defined on an interval).

The method is well illustrated in the following examples.

1. If $y = \tan^{-1}\sqrt{\dfrac{1-x}{1+x}}(|x|<1)$, then we shall find $\dfrac{dy}{dx}$.

   Substituting $x = \cos u (u \in (o, \pi))$ in y we get

   $\dfrac{1-x}{1+x} = \dfrac{1-\cos u}{1+\cos u} = \dfrac{2\sin^2(u/2)}{2\cos^2(u/2)} = \tan^2(u/2)$

   So that $\sqrt{\dfrac{1-x}{1+x}} = \tan\dfrac{u}{2}$

   And $y = \tan^{-1}\left(\tan\left(\dfrac{u}{2}\right)\right) = \dfrac{u}{2}$

   Therefore, $\dfrac{dy}{dx} = \dfrac{dy}{dx}.\dfrac{dx}{dy} \Rightarrow \dfrac{1}{2} = \dfrac{dy}{dx}(-\sin u)$

   Hence, $\dfrac{dy}{dx} = \dfrac{-1}{2\sin u} = \dfrac{-1}{2\sqrt{1-x^2}}$

   Observe that $\tan^{-1} x, \sqrt{\dfrac{1-x}{1+x}}$ and $\cos u$ are the functions that stand for $f(x), g(x)$ and $h(u)$ respectively, mentioned in the method.

2. If $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)(|x|<1)$ then we shall that find $\dfrac{dy}{dx}$

   Substituting $x = \tan u$

We get $\dfrac{2x}{1-x^2}=\dfrac{2\tan u}{1-\tan^2 u}=\tan 2u$

And $y=\tan^{-1}(\tan 2u)=2u$

Therefore from $\dfrac{dy}{dx}=\dfrac{dy}{dx}.\dfrac{dx}{du}$

We get from $2=\dfrac{dy}{dx}.\sec^2 x$

Therefore $\dfrac{dy}{dx}=2\cos^2 u=\dfrac{2}{1+\tan^2 u}=\dfrac{2}{1+x^2}$.

10.3.2 Logarithmic Differentiation:

Use of the logarithms will be of great help in finding the derivatives of function of the form $y=f(x)^{g(x)}, f: A\to(0,\infty), g: A\to R$ (A an interval).

Write $y=h(x)=f(x)^{g(x)}$. Then $\log h(x)=g(x)\log f(x)$.

Differentiating both sides with respect to x, we get $\dfrac{h'(x)}{h(x)}=g'(x)\log f(x)+g(x)\dfrac{f'(x)}{f(x)}$

$$\text{Therefore } h'(x)=h(x)\left[g'(x)\log f(x)+g(x)\frac{f'(x)}{f(x)}\right]$$

$$h'(x)=f(x)^{g(x)}\Rightarrow h'(x)=f(x)^{g(x)}\left[g'(x)\log f(x)+g(x)\frac{f'(x)}{f(x)}\right]$$

This method is well illustrated in the following examples.

1. If $y=x^x (x>0)$, we shall find $\dfrac{dy}{dx}$

   Taking logarithm on both sides of $y=x^x$, we obtain $\log y=x\log x$

   Differentiating with respect to x, we get $\dfrac{y'}{y}=x.\dfrac{1}{x}+\log x=1+\log x$

   Therefore $\dfrac{dy}{dx}=y'=y(1+\log x)=x^x(1+\log x)$

2. If $y=(\tan x)^{\sin x}(0<x<\dfrac{\pi}{2})$. Compute $\dfrac{dy}{dx}$

   Taking logarithms on both sides of $y=(\tan x)^{\sin x}$, we get

   $\log y=\sin x.\log(\tan x)$

   Differentiating with respect to x, we get $\dfrac{y'}{y}=\dfrac{\sin x}{\tan x}.\sec^2 x+\cos x.\log(\tan x)=\sec x+\cos x.\log(\tan x)$

Hence $\frac{dy}{dx} = (\tan x)^{\sin x}[\sec x + \cos x \log(\tan x)]$.

10.3.3 Parametric Differentiation:

Let A, B, C be intervals, $f: A \to B, g: A \to C, f$ a bijection, f^{-1}, g be differentiable.

Then, writing $x = f(t), y = g(t)$ we get $y = (g \circ f)^{-1}(x) = \varphi(x)$

$x = f(t), y = g(t)$ are called the parametric equations of the function $y = \varphi(x)$

$$y = g(f^{-1}(x)) \Rightarrow \frac{dy}{dx} = g'(f^{-1}(x))(f^{-1}(x))' = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dt}{dx}} = \frac{g'(t)}{f'(t)}$$

The following examples illustrate parametric differentiation.

1. If $x = a\cos^3 t, y = a\sin^3 t$, find $\frac{dy}{dx}$

   Here $\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ & $\frac{dy}{dx} = 3a\sin^2 t.\cos t$

   Therefore $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$

2. If $y = e^t + \cos t, x = \log t + \sin t$, find $\frac{dy}{dx}$

   Here $\frac{dy}{dt} = e^t - \sin t$ and $\frac{dx}{dt} = \frac{1}{t} + \cos t$

   Therefore $\frac{dy}{dx} = \frac{t(e^t - \sin t)}{(1 + t\cos t)}$.

3. To find the derivatives of $f(x) = x^{\sin^{-1} x}$ with respect to $g(x) = \sin^{-1} x$

   We have to compare $\frac{df}{dg}$.

   Now $f(x) = x^{\sin^{-1} x} \Rightarrow \log f(x) = \sin^{-1} x.\log x$ so that

   $$\frac{f'(x)}{f(x)} = \left[\frac{1}{x}\sin^{-1} x + \frac{\log x}{\sqrt{1-x^2}}\right] \Rightarrow f'(x) = x^{\sin^{-1} x}\left[\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$$

   $$g(x) = \sin^{-1} x \Rightarrow g'(x) = \frac{1}{\sqrt{1-x^2}}$$

Therefore $\dfrac{df}{dg}=\dfrac{f'(x)}{g'(x)}=\sqrt{1-x^2}\,.x^{\sin^{-1}x}\left[\dfrac{\sin^{-1}x}{x}+\dfrac{\log x}{\sqrt{1-x^2}}\right].$

10.3.4 Differentiation of implicit function:

A function defined on a set $A(\subseteq R)$ is usually denoted by $y=f(x)$. A function which can be put in this form is said to be in implicit form. Sometimes, such a form of a function may not be possible. But f can be defined in terms of a function F which is defined on R^2 by the equation of the form $F(x,y)=0$ For example, $y=f(x)$ defined by $x^2-6xy+y^2=0$ is a function which can't be given in explicit form.

A function $y=f(x)$, defined by $F(x,y)=0.$ is called an implicit function. In order to differentiate such functions, we differentiate F with respect to x (treating y as a function of x) and equate it to zero and thereby we get $\dfrac{dy}{dx}$.

The following examples illustrate this process.

1. If $x^3+y^3-3axy=0$, find $\dfrac{dy}{dx}$

   Let the given equation define the function
   $y=f(x)$ That is $x^3+(f(x))^3-3axf(x)=0$
   Differentiating both sides of this equation with respect to x, we get
   $3x^3+3(f(x))^2f'(x)-[3a.f(x)+3axf'(x)]=0$
   Hence $3x^3+3y^2f'(x)-[3ay+3axf'(x)]=0$

   Therefore $f'(x)=\dfrac{dy}{dx}=\dfrac{ay-x^2}{y^2-ax}$

2. If $2x^2-3xy+y^2+x+2y-8=0$, find $\dfrac{dy}{dx}$

   Treating y as a function of x and then differentiating with respect to $x.$ We get $4x-3y-3xy'+2yy'+1+2y'=0.$

   Therefore $\dfrac{dy}{dx}=y'=\dfrac{3y-4x-1}{2y-3x+2}.$

10.3.5 Solved problems:

1. Problem: If $y=\tan^{-1}\left(\cos\sqrt{x}\right)$ then find $\dfrac{dy}{dx}$

Solution: Substitute $t=\sqrt{x}, u=\cos\sqrt{x}$.

Then $y=\tan^{-1}u$ and $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dt}\times\dfrac{dt}{dx}$

$$= \frac{1}{1+u^2} \times -\sin t \times \frac{1}{2\sqrt{x}}$$

$$= -\frac{\sin\sqrt{x}}{2\sqrt{x}(1+\cos^2\sqrt{x})}$$

2. Problem: If $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] \forall 0 < |x| < 1$ then find $\frac{dy}{dx}$

Solution: Substituting $x^2 = \cos 2\theta$, we get

$$y = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right) = \frac{\pi}{4}+\theta$$

Therefore $\quad y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)$

Hence $\quad \frac{dy}{dx} = \frac{1}{2}\frac{(-1)}{\sqrt{1-x^4}} \times 2x = \frac{-x}{\sqrt{1-x^4}}$

3. Problem: If $x = a\left[\cos t + \log\tan\frac{t}{2}\right], y = a\sin t$ then find $\frac{dy}{dx}$

Solution: Here, $\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)}.\sec^2\left(\frac{t}{2}\right).\frac{1}{2}\right] = \frac{a\cos^2 t}{\sin t}$

And $\quad \frac{dy}{dt} = a\cos t$ so that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \tan t$.

4. Problem: If $x^y = e^{x-y}$ then show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution: Taking logarithms on both sides of $x^y = e^{x-y}$ we get $y\log x = x - y$

That is, $y = \dfrac{x}{1+\log x}$

Therefore $\dfrac{dy}{dx} = \dfrac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2} = \dfrac{\log x}{(1+\log x)^2}$

5. Problem: If $\sin y = x\sin(a+y)$, show that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$, a is not a multiple of π

Solution: $\sin y = x\sin(a+y) \Rightarrow x = \dfrac{\sin y}{\sin(a+y)}$

Differentiating both the sides with respect to x, we get

$$1 = \frac{\sin(a+y).\cos y - \sin y.\cos(a+y)}{\sin^2(a+y)}.\frac{dy}{dx}$$

Hence $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin(a+y-y)} = \dfrac{\sin^2(a+y)}{\sin a}$

Exercise 10(c)

1. Find the derivatives of the following functions.

$(i)\ \sin^{-1}(3x-4x^3)$ $(ii)\ \cos^{-1}(4x^3-3x)$ $(iii)\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ $(iv)\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)$

$(v)\ \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ $(vi)\ \sin(\cos(x^2))$ $(vii)\sin(\tan^{-1}(e^{-x}))$

2. Differentiate $f(x)$ with respect to $g(x)$ for the following.

   $(i)\ f(x)=e^x, g(x)=\sqrt{x}$ $(ii)\ f(x)=e^{\sin x}, g(x)=\sin x$

   $(iii)\ f(x)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), g(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

3. If $y = e^{a\sin^{-1}x}$ then prove that $\dfrac{dy}{dx} = \dfrac{ay}{\sqrt{1-x^2}}$

4. Find the derivative of the following functions

   $(i)\ \tan^{-1}\left(\dfrac{3a^2x-x^3}{a(a^2-3x^2)}\right)$ $(ii)\ \tan^{-1}(\sec x+\tan x)$ $(iii)\tan^{-1}\dfrac{\left[\sqrt{1+x^2}-1\right]}{x}$

   $(iv)(\log x)^{\tan x}$ $(v)(x^x)^x$ $(vi)\ 20^{\log(\tan x)}$ $(vii)x^x+e^{e^x}$ $(viii)x\log x.\log(\log x)$

5. Find $\dfrac{dy}{dx}$ for the following functions

$(i)\, x = 3\cos t - 2\cos^3 t,\ y = 3\sin t - 2\sin^3 t \quad (ii)\, x = \dfrac{3at}{1+t^3},\ y = \dfrac{3at^2}{1+t^3}$

$(iii)\, x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

6. Differentiate $f(x)$ with respect to $g(x)$ for the following.

$(i)\, f(x) = \log_a x,\ g(x) = a^x \quad (ii)\ f(x) = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right),\ g(x) = \sqrt{1-x^2}$

$(iii)\ f(x) = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right),\ g(x) = \tan^{-1} x$

7. Find the derivative of the function y defined implicitly by each of the following equations.

$(i)\ x^4 + y^4 - a^2xy = 0\ (ii)\ y = x^y\ (iii)\, y^x = x^{\sin y}$

8. Establish the following

(i) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$

(ii) If $y = x\sqrt{a^2+x^2} + a^2\log(x + \sqrt{a^2+x^2})$ then $\dfrac{dy}{dx} = 2\sqrt{a^2+x^2}$

(iii) If $x^{\log y} = \log x$ then $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{1-\log x \log y}{\log^2 x}\right]$

(iv) If $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) - \tan^{-1}\left(\dfrac{4x-4x^3}{1-6x^2+x^4}\right)$ then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

(v) If $x^y = y^x$ then $\dfrac{dy}{dx} = \dfrac{y(x\log y - y)}{x(y\log x - x)}$

(vi) If $x^{2/3} + y^{2/3} = a^{2/3}$ then $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$

9. Find $\dfrac{dy}{dx}$ of each of the following functions

$(i)\, y = \dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1-2x)^{-6/7}} \quad (ii)\, y = \dfrac{x^4\sqrt[3]{x^2+4}}{\sqrt{4x^2-7}}\ (iii)\, y = \dfrac{(a-x)^2(b-x)^3}{(c-2x)^3}$

$(iv)\, y = \dfrac{x^3.\sqrt{2+3x}}{(2+x)(1-x)}$

10. Find the derivatives of the following functions

$(i)\,(\sin x)^{\log x} + x^{\sin x} \quad (ii)\, x^{x^x} \quad (iii)\, x^x + (\cot x)^x \quad (iii)\,(\sin x)^x + x^{\sin x}$

11. Establish the following

(i) If $x^y + y^x = a^b$ then $\dfrac{dy}{dx} = -\left[\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$

(ii) If $f(x) = \sin^{-1}\sqrt{\dfrac{x-\beta}{\alpha - x}}$ & $g(x) = \tan^{-1}\sqrt{\dfrac{x-\beta}{\alpha - x}}$ then $f^{'}(x) = g^{'}(x)(\beta < x < \alpha)$

(iii) If $a > b > 0$ and $0 < x < \pi$; then $f(x) = (a^2 - b^2)^{-1/2}.\cos^{-1}\left(\dfrac{a\cos x + b}{a + b\cos x}\right)$

12. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Using product rule.

(ii) Obtaining a single polynomial expanding the product

(iii) Logarithmic differentiation

KEY CONCEPTS

I. The derivative of a function *f* at *x* = *a* is denoted by

$$f^{'}(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{h\to a}\frac{f(x) - f(a)}{x - a}$$ both exist and are equal.

II. Every differentiable function is continuous but the converse is not true.

III. let *u* and *v* be functions of *x* whise derivatives exixts. Then

(i) $\dfrac{d(c)}{dx} = 0$

(ii). $\dfrac{d(ku)}{dx} = k\dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$

(iv). $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

(v) $\dfrac{d}{dx}(\dfrac{1}{u}) = -\dfrac{1}{u^2}\dfrac{du}{dx}$

(vi) $\frac{d}{dx}(\frac{u}{v})=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

IV. Some of the standered derivatives in appropriate domains

1. $\frac{d}{dx}(\sin x)=\cos x$

2. $\frac{d}{dx}(x^n)=nx^{n-1}$

3. $\frac{d}{dx}(e^x)=e^x$.

4. $\frac{d}{dx}(\log_e x)=\frac{1}{x}$

5. $\frac{d}{dx}(a^x)=a^x\log_e a$

6. $\frac{d}{dx}(\sin x)=\cos x$

7. $\frac{d}{dx}(\cos x)=-\sin x$

8. $\frac{d}{dx}(\tan x)=\sec^2 x$

9. $\frac{d}{dx}(\cot x)=-\cos ec^2 x$

10. $\frac{d}{dx}(\sec x)=\sec x\tan x$

11. $\frac{d}{dx}(\cos ecx)=-\cos ecx\cot x$ 7.

12. $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$

13. $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$.

14. $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$.

15. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

17. $\frac{d}{dx}(co\sec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$.

18. $\frac{d}{dx}(\sinh x) = \cosh x$

19. $\frac{d}{dx}(\cosh x) = \sinh x$.

20. $\frac{d}{dx}(\tanh x) = \sec h^2 x$.

21. $\frac{d}{dx}(\sec hx) = -\sec hx \tanh x$

22. $\frac{d}{dx}(co\sec hx) = -\cos echx \coth x$.

23. $\frac{d}{dx}(\coth x) = -\cos ec^2 x$

24. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

25. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$.

26. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

27. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

28. $\frac{d}{dx}(\cos ec\text{h}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$.

29. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Answers

Exercise 10(a)

1. (i) $\frac{1}{2\sqrt{x}}+\frac{3}{2}x^{-\frac{1}{4}}+\frac{5}{2}x^{-\frac{1}{6}}$ (ii) $\frac{1}{\sqrt{2x-3}}-\frac{3}{2\sqrt{7-3x}}$ (iii) $20x^4-36x^2+2x$

(iv) $\frac{3}{2}\sqrt{x}-6x-\frac{1}{2x\sqrt{x}}$ (v) $\frac{5}{2}x\sqrt{x}+2x-6\sqrt{x}+\frac{1}{\sqrt{x}}-4$

(vi) $(ax+b)^n(cx+d)^m\left(\frac{na}{ax+b}+\frac{mb}{cx+d}\right)$ (vii) $5\cos x+e^x(\log x+\frac{1}{x})$

(viii) $5^x\log 5+\frac{1}{x}+(x^3+3x^2)e^x$ (ix) $e^x+\cos 2x$ (x) $\frac{apx^2+2pbx+bq-ar}{(ax+b)^2}$

(xi) $\frac{1}{x\log x}\log_7 e$ (xii) $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$ (xiii) $e^{2x}\left(2\log(3x+4)+\frac{3}{3x+4}\right)$

(xiv) $2e^{2x}(x^2+x+4)$ (xv) $\frac{ad-bc}{(cx+d)^2}$ 2. $f'(1)=5050$

4. (i) $3x^2$ (ii) $4x^3$ (iii) $2ax+b$ (iv) $\frac{1}{2\sqrt{x+1}}$ (v) $2\cos 2x$ (vi) $-a\sin ax$

(vii) $2\sec^2 2x$ (viii) $-\operatorname{cosec}^2 x$

5 (i) $\frac{-3\sqrt{x}}{(1+x\sqrt{x})^2}$ (ii) $x^{n-1}n^x\left(n\log(nx)+x\log n\log(nx)+1\right)$

(iii) $ax^{2n-1}\left(2n\log x+1\right)+bx^{n-1}e^{-x}\left(n-x\right)$ (iv) $\left(\frac{1}{x}-x\right)^2 e^x\left(\frac{1}{x}-\frac{3}{x^2}-3-x\right)$

7. not differentiable at 1 and 3. 8. not differentiable at 2

Exercise 10(b)

1. (i) $-n\cot^{n-1}x\operatorname{cosec}^2 x$ (ii) $-4\operatorname{cosec}^4 x\cot x$ (iii) $e^x\sec^2(e^x)$ (iv) $2\tan x\sec^2 x$

(v) $\sin^{m-1}x\cos^{n-1}x(m\cos^2 x-n\sin^2 x)$ (vi) $m\cos mx\cos nx-n\sin mx\sin nx$ (vii) $\frac{x}{1+x^2}+\tan^{-1}x$

(viii) -1 (ix) $10\operatorname{cosec}10x$ (x) $\frac{3}{\sqrt{16-9x^2}}$ (xi) $\frac{1}{x(1+(\log x)^2)}$ (xii) $\frac{4-2x^2}{x^4+3x^2+4}$

(xiii) $\frac{e^x}{\sin^{-1}(e^x)\sqrt{1-e^{2x}}}$ (xiv) $2(\sin^{-1}x)(\sin x)\left((\cos x.\sin^{-1}x)+\frac{\sin x}{\sqrt{1-x^2}}\right)$

$(xv)\ \dfrac{-1}{(\sin x+\cos x)^2}$ $(xvi)\ \dfrac{1+3x^2-2x^4}{(1-x^2)^{\frac{3}{2}}}$ $(xvii)\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$(xvii)-\sin(\log x+e^x)(\dfrac{1}{x}+e^x)$

$(xix)\cos a\sec^2 x\ (xx)\dfrac{3\cos ec3x\cot 3x}{1+\cos ec^2 3x}$

2 $(i)\ \dfrac{1}{2\sqrt{x+x^2}}$ $(ii)\ \dfrac{1}{2\sqrt{x}(1-x)}$ $(iii)\ \dfrac{1}{x\cosh y}$ $(iv)\dfrac{-e^{-y}}{1+x^2}$ $(v)\dfrac{e^x}{\sin 2y}$

$(vi)\,2\sqrt{y}(1+\sqrt{y})$

3 $(i)\ \dfrac{\sin(\log(\cot x))}{\sin x\cos x}(ii)\ \dfrac{\pm\sqrt{2}}{(1+x)\sqrt{1+x^2}}(iii)\ 4x\cos ec2(1-x^2)(iv)-2x\sin(x^2)\cos(\cos(x^2))$

$(v)\dfrac{e^x}{1+e^{2x}}\cos(\tan^{-1}(e^x))(vi)\ \dfrac{a\cos(ax+b)\cos(cx+d)+c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$

$(vii)\ \sin x.(\tan^{-1}x)^2$

4 $(i)\ \dfrac{\sqrt{a^2-b^2}}{a+b\sin x}$ $(ii)\ \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$ $(iii)\dfrac{-\sin x}{2\cos^2 x+2\cos x+1}$

Exercise 10(c)

1 $(i)\dfrac{3}{\sqrt{1-x^2}}$ $(ii)\ -\dfrac{3}{\sqrt{1-x^2}}$ $(iii)\dfrac{2}{1+x^2}$ $(iv)\dfrac{-1}{1+x^2}$ $(v)\ \dfrac{1}{2}$

$(vi)\ -2x\sin(x^2)\cos(\cos(x^2))\ (vii)\dfrac{-e^{-x}\cos(\tan^{-1}(e^{-x}))}{1+e^{-2x}}$

2. $(i)\ 2\sqrt{x}e^x$ $(ii)\ e^{\sin x}$ $(iii)1$

4 $(i)\ \dfrac{3a}{a^2+x^2}(ii)\ \dfrac{1}{2}(iii)\dfrac{1}{2(1+x^2)}(iv)(\log x)^{\tan x}\left(\dfrac{\tan x}{x\log x}+\sec^2 x\log(\log x)\right)$

$(v)\ x^{x^2+1}\log ex^2\ (vi)2\log 20.\cos ec2x.\ 20^{\log(\tan x)}\ (vii)x^x(1+\log x)+e^x e^{e^x}$

$(viii)\log(e\log x)+\log x.\log(\log x)$

5 $(i)\cot t$ $(ii)\dfrac{t(2-t^3)}{1-2t^3}$ $(iii)\tan t$

6 $(i)\dfrac{1}{xa^x(\log a)^2}$ $(ii)\dfrac{2}{|x|}$ $(iii)\dfrac{1}{2}$

7 $(i)\ \dfrac{a^2y-4x^3}{4y^3-a^2x}$ $(ii)\dfrac{y^2}{x(1-\log x)}$ $(iii)\dfrac{y(\sin y-x\log y)}{x(x-y\cos y\log x)}$

9 $(i)\dfrac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1-2x)^{-6/7}}\left[\dfrac{5}{1-6x}+\dfrac{6}{1+7x}-\dfrac{4}{3(1-2x)}-\dfrac{9}{4(1+3x)}\right]$

$(ii)\dfrac{x^4\sqrt[3]{x^2+4}}{\sqrt{4x^2-7}}\left[\dfrac{4}{x}+\dfrac{2x}{3(4+x^2)}-\dfrac{4x}{4x^2-7}\right]$ $(iii)\dfrac{(a-x)^2(b-x)^3}{(c-2x)^3}\left[\dfrac{6}{c-2x}-\dfrac{3}{b-x}-\dfrac{2}{a-x}\right]$

$(iv)\dfrac{x^3.\sqrt{2+3x}}{(2+x)(1-x)}\left[\dfrac{3}{x}+\dfrac{3}{2(2+3x)}+\dfrac{1}{1-x}-\dfrac{1}{2+x}\right]$

10 $(i)(\sin x)^{\log x}\left[\dfrac{\log(\sin x)}{x}+\cot x\log x\right]$ $(ii)\ x^{x^x+x-1}(1+x\log x\log(ex))$

$(iii)\ x^x(1+\log x)+(\cot x)^x\left[\log(\cot x)-\dfrac{2x}{\sin 2x}\right]$

$(iii)\ (\sin x)^x\left[x\cot x+\log(\sin x)\right]+x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]$ 12 yes

11. APPLICATIONS OF DERIVATIVES

Introduction:

In chapter 10, we have studied the concept of the derivative of a function. In this chapter, we will study some applications of derivatives. In fact, the derivative plays a vital role in solving some problems such as errors and approximation, finding maxima and minima (extreme values) of a function. We shall also discuss the geometrical interpretation of the derivative and the methods of finding the equations of the tangent and the normal at a point on a given curve.

11.1 Errors and approximation:

The word infinitesimal is used in the sense that it is extremely small or very very small. In other words, it is so small that it cannot be distinguished from zero by any available means. Roughly we can say that an infinitesimal it close to zero but it is not equal to zero. The infinitesimal in the variable x is denoted by Δx and the infinitesimal in the variable y is denoted by Δy. Then infinitesimal Δx and Δy are referred as change in x and change in y respectively.

If a dependent variable y depends on x by a functional relation $y = f(x)$ then change in y is given by $\Delta y = f(x+\Delta x) - f(x)$

Where the variable x is changed from x to $x+\Delta x$.

11.1.1 Notation: $\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}$ Denotes the value of the derivative of the function $f(x, y)$ at (x_0, y_0).

If the function y is in explicit form $y = f(x)$ then we write $\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}$ as $f^{'}(x_0)$.

11.1.2 Formula for approximation value of Δy : We define $f^{'}(x)\Delta x$ or $\dfrac{dy}{dx}\Delta x$ as differential of y and is denoted by dy,

i.e..,
$$dy = f^{'}(x)\Delta x \qquad (1)$$

$$dy = \frac{dy}{dx}\Delta x \qquad (2)$$

If we take $y = f(x) = x$ in (1), we get

$$dy = \Delta x \qquad (3)$$

And we call dx as differential of x.

Though dx and Δx are equal, $\Delta y \,\&\, dy$ need not be equal. In case $y = f(x)$ represents a line then $\Delta y \,\&\, dy$ are equal. Geometrically dy denotes the change in y along the tangent line where as Δy is the change along the curve (in fig).

In the above fig, PT is the tangent to the curve $y = f(x)$ at $P(x_0, f(x_0))$ and $Q(x_0 + \Delta x, f(x_0 + \Delta x))$ is a neighbouring point of P lying on the curve. The line segments PR, RS and RQ are respectively equal to $\Delta x, dy \,\&\, \Delta y$.

If Δx is an infinitesimal then $\frac{\Delta y}{\Delta x} = f^{'}(x)$ (4)

Since $\frac{\Delta y}{\Delta x}$ is approximately equal to $f'(x)$ there exists an infinitesimal ε such that

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon \qquad (5)$$

Where ε depends on x and Δx. Equation (5) can be expressed as

$$\Delta y = f'(x)\Delta x + \varepsilon \Delta x \qquad (6)$$

Since ε and Δx are infinitesimal, their product is very small and nearer to zero [like the product of (0.001).(0.000001)=0.00000001]. Therefore we take $f'(x)\Delta x$ as an approximate value of Δy. Thus

$$\Delta y \approx f'(x)\Delta x \qquad (7)$$

$$\Delta y \approx f'(x)dx \qquad (8)$$

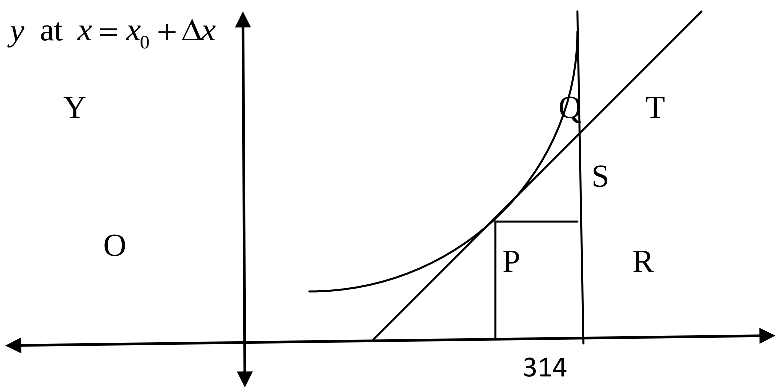
In view of (1), the equation (7) can be written as

$$\Delta y \approx dy$$

We can also have another formula from equation (8) i.e..,

$$f(x + \Delta x) \approx f(x) + f^{'}(x)dx \qquad (9)$$

Since $\Delta y = f(x + \Delta x) - f(x)$.The equation (9) can be used to find an approximate value of y at $x = x_0 + \Delta x$



11.1.3 Note: When x changes from x_0 to $x_0 + \Delta x$ then change in y is given by

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \qquad (10)$$

And we call Δy given by (10) as the change in y at $x = x_0$

11.1.4 Definition: If a number A is very close to a number B but it is not equal to B then A is called an approximate value of B. For example 3.141592 is an approximate value of Π=3.14159263589.........

If K(e) is an exact value of a certain entity (length of a side, square root of a number) and K(a) is an approximate value of K(e) then the difference of these two is defined as an error i.e.., K(e)-K(a) is the error. If Δx is considered as an error in x then the error in $y = f(x)$ is Δy . The exact error can be computed from equation (1) of 10.1.2 and the approximation of Δy can be computed from the equation (8) of 10.1.2.

Definition (Absolute error, Relative error and Percentage error)

If y is any variable then

(i) Δy is called an absolute error in y

(ii) $\dfrac{\Delta y}{y}$ is called a relative error in y .

(iii) $\dfrac{\Delta y}{y} \times 100$ is called percentage error in y.

If $y = f(x)$ is a differentiable function and Δx is an error in x then the approximation of absolute error, relative error and percentage error in y are respectively as given below

$$\Delta y \approx f'(x)\Delta x \qquad (1)$$

$$\frac{\Delta y}{y} \approx \left(\frac{f'(x)}{f(x)}\right)\Delta x \qquad (2)$$

And

$$\frac{\Delta y}{y} \times 100 \approx \left(\frac{f'(x)}{f(x)}\right) \times 100 \times \Delta x \qquad (3)$$

11.1.5 Solved Problems:

1. Problem: Find $dy \,\&\, \Delta y$ of $y = x^2 + x$ at $x = 10$ when $\Delta x = 0.1$

Solution: As change in $y = f(x)$ is given by $\Delta y = f(x + \Delta x) - f(x)$, this change at $x = 10$ with $\Delta x = 0.1$.

$$\Delta x = f(10.1) - f(10) = \{(10.1)^2 + 10.1\} - \{10^2 + 10\} = 2.11$$

Since $dy = f^{'}(x), \Delta x, dy$ at $x = 10$ with $\Delta x = 0.1$ is

$$dy = \{(2)(10)+1\}0.1 = 2.1 \quad (\text{since } \frac{dy}{dx} = 2x+1)$$

2. Problem: Find $\Delta y \,\&\, dy$ for the function $y = \cos(x)$ at 60° with $\Delta x = 1^\circ$

Solution: For the given $\Delta y \,\&\, dy$ at $x = 60^\circ$ with $\Delta x = 1^\circ$ are

$$\Delta y = \cos(60^\circ + 1^\circ) - \cos(60^\circ) \qquad (1)$$

And $$dy = -\sin(60^\circ)(1^\circ) \qquad (2)$$

$$\cos(60^\circ) = 0.5,\ \cos(61^\circ) = 0.4848,\ \sin(60^\circ) = 0.8660,\ 1^\circ = 0.0174 \text{ radians}$$

Therefore, $$\Delta y = -0.0152 \,\&\, dy = -0.0150$$

3. Problem: If the radius of a sphere is increased from 7cm to 7.02 cm then find the approximate increase in the volume of the sphere.

Solution: Let r be the radius of a sphere and V be its volume. Then

$$V = \frac{4\pi r^3}{3} \qquad (1)$$

Here V is a function of r. As the radius is increased from 7*cm* to 7.02, we can take r = 7cm and $\Delta r = 0.02$ cm. Now we have to find the approximate increase in the volume of the sphere.

$$\therefore \Delta V \approx \frac{dV}{dr}\Delta r = 4\pi r^2 \Delta r$$

Thus, the approximation increase in the volume of the sphere is

$$\frac{4(22)(7)(7)(0.02)}{7} = 12.32 cm^3$$

4. Problem: If $y = f(x) = kx^n$ the show that the approximate relative error (or increase) in y in n times the relative error (or increase) in x where n and k are constants.

Solution: The approximate relative error (or increase) in y by the equation (2) of 10.1.4 is $\left(\frac{f^{'}(x)}{f(x)}\right)\Delta x = \frac{knx^{n-1}}{kx^n}\Delta x = n\left(\frac{\Delta x}{x} = n\right)$ relative error (or increase) in x.

Hence the approximate relative error in $y = kx^n$ is n times the relative error in x.

5. Problem: If an error of 0.01 cm is made in increasing the perimeter of a circle and the perimeter is measured as 44cm then find the approximate error and relative error in its area.

Solution: Let $r, p \,\&\, A$ be the radius, perimeter and area of the circle respectively. Given that $p = 44cm \,\&\, \Delta p = 0.01$. We have to find the approximation of $\Delta A \,\&\, \frac{\Delta A}{A}$. Note that $A = \pi r^2$ which is a function of r. As $p \,\&\, \Delta p$ are given to transform $A = \pi r^2$ into the form $A = f(p)$. This can be achieved by using the relation, perimeter $2\pi r = p$.

$$A = \pi \left(\frac{p}{2\pi} \right)^2 = \frac{p^2}{4\pi}$$

Hence the approximate error in $A = \frac{dA}{dp} \Delta p = \frac{2p}{4\pi} \Delta p = \frac{p}{2\pi} \Delta p$

The approximation error in A when $p = 44 \,\&\, \Delta p = 0.01 = \frac{44}{2\pi}(0.01) = 0.07$

The approximate relative error = $\frac{\left(\frac{dA}{dp}\right)}{A} . \Delta p = \frac{\left(\frac{p}{2\pi}\right)}{\left(\frac{p^2}{4\pi}\right)} . \Delta p = 2\frac{\Delta p}{p} = \frac{2(0.01)}{44} = 0.0004545$.

Exercise 11(a)

1. Find $\Delta y \,\&\, dy$ for the following functions for the values of $x \,\&\, \Delta x$ which are shown against each of the functions.

   (i) $y = x^2 + 3x + 6, x = 10 \,\&\, \Delta x = 0.01$

   (ii) $y = e^x + x, x = 5 \,\&\, \Delta x = 0.02$

   (iii) $y = 5x^2 + 6x + 6, x = 2 \,\&\, \Delta x = 0.001$

   (iv) $y = \frac{1}{x+2}, x = 8 \,\&\, \Delta x = 0.02$

   (v) $y = \cos(x), x = 60^\circ \,\&\, \Delta x = 1^\circ$

2. Find the approximations of the following
   (i) $\sqrt{82}$ (ii) $\sqrt[3]{65}$ (iii) $\sqrt{25.001}$ (iv) $\sqrt[3]{7.8}$ (v) $\sin(62^\circ)$ $(vi)\cos(60^\circ 5')$ $(vii)\sqrt[4]{17}$

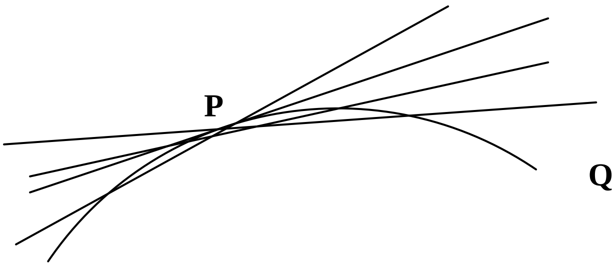
3. If the increase in the side of a square is 4% then find the approximate percentage of increase in the area of the square.

4. The radius of a sphere is measured as 14cm. Later it was found that there is an error 0.02 cm in measuring the radius. Find the approximate error in surface area of the sphere.
5. The diameter of a sphere is measured to be 40cm. IF an error of 0.02 cm is made in it, then find approximate errors in volume and surface area of the sphere.
6. The time t of a complete oscillation of a simple pendulum of length l is given by $t = 2\pi\sqrt{\frac{l}{g}}$ where g is gravitational constant. Find the approximate percentage of error in t when the percentage of error in l is 1%.

11.2 Geometrical interpretation of the derivative:

In this section we first recall the definition of tangent at a point to a curve. Then we give the geometrical interpretation of the derivative.

11.2.1 Definition: Let P be a point on a curve (fig). Let Q be a neighbouring point P on the curve. The line through P and Q is a secant of the curve. The limiting position of the secant PQ as Q moves nearer to P along the curve is called the tangent to the curve at the point P.



11.2.2 Geometrical interpretation of derivative: Let APQ denote the curve $y = f(x)$ defined on an interval. Let P be a point on the curve and $P = (c, f(c))$

If we let Q to be a neighbouring point of P on the curve (see fig), then Q can be taken as $Q = (c + \delta c, f(c + \delta c))$ .Let the tangent at P to the curve which is not parallel to X-axis in general, meet the X-axis at T and make an angle ψ with X-axis. Let the chord drawn through P,Q meet X-axis in S and make an angle θ. Let L, M be the feet of the perpendiculars drawn from P,Q respectively on the X-axis. Then $PL = f(c)$ & $QM = f(c + \delta c)$. Since, PR is parallel to OM we have $Q\hat{P}R = \theta$.

$$\text{Then, } \quad \tan\theta = \frac{QR}{PR} = \frac{QM - RM}{OM - OL} = \frac{QM - PL}{OM - OL}$$

$$= \frac{f(c + \delta c) - f(c)}{c + \delta c - c} = \frac{f(c + \delta c) - f(c)}{\delta c}$$

As the point Q approaches P, the limiting position of the chord PQ is the tangent PT at P, i.e.., if $Q \to P$ then $QR \to 0, PR \to 0, \theta \to \psi$ and chord $\overleftrightarrow{PQ}$ approaches $\overleftrightarrow{PT}$

Therefore $$f^{'}(c) = \lim_{\delta c \to 0} \frac{f(c+\delta c) - f(c)}{\delta c} = \lim_{Q \to P} \frac{QR}{PR} = \lim_{\theta \to \psi} \tan\theta = \tan\psi$$

Observe that $\tan\psi$ is the slope of the tangent *PT*. Thus, the summary of the above discussion is that the derivative of *f(x) at c* is the slope of the tangent to the curve *y=f(x)* at the point *(c, f(c))*

11.3 Equations of tangent and normal to a curve:

In the previous section 10.2 we have seen that $\frac{dy}{dx}$ represents the slope of the tangent at a point *(x, y)* on the curve *y=f(x).* Using this concept, it is easy to find the equations of tangent and normal.

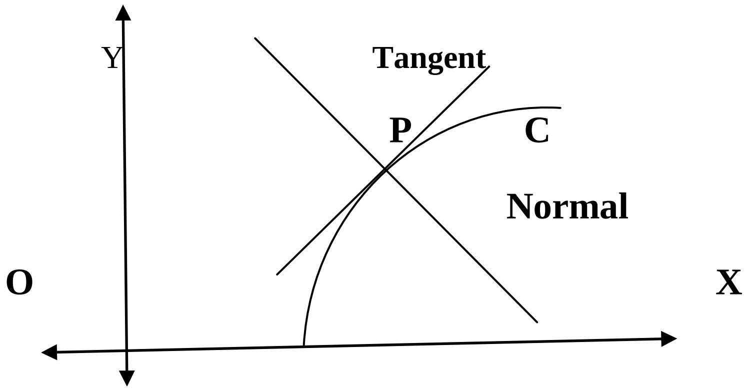
11.3.1 Equation of tangent: Let *y=f(x)* be a curve and *P (a, b)* be a point on it. Then we know that the slope *m* of the tangent at *P* is

$$m = f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{(a,b)}$$

Therefore, the equation of the tangent to the curve at (a,b) is

$$y - b = m(x - a)$$
$$y - b = f^{'}(a)(x - a)$$

11.3.2 Definition: Let *P* be a point on curve *C*. The straight line passing through *P* and perpendicular to the tangent to the curve at *P* is called the normal to the curve *C* at *P* (see fig).



11.3.3 Equation of normal: Since, the slope of the tangent to the curve *y=f(x)* at *P(a,b)* is $f'(a)$, the slope of the normal at *P* is $\frac{-1}{f'(a)}$ if $f'(a)$ not equal to zero. If $f'(a) = 0$ the tangent to the curve at *P* is parallel to the X-axis and therefore the normal is parallel to the Y-axis.

Thus the equation of the normal is

$$y-b=\frac{-1}{f'(a)}(x-a) \text{ if } f'(a)\neq 0 \qquad \text{and}$$

$$x=a \text{ if } \qquad f'(a)=0$$

11.3.4 Note: The curve *y=f(x)* is said to have a

(i) Horizontal tangent at a point *(a, f(a))* on the curve when $f'(a)=0$

(ii) Vertical tangent at a point *(a, f(a))* on the curve when

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\infty \text{ or } -\infty$$

11.3.5 Solved Problems:

1. Problem: Find the slope of the tangent to the following curves at the points as indicated.

(i) $y=5x^2$ At (-1, 5)

(ii) $y=\frac{1}{x-1}(x\neq 1)$ at $\left(3,\frac{1}{2}\right)$

(iii) $x=a\sec\theta, y=a\tan\theta$ at $\theta=\frac{\pi}{6}$

(iv) $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at (a,b)

Solution:

(i) $y=5x^2$ then $\frac{dy}{dx}=10.x$

Therefore the slope of the tangent at the given point is $\left.\frac{dy}{dx}\right|_{(-1,5)}=-10$.

(ii) $y=\frac{1}{x-1}(x\neq 1)$ then $\frac{dy}{dx}=\frac{-1}{(x-1)^2}$

Therefore the slope of the tangent at $\left(3,\frac{1}{2}\right)$ is $\left.\frac{dy}{dx}\right|_{(3,1/2)}=\frac{-1}{(3-1)^2}=\frac{-1}{4}$

(iii) $x=a\sec\theta, y=a\tan\theta$

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)=\frac{a\sec^2\theta}{a\sec\theta\tan\theta}=\cos ec\theta$$

Slope of the tangent of the point with $\theta=\frac{\pi}{6}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}=\cos ec\left(\frac{\pi}{6}\right)=2$$

(iv) $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating both sides w.r.t x,

$$n\left(\frac{x}{a}\right)^{n-1}.\frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1}.\frac{1}{b}.\frac{dy}{dx} = 0$$

i.e.., $\frac{dy}{dx} = -\left(\frac{b}{a}\right)^n\left(\frac{x}{y}\right)^{n-1}$

Slope of the tangent at (a, b) $\left.\frac{dy}{dx}\right|_{(a,b)} = \frac{-b}{a}$.

2. Problem: Find the equations of the tangent and the normal to the curve $y = 5x^4$ at the point (1, 5).

Solution: $y = 5x^4$ implies that $\frac{dy}{dx} = 20x^3$

Slope of the tangent to the curve at (1, 5) is $\left.\frac{dy}{dx}\right|_{(1,5)} = 20(1)^3 = 20$

The slope of the normal to the curve at (1, 5) is $\frac{-1}{20}$

Equations of the tangent and the normal to the curve at (1, 5) are

$y - 5 = 20(x-1)$ & $y - 5 = \frac{-1}{20}(x-1)$ respectively.

i.e.., $y = 20x - 15$ & $20y = 101 - x$ respectively

3. Problem: Find the equations of the tangents to the curve $y = 3x^2 - x^3$ =0, where it meets the X- axis.

Solution: Putting $y = 3x^2 - x^3$ =0 , we get the points of intersection of the curve and X-axis, i.e.., $y = 0$. They are given by

$$3x^2 - x^3 = 0 \text{ or } x^2(3-x) = 0 \text{ i.e.., } x = 0, x = 3.$$

Thus the curve crosses the X axis at the points O (0, 0) and A (3, 0).

$$\frac{dy}{dx} = 6x - 3x^2$$

that implies slope of the tangent at O(0, 0) to the curve is

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 0$$

Tangent at O (0, 0) is $y-0=0(x-0)$

i.e.., X-axis is the tangent to the curve at (0, 0).

Now the slope of the tangent at A(3, 0) to the curve is $\left.\frac{dy}{dx}\right|_{(3,0)} = 6(3)-3(3)^2 = -9$

Therefore the tangent at (3, 0) is $y-0=-9(x-3),\ y+9x=27.$

4. Problem: Find the points at which the curve $y = sinx$ has horizontal tangents.

Solution: $y = sinx$

$$\therefore \frac{dy}{dx} = \cos x$$

A tangent is horizontal if and only if its slope is zero.

Therefore $\cos x = 0$

Hence $x = (2n+1)\frac{\pi}{2}; n \in Z$

Hence the given curve has horizontal tangent at point (x_0, y_0)

$\Leftrightarrow x_0 = (2n+1)\frac{\pi}{2}$ and $y_0 = (-1)^n \forall \in Z$ (See fig)

5. Problem: Show that the tangent at any point θ on the curve $x = c\sec\theta, y = c\tan\theta$ is $y\sin\theta = x - c\cos\theta$

Solution: Slope of the tangent at any point θ (i.e.., at $(c\sec\theta, c\tan\theta)$) on the curve is

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{c\sec^2\theta}{c\sec\theta\tan\theta} = \cos ec\theta$$

Therefore the equation of the tangent is

$$y - c\tan\theta = \cos ec\theta(x - c\sec\theta)$$

i.e.., $y\sin\theta = x - c\cos\theta$

Exercise 11(b)

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
2. Find the slope of the tangent to the curve $y = \frac{x-1}{y-2}, x \neq 2$ at $x = 10$.
3. Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point whose x^- coordinate is 2.
4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x^- coordinate is 3.
5. Find the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.
6. Find the slope of the normal to the curve $\theta = \frac{\pi}{4}$ at $\theta = \frac{\pi}{2}$
7. Find the points at which the tangents to the curve $y = x^3 - 3x^2 - 9x + 7$ are parallel to the x^- axis.
8. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).
9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.
10. Find the equation of tangent and normal to the following curves at the points indicated against:
    - (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0,5)
    - (ii) $y = x^3$ at (1,1)
    - (iii) $y = x^2$ at (0,0)
    - (iv) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$
    - (v) $y = x^2 - 4x + 2$ at (4,2)
    - (vi) $y = \frac{1}{1+x^2}$ at (0,1)
11. Find the equations of tangent and normal to the curve xy=10 at (2, 5).
12. Find the equation of tangent and normal to the curve $y = x^3 + 4x^2$ at (-1, 3).
13. If the slope of the tangent to the curve $x^3 - 2xy + 4y = 0$ at a point on it is $-3/2$, then find the equations of tangent and normal at that point.
14. If the slope of the tangent to the curve $y = x\log x$ at a point on it is $3/2$, the find the equations of tangent and normal at the point.
15. Find the tangent and normal to the curve $y = 2e^{\frac{-x}{3}}$ at the point where the curve meets the Y-axis.
16. Show that the tangent at $P(x_1, y_1)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is
    $yy_1^{\frac{-1}{2}} + xx_1^{\frac{-1}{2}} = a^{\frac{1}{2}}$

17. At what point on the curve $x^2 - y^2 = 2$ the slopes of tangents are equal to 2?
18. Show that the curves $x^2 + y^2 = 2 \, \& \, 3x^2 + y^2 = 4x$ have a common tangent at the point (1, 1).
19. At a point (x_1, y_1) on the curve $x^3 + y^3 = 3axy$ show that the tangent $\left(x_1^2 - ay_1\right)x + \left(y_1^2 - ax_1\right)y = ax_1y_1$
20. Show that the tangent at the point P (2, -2) on the curve $y(1-x) = x.$ makes intercepts of equal length on the co-ordinates axes and the normal at P passes through the origin.
21. If the tangent at any point on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepts the coordinate axes in A and B then show that the length AB is constant.
22. If the tangent at any point P on the curve $x^m y^n = a^{m+n} (mn \neq 0)$ meets the coordinate axes in A, B then show that AP: BP is a constant.

11.4 Lengths of tangent, normal, sub tangent and subnormal:

In this section we define the length of tangent, normal, sub tangent and sub normal and derive formulae to find these lengths.

11.4.1 Definition: Suppose *P=(a, f(a))* is a point on the curve *y=f(x)*. Let the tangent and normal to the curve at *P* meet the X-axis in *L* and *G* respectively. Let *M* be the foot of the perpendicular drawn from *P* onto the X-axis.

Then

(i) *PL* is called the length of the tangent.
(ii) *PG* is called the length of the normal
(iii) *LM* is called the length of the sub tangent.
(iv) *MG* is called the length of sub normal.

If $\angle PLM = \varphi$ then, $\angle MPG = \varphi$ In general if , $\theta \neq 0 \, \& \, \varphi \neq \frac{\pi}{2}$

We can find simple formulae for the above four lengths.

(i) Length of the tangent = $PL = PM \cos ec\varphi$

$$\left| f(a) \cos ec\varphi \right| = \left| \frac{f(a)\sqrt{1+(f'(a))^2}}{f'(a)} \right| \qquad (f'(a) \neq 0 \text{ as } \varphi \neq 0)$$

(ii) Length of the normal = $PG = PM \left|\sec\varphi\right|$

$$= \left| f(a) \sec\varphi \right| = \left| f(a)\sqrt{1+(f'(a))^2} \right|$$

(iii) Length of the sub tangent = $LM = \left| \frac{f(a)}{\tan\varphi} \right| = \left| \frac{f(a)}{f'(a)} \right|$

(iv) Length of the subnormal $= MG = |f(a)\tan\varphi| = |f(a)f'(a)|$

In case of implicit we write $\left(\dfrac{dy}{dx}\right)_{(0,f(a))}$ instead of $f'(a)$ in the above formulae.

In case of general point (x, y) on a curve, the above formulae can be remembered as

(i) Length of tangent $= \left|\dfrac{y\sqrt{1+(y')^2}}{y'}\right|$

(ii) Length of normal $= \left|y\sqrt{1+(y')^2}\right|$

(iii) Length of sub tangent $= \left|\dfrac{y}{y'}\right|$

(iv) Length of subnormal $= |yy'|$

11.4.2 Solved Problems:

1. Problem: Show that the length of the subnormal at any point on the curve $y^2 = 4ax$ is a constant.

Solution: Differentiating $y^2 = 4ax$ with respect to x, we have

$$2yy' = 4a \Rightarrow y' = \frac{2a}{y} \text{ i.e.., } yy' = 2a$$

Therefore the length of the subnormal at any point (x, y) on the curve

$= |yy'| = |2a|$, a constant.

2. Problem: Show that the length of the sub tangent at any point on the curve $y = a^x (a > 0)$ is a constant.

Solution: Differentiating $y = a^x$ w.r.t x, we have $y' = a^x \log a$

Therefore the length of the sub tangent at any point (x, y) on the curve is

$$= \left|\frac{y}{y'}\right| = \left|\frac{a^x}{a^x \log a}\right| = \frac{1}{\log a} = \text{ constant.}$$

3. Problem: Show that the square of the length of sub tangent at any point on the curve $by^2 = (x+a)^3 (b \neq 0)$ varies with the length of the subnormal at that point.

Solution: Differentiating $by^2 = (x+a)^3$ w.r.t x, we get $2byy' = 3(x+a)^2$

The length of the subnormal at any point (x, y) on the curve

$$= |yy'| = \left|\frac{3}{2b}(x+a)^2\right| \qquad (1)$$

The square of the length of the sub tangent

$$= \left|\frac{y}{y'}\right|^2 = \frac{y^2}{y'^2}$$

$$= \frac{(x+a)^2}{b\left[\frac{3(x+a)^2}{2by}\right]^2} = \frac{(x+a)^3}{b} \times \frac{4 \times b^2 \times y^2}{9(x+a)^4}$$

$$= \frac{(x+a)^3}{b} \times \frac{4}{9} \times b^2 \times \frac{(x+a)^3}{b} \times \frac{1}{(x+a)^4} \qquad (\because by^2 = (x+a)^3)$$

$$= \frac{4}{9}(x+a)^2 \qquad (2)$$

Therefore the square of the length of sub tangent at any point on the curve varies with the length of the subnormal at that point.

4. Problem: Find the value of k so that the length of the subnormal at any point on the curve $y = a^{1-k}x^k$ is constant.

Solution: Differentiating $y = a^{1-k}x^k$ w.r.t x, we get $y' = ka^{1-k}x^{k-1}$

Length of subnormal at any point P (x, y) on the curve

$$= |yy'| = |yka^{1-k}x^{k-1}|$$

$$= |ka^{1-k}x^k a^{1-k}x^{k-1}| = |ka^{2-2k}x^{k-1}|$$

$$= |ka^{2-2k}x^{2k-1}|$$

In order to make these values a constant, we should have $2k - 1 = 0 \Rightarrow k = \frac{1}{2}$

Exercise 11(c)

1. Find the lengths of sub tangent and subnormal at a point on the curve $y = b\sin\frac{x}{a}$
2. Show that the length of the subnormal at any point on the curve $xy = a^2$ varies as the cube of the ordinate of the point.

3. Show that at any point (x, y) on the curve $y = be^{\frac{x}{a}}$, the length of the sub tangent is a constant and the length of the subnormal is $\frac{y^2}{a}$.
4. Find the value of *k* so that the length of the subnormal at any point on the curve $xy^k = a^{k+1}$ is constant.
5. At any point *t* on the curve $x = a(t + \sin t), y = a(1 - \cos t)$, find the lengths of tangent , normal, sub tangent and sub normal.
6. Find the lengths of normal and subnormal at a appoint on the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$
7. Find the lengths of sub tangent , subnormal at a point *t* on the curve

$$x = a(\cos t + t\sin t)$$
$$y = a(\sin t - t\cos t)$$

11.5 Angle between two curves and condition for orthogonality of curves:

If two curves $C_1 \& C_2$ intersect at a point *P*, then the angle between the tangents to the curves at *P* is called the angle between the curves at *P* (see fig).

In general, there are two angles between these two tangents; if both of these angles are not equal, then one is an acute angle and the other obtuse.

It is customary to consider the acute angle to be the angle between the curves.

Let *y=f(x), y=g(x)* denote the curves $C_1 \& C_2$ and let these two curves intercept at the point $P(x_{0,}y_0)$.

Let $m_1 = f'(x)|_p \& m_2 = g'(x)|_p$ be the slopes of tangents at *P* to curves $C_1 \& C_2$ respectively.

(i) In case $m_1 = m_2$, the curves have a common tangent at *P*. Then the angle between the curves is zero. In this case we say that the curves touch each other at *P*. This include $m_1 = m_2$ =0 also.

(ii) If $m_1 m_2 = -1$ then the tangent at *P* to the curves are perpendicular. In this case the curves are said to cut each other orthogonally at *P*.

(iii) $m_1 m_2 \neq -1 \& m_1 \neq m_2$ and φ is the acute angle between the curves at *P*,then

$$\tan\varphi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

(iv) If either of $m_1 and m_2$ say $m_2 = 0$, then the angle between the curves is $\varphi = \tan^{-1}(m_1)$.

11.5.1 Solved Problems:

1. Problem: Find the angle between the curves $xy = 2 \,\&\, x^2 + 4y = 0$.

Solution: Let us first find the points of intersection of $xy = 2 \,\&\, x^2 + 4y = 0$

Putting $y = \frac{-x^4}{4}$ in $xy = 2$, we get $x^3 = -8$

i.e.., $x = -2 \Rightarrow y = \frac{-x^2}{4} = -1$

Therefore the point of intersection of the curves is P (-2, -1)

$$xy = 2 \Rightarrow y' = \frac{-2}{x^2}$$

$$x^2 + 4y = 0 \Rightarrow y' = \frac{-x}{2}$$

Slope of the tangent to the curve $xy = 2$ at P is

$$m_1 = y'|_{(-2,-1)} = \frac{-2}{(-2)^2} = \frac{-1}{2}$$

Slope of the tangent to the curve $x^2 + 4y = 0$ at P is

$$m_2 = \frac{-x}{2}|_{(-2,-1)} = 1$$

Let φ be the angle between the curves at P. Then

$$\tan\varphi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 + \left(-\frac{1}{2} \times 1\right)}\right| = 3$$

Therefore $\varphi = \tan^{-1} 3$.

2. Problem: Find the angle between the curve $2y = e^{\frac{-x}{2}}$ and Y – axis.

Solution: Equation of Y axis is $x = 0$. The point of intersection of the curve

$2y = e^{\frac{-x}{2}}$ and $x = 0$ is $P\left(0, \frac{1}{2}\right)$

The angle ψ made by the tangent to the curve $2y = e^{\frac{-x}{2}}$ at P with X axis is given by

$$\tan\psi = \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = \left.\frac{-1}{4}e^{\frac{-x}{2}}\right|_{\left(0,\frac{1}{2}\right)} = \frac{-1}{4}$$

Further if φ is the angle between the Y axis and the tangent at P to the curve $2y = e^{\frac{-x}{2}}$ then we have

$$\tan\varphi = \left|\tan\left(\frac{\pi}{2} - \psi\right)\right| = |\cot\psi| = 4$$

Therefore the angle between the curve and the Y axis is $\tan^{-1} 4$.

3. Problem: Show that the condition for the orthogonality of the curve $ax^2 + by^2 = 1 \,\&\, a_1x^2 + b_1y^2 = 1$ is $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$.

Solution: Let the curves $ax^2 + by^2 = 1 \,\&\, a_1x^2 + b_1y^2 = 1$ intersect at $P(x_1, y_1)$ so that $ax_1^2 + ay_1^2 = 1$ and $a_1x^2 + b_1y^2 = 1$,

From which we get (by cross multiplication rule.)

$$\frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a - a_1} = \frac{1}{ab_1 - a_1b}$$

(1)

Differentiating $ax^2 + by^2 = 1$ w.r.t x, we get

$$\frac{dy}{dx} = \frac{-ax}{by}$$

Hence if m_1 is the slope of the tangent at $P(x_1, y_1)$ to the curve

$$ax^2 + by^2 = 1 \text{ then } m_1 = \frac{-ax_1}{by_1}$$

Similarly the slope m_2 of the tangent at P to $a_1x^2 + b_1y^2 = 1$ is given by

$$m_2 = \frac{-a_1x_1}{b_1y_1}$$

Since the curves cut orthogonally, we have $m_1 m_2 = -1$

i.e.., $$\frac{aa_1 x_1^2}{bb_1 y_1^2} = -1 \text{ or } \frac{x_1^2}{y_1^2} = \frac{-bb_1}{aa_1} \quad (2)$$

now from (1) and (2) the condition for the orthogonality of the given curves is

$$\frac{b_1 - b}{a - a_1} = \frac{-bb_1}{aa_1} \text{ or } (b-a)a_1 b_1 = (b_1 - a_1)ab \text{ or } (b-a)a_1 b_1 = (b_1 - a_1)ab$$

Exercise 11(d)

1. Find the angles between the curves given below
   (i) $x + y + 2 = 0; x^2 + y^2 - 10y = 0$
   (ii) $y^2 = 4x; x^2 + y^2 = 5$
   (iii) $x^2 + 3y = 3; x^2 - y^2 + 25 = 0$
   (iv) $x^2 = 2(y+1); y = \frac{8}{x^2 + 4}$
   (v) $2y^2 - 9x = 0; 3x^2 + 4y = 0$ (in the (iv) ques)
   (vi) $y^2 = 8x; 4x^2 + y^2 = 32$
   (vii) $x^2 y = 4; y(x^2 + 4) = 8$
   (viii) Show that the curves $6x^2 - 5x + 2y = 0 \,\&\, 4x^2 + 8y^2 = 3$ touch each other at $\left(\frac{1}{2}, \frac{1}{2}\right)$

11.6 Derivative as a rate of change:

In this section we learn how the derivative can be used to determine the rate of change of a variable. We also discuss their application to the physics and social studies

11.6.1 Average rate of change: If $y = f(x)$ then the average rate of change in y between $x = x_1 \,\&\, x = x_2$ is defined as $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Geometrically the average rate of change of y w.r.t x is the slope of secant line joining $(x_1, f(x_1)) \,\&\, (x_2, f(x_2))$ which are the points lying on the graph of $y = f(x)$.

The units of average rate of change of a function are the units of y per unit of the variable x.

11.6.2 Instantaneous rate of change of a function f at $x = x_0$: If $y = f(x)$ then instantaneous rate of change of a function f at $x = x_0$ is defined as

$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ which is equal to $f'(x_0)$. i.e.., instantaneous rate of change of the function f at x is $f'(x)$.

11.6.3 Note: Instantaneous rate of change of the function f at x is $f'(x)$

11.6.4 Rectilinear motion: The motion of a particle in a line is called rectilinear motion. It is customary to represents the line of motion by a coordinate axis. We choose a reference point (origin), appositive direction (to the right of origin) and a unit of distance on the line.

The rectilinear motion is described by $s = f(t)$ where $f(t)$ is the rule connecting *s and t*. Here s is the coordinate of the particle for the amount of time t that elapsed since the motion began.

If a particle moves according to the rule $s = f(t)$ where *s* is the displacement of the particle at time *t*, then $\frac{\Delta s}{\Delta t}$ is the average rate of change in *s* between *t* and $t + \Delta t$.

i.e.., $$\frac{\Delta s}{\Delta t} = \frac{f(x + \Delta t) - f(t)}{\Delta t}$$

since the rate of change o displacement in the velocity , we call $\frac{\Delta s}{\Delta t}$ as the average velocity of the function $s = f(t)$ between the time t and $t + \Delta t$.

11.6.5 Note: If $s = f(t)$ then the average velocity between $t = t_1$ and $t = t_2$ is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

11.6.6 Instantaneous velocity: Suppose a taxi-car travelled 400kms in 8hours. Then its average velocity in 8hours is 50km/hr. The average velocity 50km/hr of the taxi-car does not imply that the car at each point of its travelled path has the velocity 50km/hr. The velocity of taxi-car at a given instant during movement of the car is shown on its speedometer.

Expression for velocity

If $s = f(t)$ then instantaneous rate of change of function f or s at $t = t_0$ is $\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t)}{\Delta t}$ and it is equal to $f'(t_0)$ or $\left.\frac{ds}{dt}\right|_{t=t_0}$. The rate of change of displacement in a unit time is the velocity. Therefore $f'(t_0)$ [or $\left.\frac{ds}{dt}\right|_{t=t_0}$] represents the instantaneous velocity of the particle at time $t = t_0$. Further $f'(t)$ (or $\left(\frac{ds}{dt}\right)$) represents the instantaneous velocity at any time t.

11.6.7 Note: The acceleration of a particle at time $t = t_0$, moving with $s = f(t)$ is given by $\left(\frac{d^2s}{dt^2}\right)$ at $t = t_0$ since the acceleration is the rate of change of velocity.

11.6.8 Note: 1. The acceleration of a particle at any time t, moving with $s = f(t)$ is given by $\left(\frac{d^2s}{dt^2}\right)$.

2. If $y = f(x)$ and x and y are functions of t then $\frac{dy}{dt} = f'(x)\frac{dx}{dt}$

11.6.9 Solved Problems:

1. Problem: Find the average rate of change of $s = f(t) = 2t^2 + 3$ between $t = 2$ and $t = 4$.

Solution: The average rate of change of s between $t = 2$ and $t = 4$ is

$$\frac{f(4)-(2)}{4-2} = \frac{35-11}{4-2} = 12$$

2. Problem: Find the rate of change of area of a circle w.r.t radius when $r = 5$.

Solution: Let A be the area of the circle with radius r. Then $A = \pi r^2$. Now the rate of change of area A w.r.t r is given by $\frac{dA}{dr} = 2\pi r$. When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$

Thus, the area of the circle is changing at the rate of $10\pi cm^2 / cm$.

3. Problem: The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing at a rate of the edge is 10 centimetres?

Solution: Let x be the length of the edge of the cube, V be its volume and S be its surface area. Then $V = x^3$ & $S = 6x^2$. Given that rate of change of volume is $9cm^3 / \sec$.

Therefore , $\frac{dV}{dt} = 9cm^3 / \sec$

Now differentiating V w. r t t we get ,

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow 9 = 3x^2 \frac{dx}{dt}$$

i.e.., $$\frac{dx}{dt}=\frac{3}{x^2}$$

Differentiating S w.r.t t we get,

$$\frac{dS}{dt}=12x\times\frac{dx}{dt}=12x\times\frac{3}{x^2}=\frac{36}{x}$$

Hence, when x = 10cm, $\frac{dS}{dt}=\frac{36}{10}=3.6cm^2/\sec$

4. Problem: A particle is moving in a straight line so that after its distance is 's' (in cms) from a fixed point on the line is given by $s=f(t)=8t+t^3$. Find (i) the velocity at time t = 2sec (ii) the initial velocity (iii) acceleration at t = 2sec.

Solution: The distance s and time t are connected by the relation

$$s=f(t)=8t+t^3$$

Therefore, velocity is given by $v=8+3t^2$

And the acceleration is given by $a=\frac{d^2s}{dt^2}=6t$

(i) The velocity at $t = 2$ is 8 + 3 (4) = 20cm/sec

(ii) The initial velocity (t = 0) is 8cm/sec

(iii) The acceleration at t = 2 is 6(2) =12cm/ $\sec^2$.

5. Problem: A container in the shape of an inverted cone has height 12cm and radius 6cm at the top. If it is filled with water at the rate of $12cm^3/\sec$.,what is the rate of change in the height of water level when the tank is filled 8cm?

Solution: Let OC be height o water level at 't' sec (fig). The triangles OAB and OCD are similar triangles. Therefore

$$\frac{CD}{AB}=\frac{OC}{OA}$$

Let OC=h and CD =r. Given that AB=6cm, OA=12cm.

$$\frac{r}{6}=\frac{h}{12} \quad \text{i.e.., } r=\frac{h}{2} \qquad (1)$$

Volume of the cone V is given by $V=\frac{\pi r^2 h}{3}$ (2)

Using (1), we have $V = \frac{rh^3}{12}$ (3)

Differentiating (3) w.r.t t we get $\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

Hence $\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$

When h = 8cm, the rate of rise of the water level (height) is $\left(\frac{dh}{dt}\right)_{h=8}$

i.e.., $\left(\frac{1}{\pi}\right)\frac{4}{8^2}(12) = \frac{3}{4\pi}$ cm/sec.

Hence, the rate of the change of water level is $\frac{3}{4\pi}$ cm/sec when the water level of the tank is 8cm.

6. Problem: A particle is moving along a line according to $s = f(t) = 4t^3 - 3t^2 + 5t - 1$ where s is measured in meters and t is measured in seconds. Find the velocity and acceleration at time t. At what time the acceleration is zero.

Solution: since $s = f(t) = 4t^3 - 3t^2 + 5t - 1$, the velocity at time t is

$$v = \frac{ds}{dt} = 12t^2 - 6t + 5$$

And the acceleration at time t is $a = \frac{d^2s}{dt^2} = 24t - 6$

The acceleration is 0 if $24t - 6 = 0$

i.e.., $t = \frac{1}{4}$

The acceleration of the particle is zero at $t = \frac{1}{4}$ sec.

7. Problem: The total cost C (x) in rupees associated with production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 500$. Find the marginal cost when 3 units are produced (marginal cost is the rate of change of total cost).

Solution: Let M represent the marginal cost. Then

$$M = \frac{dC}{dx}$$

Hence, $$M = \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 500)$$

$$= 0.005(3x^2) - 0.02(2x) + 30$$

The marginal cost at $x = 3$ is

$$(M)_{x=3} = 0.005(27) - 0.02(2x) + 30$$

Hence the required marginal cost is Rs 30.02 to produce 3 units.

8. Problem: The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when $x = 5$ (marginal revenue is the rate of change of total revenue).

Solution: Let m denote the marginal revenue. Then

$$m = \frac{dR}{dx} \text{ (since the total revenue is R}(x))$$

Given that $$R(x) = 3x^2 + 36x + 5$$

$$\therefore m = 6x + 36$$

The marginal revenue at $x = 5$ is $\left[m = \frac{dR}{dx}\right]_{x=5} = 30 + 36 = 66$.

Hence the required marginal revenue is Rs.66

Exercise 11(e)

1. At time *t* the distance *s* of a particle moving in a straight line is given by $s = -4t^2 + 2t$. Find the average velocity between $t = 2$ sec and $t = 8$ sec.
2. If $y = x^4$ then find the average rate of change of y between $x = 2$ and $x = 4$.
3. A particle moving along a straight line has the relation $s = t^3 + 2t + 3$ connecting the distance s described by the particle in time t. Find the velocity and acceleration of the particle at $t = 4$ sec.
4. The distance time formula for the motion of a particle along a straight line is $s = t^3 - 9t^2 + 24t - 18$. Find when and where the velocity is zero.
5. The displacement *s* of a particle travelling in a straight line in t seconds is given by $s = 45t + 11t^2 - t^3$.Find the time when the particle comes to rest.
6. The volume of a cube is increasing at the rate of $8\,cm^3/\sec$. How fast is the surface area increasing when the length of an edge is $12\,cm^2$.

7. A stone is dropped into a quiet lake and ripples move in circles at the speed of 5cm/sec. At the instant when the radius of circular ripple 8cm, how fast is the enclosed area increases?
8. The radius of a circle is increasing at the rate of 0.7cm/sec. What is the rate o increase of its circumference?
9. A ballo0n which always remains spherical on inflation is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of balloon increases when the radius is 15cm.
10. The radius of an air bubble is increasing at the rate of ½ cm/sec. At what rate is the volume of the bubble increasing when the radius is 1cm?
11. Assume that an object is launched upward at 980m/sec. Its position would be given by $s = -4.9t^2 + 980t$. Find the maximum height attained by the object.
12. Let a kind of bacteria grow in such a way that t sec there are $t^{3/2}$ bacteria. Find the rate of growth at time $t = 4$ hours.
13. Suppose we have a rectangular aquarium with dimensions of length 8cm, width 4m and height 3m. Suppose we are filling the tank with water at the rate o 0.4 $m^3/\sec$. How fast is the height of water changing when the water level is 25m?
14. A container is in the shape of an inverted cone has height 8m and radius 6m at the top. If it is filled with water, at the rate of $2\,m^3/\min ute$, how fast is the height of water changing when the level is 4m?
15. The total cost C(x) in rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.
16. The total revenue in rupees received from the sale of x units of a produce is given by $R(x) = 13x^2 + 26x + 15$.Find the marginal revenue when x=7.
17. A point P is moving on the curve $y = 2x^2$. The x coordinate of P is increasing at the rate of 4 units per second. Find the rate at which the y increasing when the point is at (2, 8).

Key Concepts

1. Δx is small change in x.

Δy is small change in y corresponding to Δx in x when $y = f(x)$.

2. Differential of y is denoted by dy.

3. If y is any variable then

(iv) Δy is called an absolute error in y

(v) $\frac{\Delta y}{y}$ is called a relative error in y.

(vi) $\frac{\Delta y}{y}\times 100$ is called percentage error in *y*.

4. (i) the slope *m* of the tangent at *P* is $m = f'(a)$ or $\left.\frac{dy}{dx}\right|_{(a,b)}$

(i) the equation of the tangent at P (a,b) is $y-b = m(x-a)$

(ii) The equation of the normal is $y-b = \frac{-1}{m}(x-a)$

(iii) Length of the tangent $= \left|\frac{f(a)\sqrt{1+(f'(a))^2}}{f'(a)}\right|$ $(f'(a)\neq 0)$

(iv) Length of the normal $= \left|f(a)\sqrt{1+(f'(a))^2}\right|$

(v) Length of the sub tangent $= \left|\frac{f(a)}{f'(a)}\right|$

(vi) Length of the subnormal $= \left|f(a)f'(a)\right|$

5. *y=f(x), y=g(x)* denote the curves $C_1 \& C_2$ and let these two curves intersecting at the point P .

Let $m_1 = f'(a)|_p \& m_2 = g'(a)|_p$ be the slopes of tangents at P to curves

(v) In case $m_1 = m_2$, the curves have a common tangent at P. Then the angle between the curves is zero. In this case we say that the curves touch each other at P. This include $m_1 = m_2$ =0 also.

(vi) If $m_1m_2 = -1$ then the tangent at P to the curves are perpendicular. In this case the curves are said to cut each other orthogonally at P.

(vii) $m_1m_2 \neq -1 \& m_1 \neq m_2$ and φ is the acute angle between the curves at P ,then

$$\tan\varphi = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|.$$

(viii) If either of $m_1 and m_2$ say $m_2 = 0$, then the angle between the curves is

6. (i) $\frac{dy}{dx}$ can be viewed as the rate of change of *y* with respect to *x*.

(ii) Velocity $v = \frac{ds}{dt}$

(iii) acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

ANSWERS

Exercise 11(a)

1 (i) $\Delta y = 0.2301, dy = 0.23$ (ii) $\Delta y = e^5(e^{0.02} - 1) + 0.02, dy = (e^5 + 1)0.02$

(iii) $\Delta y = 0.026005, dy = 0.026$ (iv) $\Delta y = -0.0001996, dy = -0.0002$

(v) $\Delta y = -0.0152, dy = -0.01516$

2 (i) 9.056 (ii) 400208 (iii) 5.0001 (iv) 1.9834 (v) 0.8834 (vi) 0.4987 (vii) 2.03125

3. 8 4. 7.04 5. $16\pi, 1.6\pi$ 6. $\frac{1}{2}$

Exercise 11(b)

1. 764 2. $\frac{-1}{64}$ 3. 11 4. 24 5. 1 6. $\frac{-a}{2b}$ 7. $(3,-10), (-1,2)$

8. $(3,1)$ 9. $(2,-9)$ 10. (i) $10x + y = 5, x - 10y + 50 = 0$ (ii). $3x - y + 2 = 0, x + 3y - 4 = 0$

(iii). $y = 0, x = 0$ (iv). $x + y - \sqrt{2} = 0, x = y$ (v). $4x - y - 14 = 0, x + 4y - 12 = 0$

(vi). $y - 1 = 0, x = 0$ 11. $5x + 2y - 20 = 0, 2x - 5y + 21 = 0$

12. $5x + y + 2 = 0, x - 5y + 16 = 0$ 13. at $\left(1, -\frac{1}{2}\right)$ $3x + 2y - 2 = 0, 4x - 6y - 7 = 0$

and at $\left(3, \frac{9}{2}\right)$ $3x + 2y - 18 = 0, 4x - 6y + 15 = 0$ 14. $3x - 2y - 2\sqrt{e} = 0, 4x + 6y - 7\sqrt{e} = 0$

15. $2x + 3y - 6 = 0, 3x - 2y + 4 = 0$ 17. $\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right), \left(-2\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$

Exercise 11(c)

1. $\left|a \tan \frac{x}{a}\right|, \left|\frac{b^2}{2a} \sin \frac{2x}{a}\right|$ 4. -2 5. $2a \sin \frac{t}{2}, 2a \sin \frac{t}{2} \tan \frac{t}{2}, a \sin t, 2a \sin^2 \frac{t}{2} \tan \frac{t}{2}$.

6. $a \cosh^2 \frac{x}{a}, \frac{a}{2} \sinh \frac{2x}{a}$. 7. $a(\sin t - t \cos t) \cot t, a(\sin t - t \cos t) \tan t$.

Exercise 11(d)

1. (i) $\tan^{-1} \frac{1}{7}$ (ii) $\tan^{-1} 3$ (iii) $\tan^{-1} \frac{22\sqrt{6}}{69}$ (iv) $\frac{\pi}{2}$ (v) $\tan^{-1} \frac{9}{13}$ (vi) $\tan^{-1} 3$ (vii) $\tan^{-1} \frac{1}{3}$

Exercise 11(e)

1. $-38\text{units}/\text{sec}$ 2. 120 3. $50\text{units}/\text{sec}, 24\text{units}/\text{sec}^2$

4. $t = 2\text{sec}, s = 2\text{units}, t = 4\text{sec}, s = -2\text{units}$ 5. 9 6. $\frac{8}{3}\text{cm}^2/\text{sec}$

7. $80\pi\text{cm}^2/\text{sec}$ 8. $1.4\pi\text{cm}/\text{sec}$ 9. $\frac{1}{\pi}\text{cm}/\text{sec}$ 10. $2\pi\text{cm}^3/\text{sec}$

11. 49000units 12. 180 (13). $\frac{1}{80}$ 14. $\frac{2}{9\pi}$ 15. 20.967

16. 208 17. $32\text{units}/\text{sec}$

12. LOCUS

12.1 Definition of Locus-Illustrations

12.1.1 Definition: Locus is the set of points (and only those points) that satisfy the given condition(s).

From the above definition, it follows that:

(i) Every point satisfying the given condition(s) is a point on the locus.
(ii) Every point on the locus satisfies the given condition(s).

12.1.2 Examples

1. Example: In a plane the locus of a point whose distance from a given point A is 4.

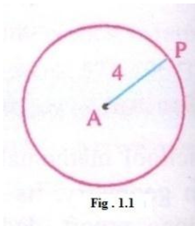


Fig . 1.1

Any point which is at a distance 4 from A Lies on the circle of radius 4 with centre A. Conversely, every point on the circle is at a distance of 4 from A. Hence, the set of all points on the circle is the locus in this example.(See Fig. 12.1).

i.e., locus = the circle, in the given plane, with centre at A and radius 4.

2. Example: Locus of a point above the X-axis whose distance from the X-axis is 2.

Let M be the point on the positive direction of the Y-axis with OM=2. Pis a point above the X-axis with PD=2, where PD is the distance of P from the X-axis (Fig.1.2).ODPM is a rectangle so that $\overleftrightarrow{MP} \parallel \overleftrightarrow{OX}$. Conversely, we can prove that any point P on the locus lies on the line parallel to X-axis 9above X-axis) which passes through M.

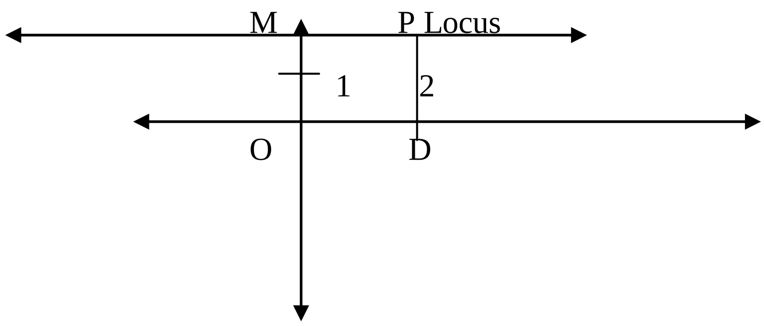


Figure 12.2

In view of the above examples, locus of a point in the plane is generally a curve in the plane. For simplicity, we call that curve itself as locus. The locus in example 1 is a circle, where as it is a straight line in Example 2.

12.1.3 Equation of Locus – Problems connected to it

It is clear that, every point on the locus satisfies the given conditions and every point which satisfies the given conditions lies on the locus.

Equation of the locus of a point is an algebraic equation in x and y satisfied by the points (x, y) on the locus alone(and by no other point).

12.1.4 Solved Problems

1. Problem: Find the equation of the locus of a point which is at a distance 5 from (-2,3) in the XOY Plane.

Solution : Let the given point be A=(-2,3) and P(x,y) be a point on the plane.

The geometric condition to be satisfied by P to be on the locus is that AP = 5 **-** (1)

i.e., $AP^2 = 5$

i.e., $\sqrt{(x+2)^2+(y-3)^2} = 5$

i.e., $x^2+4x+4+y^2-6y+9=25$

i.e., $x^2+y^2+4x-6y-12=0$ --------(2)

Let $Q(x_1,y_1)$ satisfy equation (2)

Then $x_1^2+y_1^2+4x_1-6y_1-12=0$ ----(3)

Now the distance of A from Q is AQ = $\sqrt{(x_1+2)^2+(y_1-3)^2}$

Therefore $AQ^2 = x_1^2+4x_1+4+y_1^2-6y_1+9$

$= (x_1^2+4x_1+4+y_1^2-6y_1+12)+25$

= 25 (by using (3))

Hence AQ = 5

This means that $Q(x_1,y_1)$ satisfies the geometric condition (1).

Therefore, the required equating of locus is $x^2+y^2+4x-6y-12=0$.

2. Problem: Find the locus of the third vertex of a right angled triangle, the ends of whose hypotenuse are (4, 0) and (0, 4).

Solution: Let A= (4, 0) and B = (0, 4). Let P (x, y) be point such that PA and PB are perpendicular. The $PA^2 + PB^2 = AB^2$. ---------(1)

i.e., $(x – 4)^2 + y^2 + x^2 + (y – 4)^2 = 16 + 16$

i.e., $2x^2 + 2y^2 – 8x – 8y = 0$

i.e., $x^2 + y^2 – 4x – 4y = 0$ ---------(2)

Let Q (x_1, y_1) satisfies equation (2) and Q different from A and B.

Then $x_1^2 + y_1^2 – 4x_1 – 4y_1 = 0$, $(x_1, y_1) \neq (4, 0)$ and $(x_1, y_1) \neq (0, 4)$ -------(3)

Now $QA^2 + QB^2 = (x_1 – 4)^2 + y_1^2 + x_1^2 + (y_1 – 4)^2$

$= x_1^2 - 8x_1 + 16 + y_1^2 + x_1^2 + y_1^2 - 8y_1 + 16$

$= 2(x_1^2 + y_1^2 - 48x_1 - 4y_1) + 32$

$= 32$ (by using (3))

$= AB^2$

Hence $QA2 + QB^2 = \ = AB^2$, $(Q \neq A$ and $Q \neq B$

Q (x_1, y_1) satisfies equation (1). Therefore the required equation of locus is (2).

3. Problem: Find the equation of the locus of P , if the ratio of the distance from P to A (5, -4) and B (7, 6) is 2 : 3.

Solution: Let P (x, y) be any point on the locus.

The geometric condition to be satisfied by P is $\frac{AP}{PB} = \frac{2}{3}$ -----(1)

i.e., 3AP = 2PB

i.e., $3AP^2 = 2PB^2$

i.e., $9[(x – 5)^2 + (y + 4)^2] = 4[(x – 7)^2 + (y - 6)^2]$

i.e.,$9[x^2 + 25 - 10x + y^2 + 16 + 8y] = 0 (y + 4)^2] =4[x^2 + 49 - 14x + y^2 + 36 - 12y]$

i.e., $5x^2 + 5y^2 - 34x + 120 y +29 =0$ ----------------(2)

Let Q (x_1, y_1) satisfy (2). Then $5 x_1^2 + 5y_1^2 - 34x_1 + 120 y_1 +29 =0$

Now $9AQ^2 = 9[x_1^2 + 25 -10x_1 + y_1^2 +16 + 8y_1]$

$= 5 x_1^2 + 5y_1^2 - 34x_1 + 120 y_1 +29 +4 x_1^2 + 4y_1^2 -56 x_1 - 48y_{1+} + 340$

$= 4 [x_1^2 + y_1^2 - 14x_1 -12 y_1 +49 + 36]$

$= 4[(x_1 – 7)^2 + (y_1 - 6)^2] = 4QB^2$

Thus 3 AQ = 2 QB. Thus Q (x_1, y_1) satisfy (1).

Hence the required equation of locus is $5x^2 + 5y^2 - 34x + 120 y +29 =0$

Exercise 12(a)

Short Answer Questions

1. Find the equation of a point which is at a distance 5 from A(4, –3).
2. Find the equation of Locus of a point which equidistant from the points A(–3, 2) and B(0, 4).
3. Find the equation of locus of a point equidistant from A (2, 0) and the Y – axis.
4. Find the equation of Locus of a point P such that the distance of P from the origin is twice the distance of P from A(1,2).
5. Find the equation of Locus of a point P, the square of whose distance from the origin is 4 times its Y-coordinate.
6. Find the equation of Locus of a point P such that $PA^2 + PB^2 = 2c^2$, where A = (a, 0), B = ($-a$, 0) and $0 < |a| < |c|$.

Essay Type Questions

1. Find the equation of Locus P, if the line segment joining (2, 3) and (–1, 5) subtends a right angle at P.
2. Find the equation of the Locus of P, if A = (4, 0) B = (–4, 0) and $|PA - PB| = 4$
3. Find the equation of the Locus of P, if A = (2, 3) B = (2, –3) and PA + PB = 8 .
4. A (5, 3) and B(3, –2) are two fixed points. Find the equation of the Locus of P, so that the area of triangle is 9.
5. If the distance from P to the points (2, 3) and (2, –3) is in the ratio 2:3, then find the equation of the Locus of P.
6. A(1,2) B(2, –3) and C(–2, 3) are three points, a point P moves such that $PA^2 + PB^2 = 2PC^2$. Show that the equation of the Locus of P is $7x - 7y + 4 = 0$.

Key Concepts

1. Locus is the set of points (and only those points) that satisfy the given consistent geometric condition(s).

2. An equation of a locus is an algebraic description of the locus. This can be obtained in the following way.

(i) Consider a point P (x, y) on the locus.
(ii) Write the geometric condition(s) to be satisfied by P in terms of an equation or in equation in symbols.
(iii) Apply the proper formula of coordinate geometry and translate the geometric condition(s) into an algebraic condition.
(iv) Simplify the equation so that it is free from radicals.
(v) Verify that if Q (x_1, y_1) satisfies the equation, then Q satisfies the geometric condition.
The equation thus obtained is the required equation of locus.

Answers
Exercise 1(a)
Short Answer Questions

1. $x^2 + y^2 - 8x + 6y = 0$

2. $6x + 4y = 3$

3. $y^2 - 4x + 4 = 0$

4. $3x^2 + 3y^2 - 8x - 16y + 20 = 0$

5. $x^2 + y^2 - 4y = 0$

6. $x^2 + y^2 = c^2 - a^2$

Essay Type Questions

1. $x^2 + y^2 - x - 8y + 13 = 0; (x, y) \neq (2,3) \;\&\; (x, y) \neq (-1,5)$

2. $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$

3. $16x^2 + 7y^2 - 64x - 48 = 0$ i.e., $\dfrac{(x-2)^2}{7} + \dfrac{y^2}{16} = 1$

4. $(5x - 2y - 37)(5x - 2y - 1) = 0$

5. $5x^2 + 5y^2 - 20x - 78y + 65 = 0$

13 TRANSFORMATIONS OF AXES

13.1 Translation of axes – Rules and simple problems

Transformation of axes, sometimes proves to be advantageous in solving some problems. Translation of axes is discussed in this section.

13.1.1. Definition (Translation of axes)

The transformation obtained, by shifting the origin to a given different points in the plane without the direction of coordinate axes therein is called a Translation of axes.

13.1.2.Changes in the coordinates by translation of axes :

Let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be the given coordinate axes. Suppose the origin O is shifted to O′ = (h, k) by the translation of axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ be the new axes as shown in **Fig.13.1.** Then with reference to $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ the point O′ has the coordinates (0, 0).

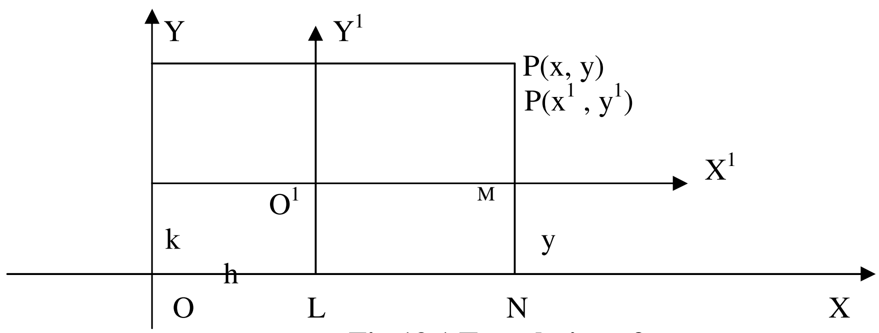


Fig.13.1 Translation of axes

Let P be a point with coordinates (x, y) in the system $\overrightarrow{OX}$ and $\overrightarrow{OY}$; and with the coordinates (x′, y′) in the new system $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$.

Then O′L = k and OL = h

Now x = ON = OL + LN

= OL+ O′M

= h + x′

And y = PN = PM + MN

= PM + O′L

= y′ + k

Thus $x = x' + h, y = y' + k$; or $x' = x - h,\ y' = y - k$.

13.1.3. Note: If the origin is shifted to (h, k) by translation of axes, the

(i) the coordinates of point P (x, y) are transformed as $P(x - h, y - k .)$ and

(ii) the equation $f(x, y) = 0$ of the curve is transformed as

$$f\ (x' + h,\ y' + k\ ) = 0\ .\ .$$

13.1.4 Examples

1. Example: When the origin is shifted to (-2,3) by translation of axes, let us find the coordinates of (1,2) with respect to new axes.

Solution: Here (h, k) = (-2, 3), Let (x, y) = (1, 2) be shifted to (x', y') by the translation of axes. Then = $(x', y') = (x - h, y - k)$ = (1 – (-2), 2 -3) = (3, -1).

2. Example:, Find the transformed equation of $2x^2 + 4xy + 5y^2 = 0$, when the origin is shifted to (3,4) by the translation of axes.

Solution: Here (h, k0 = (3, 4). On substituting $x = x' + 3$ *and* $y = y' + 4$ in the given equation, we get $2(x' + 30)^2 + 4\,(x' + 3)(y' + 4) + 5(y' + 4)^2 = 0$

Simplifying this equation , we get

$2x'^2 + 4x'y' + 5y'^2 + 28x' + 52y' + 146 = 0$

The transformed equation is $2x^2 + 4xy + 5y^2 + 28x + 52y + 146 = 0$

13.2 Rotation of axes – Rules and simple problems

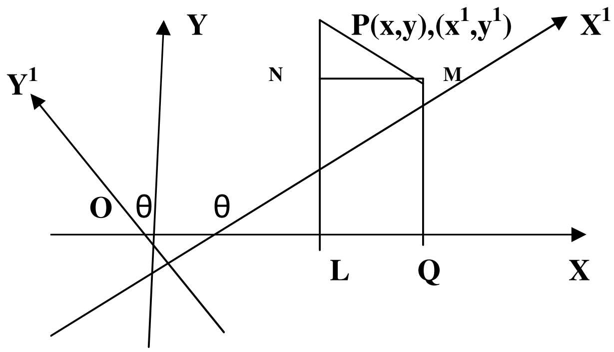
The present section is intended to discuss another transformation, namely rotation of axes.

13.2.1. Definition (Rotation of Axes)

The transformation obtained, by rotating both the coordinate axes in the plane by an equal angle, without changing the position of the origin is called a Rotation of axes.

13.2.2 Changes in the coordinates when the axes are rotated through an angle 'θ'

Let P = (x, y) with reference to the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let the axes be rotated through an angle θ in the positive direction about the origin O, to get the new system $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ as shown in **Fig.2.2.** With reference to the new axes $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$, let P = (x′, y′) . Since the angle of rotation is θ**, we have** $\underline{|XOX'} = \underline{|YOY'} = \theta$**.**



Let L, M be the feet of the perpendiculars drawn from p upon axes $\overrightarrow{OX}$ and $\overrightarrow{OX'}$.The angle between the two straight lines is equal to o the angle between their perpendiculars. Hence $\underline{|LPM} = \underline{|XOX'} = \theta$**.**

Let N be the foot of the perpendicular from M to $\overline{PL}$.

Now x = OL = OQ - LQ

= OQ – NM

$= OM\ cos\theta - PM\ sin\theta$

$= x'\,cos\,\theta - y'\,sin\theta$ ----------(1)

Also y = PL = PN + NL

$= PN + MQ$

$= PM\ cos\theta + OM\ sin\theta$

$= y'\ cos\,\theta + x'\ sin\theta$ --------(2)

Therefore $x\ =\ =\ x'\ cos\theta - y'\ sin\theta$ *and* $y\ =\ y'\ cos\theta + x'\ sin\theta.$ -----------(3)

From the above equations, the value of x′ and y′ can be found as
$x' = x\ cos\theta + y\ sin\theta$ *and* $y' = -x\ sin\theta + y\ cos\theta$ ------------(4)

The results in (3) and (4) can be remembered by the following table,

	x′	**y′**
x	**cos** θ	**-sin** θ
y	**sin** θ	**cos** θ

13.2.3. Note: When the **axes are rotated to through an angle** θ**, then**

(i) the coordinates of a point P (x, y) are transformed as
$P(x',\ y')\ =\ P\ (x\ cos\theta + y\ sin\theta, -x\ sin\vartheta + y\ cos\theta\ )$, , and

(ii) the equation $f\left(x,\ y\right) = 0$ 0f the curve is transformed as
$f\ \ (x'\ cos\theta - y'\ sin\theta,\ x'\ sin\theta + y'\ cos\theta\ )\ =\ 0$

13.2.4. Examples

1. Example: Find the coordinates of P (1, 2) with reference to the new axes, when the axes are rotated through an angle of 30^0 .

Solution: Let P (x, y) = (1, 2) and (x′, y′)be the coordinates of P in the new system.

$$x'\ =\ 1\ (cos30^0) +\ 2.(sin30^0) \quad = \frac{\sqrt{3}}{2} + 2.\ \frac{1}{2} = \frac{\sqrt{3}+2}{2}$$

$$y' = -1\ (sin30^0)\ +\ 2\ (cos30^0) \quad = \frac{-1}{2}\ +\ 2.\ \frac{\sqrt{3}}{2} = \frac{-1+2\sqrt{3}}{2}$$

Therefore, the new coordinates of P are $\left(\frac{\sqrt{3}+2}{2}\ ,\ \frac{-1+2\sqrt{3}}{2}\ \right)$

13.2.5 Note: If the origin is shifted to (h, k) and then axes are rotated t through an angle θ**,** then

(i) the coordinates of a point $P\left(x,\ y\right)$ are transformed as

$P(x',\ y')\ = (x\ cos\theta + y\ sin\theta - h, -x\ sin\,\theta +\ y\ cos\,\theta -\ k\ )$, and

(ii) the equation $f\left(x,\ y\right)\ =\ 0$ of the curve is transformed as

$f\,(x'\ cos\theta - y'\ sin\theta + h,\ x'\ sin\,\theta +\ y'\ cos\theta + k\ )$

Exercise 13(a)

Short Answer Questions

1. When the origin is shifted to $(4,-5)$ by transformation of axes, let us find the co-ordinates of the following with reference to. new axes.

   (i) (i) $(-2,4)$ $(ii)(\ 4,-5)$

2. When the origin is shifted to (2, 3) by translation of axes, the co-ordinates of a point P are changed as follows, find the coordinates of P in the original system.

   (i) $(4,5)$ (ii) $(\ -4,-3)$

3. Find the point to which the origin is to be shifted so that the point (3, 0) may change to (2, –3).
4. Find the point to which the origin is to be shifted so as to remove the first degree terms from the equation, $4x^2+9y^2-8x+36y+4=0$ . .
5. The point to which the origin is shifted and the transformed equation are given below. Find the original equation.

   (i) $(3,-4)$; $x^2+\ y^2 =\ 4$ (ii) $(-1,2)$; $x^2+\ 2y^2+16\ =0$

6. When the axes are rotated through an angle 30^o, find the new coordinates of (i) (0, 5), (ii) (–2, 4) and (iii) (0, 0).
7. When the axes are rotated through an angle 60^o, find the original co-ordinates of (i) (3, 4), (ii0 (–7, 2) and (iii) (2, 0).

8 .Find the angle through which the axes are to be rotated so as to remove the xy term in the equation $x^2+4xy-y^2-2x+2y-6=0$ . .

Essay Type Questions:

1. When the origin is shifted to the point (2, 3), the transformed equation of a curve is $x^2+3xy-y^2-2y^2+17x-7y-11=0$. Find the original equation of the curve.
2. When the origin is shifted to (–1, 2) by the translation of axes, find the transformed equation to $x^2+y^2+2x-4y+1=0$
3. When the axes are rotated through an angle 45^o , find the original equation of the curve $17x^2-16xy+17y^2=225$
4. When the axes are rotated through an angle $\frac{\pi}{4}$ find the transformed equation $3x^2+10xy+3y^2=9$

Key Concepts

1.If the origin $(0,\ 0)$ is shifted to $(h,\ k)$ by translation of axes, the

(i) the coordinates of point P (x, y) are transformed as $P(x\ -h, y\ -\ k\ .)$ and

(ii) the equation $f(x,\ y)\ =\ 0$ of the curve is transformed as

$$f\ (x'+h,\ y'+k\ )\ =\ 0\ ..$$

2. If the axes are rotated to through an angle θ, then

(i) the coordinates of a point $P\ (x,\ y)$ are transformed as

$P(x',\ y')\ =\ P\ (x\ cos\theta + \ y\ sin\theta, -x\ sin\theta + y\ cos\theta\ )$, and

(ii) the equation $f\,(x,\ y)\ =\ 0$ of the curve is transformed as

$f\ \ (x'\ cos\theta - y'\ sin\theta,\ x'\ sin\theta + \ y'\ cos\theta)\ =\ 0.$

3.If the origin is shifted to $(h,\ k)$ and then axes are rotated t through an angle θ**,** then

(i) the coordinates of a point $P\ (x,\ y)$are transformed as

$P(x',\ y')\ = (x\ cos\theta + y\ sin\theta - h, -x\ sin\theta + y\ cos\theta - k\ ),$, and

(ii) the equation $f\,(x,\ y)\ =\ 0$of the curve is transformed as

$f\ \ (x'\ cos\theta - y'\ sin\theta + h,\ x'\ sin\theta + \ y'\ cos\theta + k\ \ )$

Answers
Exercise 13(a)

Short Answer Questions

1. (i) (-6,9) (ii) (0,0)
2. (i) (6,8) (ii) (-2,6)

3. (1,3)

4. (1,-2)

5. (i) $x^2 + y^2 - 6x + 8y + 21 = 0$ (ii) $x^2 + 2y^2 + 2x - 8y + 25 = 0$

6. (i) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ (ii) $\left(2-\sqrt{3}, 1+2\sqrt{3}\right)$ (iii) 0

7.(i) $\left(\frac{3-4\sqrt{3}}{2}, \frac{3\sqrt{3}+4}{2}\right)$ (ii) $\left(\frac{-7-2\sqrt{3}}{2}, \frac{2-7\sqrt{3}}{2}\right)$ (iii) $(1, \sqrt{3})$

8. 45°

Essay Type Questions

1. $x^2 + 3xy - 2y^2 + 4x - y - 20 = 0$

2. $x^2 + y^2 - 4 = 0$

3. $25x^2 + 9y^2 = 225$

4. $8x^2 - 2y^2 = 9$

14 THE STRAIGHT LINE

14.1 Equation of Straight line – various forms – Illustrations - simple problems

In this section we discuss some basic concepts of coordinate geometry that are covered in lower classes.

14.1.1. Horizontal lines and Vertical lines

Generally the lines drawn parallel to X- axis are called horizontal lines and the lines drawn parallel to Y-axis are called vertical lines. The y- coordinate of every point on X-axis is zero. i.e., every point on X-axis satisfies the equation y= 0. Conversely, if any point has its y-coordinate is zero, then the point lies on X-axis. Therefore the equation of X-axis is y= 0. Similarly the equation of Y-axis is x= 0.

The equation y= k is the equation of horizontal line which is at a distance k units from the X- axis and lying above the X-axis.(Fig.14.1)

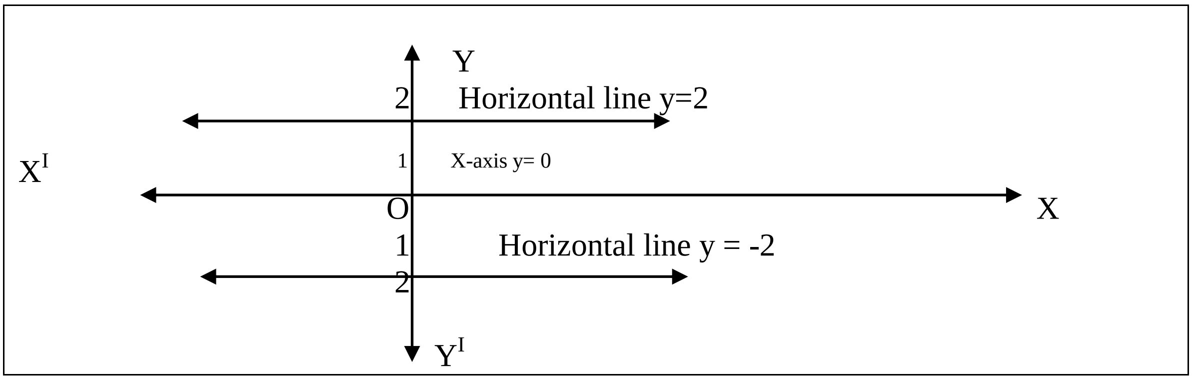


Figure 14.1

The equation y= - k is the equation of horizontal line which is at a distance k units from the X- axis and lying below the X-axis. (Fig.14.1)

In a similar way, it can be observed that the equation of the vertical line passing through the point (x_0, 0) on X-axis is $x = x_0$ 9 Here the distance of this line from Y-axis is $| x_0|$). Also the line lies to the right of the Y-axis if $x_0 > 0$ and to the left of the Y-axis if $x_0 <0$.(. (Fig.14.2)

14.1.2. The slope of a straight line.

Definition: If a non-vertical line L makes an angle θ with X-axis measured counter clock wise direction from the positive direction of the X-axis, the tan θ s called its slope of the line L. The slope of the non vertical line is generally denoted by m.

14.1.3.Note

(i). A vertical line makes aright angle with the X-axis and therefore the slope the line is not defined.

(ii). If a straight line parallel to X-axis, then θ= 0 and therefore the slope of the a horizontal line is 0

(iii) If θ is acute, tan θ is positive and if θ is obtuse then tan θ is negative.

(iv) The variation of θ is in the interval $0 \leq \theta < \pi$.

(v) If L_1 and L_2 are two non vertical lines with slopes m_1 and m_2 and if θ_1 and θ_2 are the angles in [0) *such that* $m_1 = tan\theta_1$ *and* $m_2 = tan\theta_2$ 2

$$then\ L_1 \parallel L \Leftrightarrow \theta_1 = \theta_2$$

$$\Leftrightarrow tan\theta_1 = tan\theta_2$$

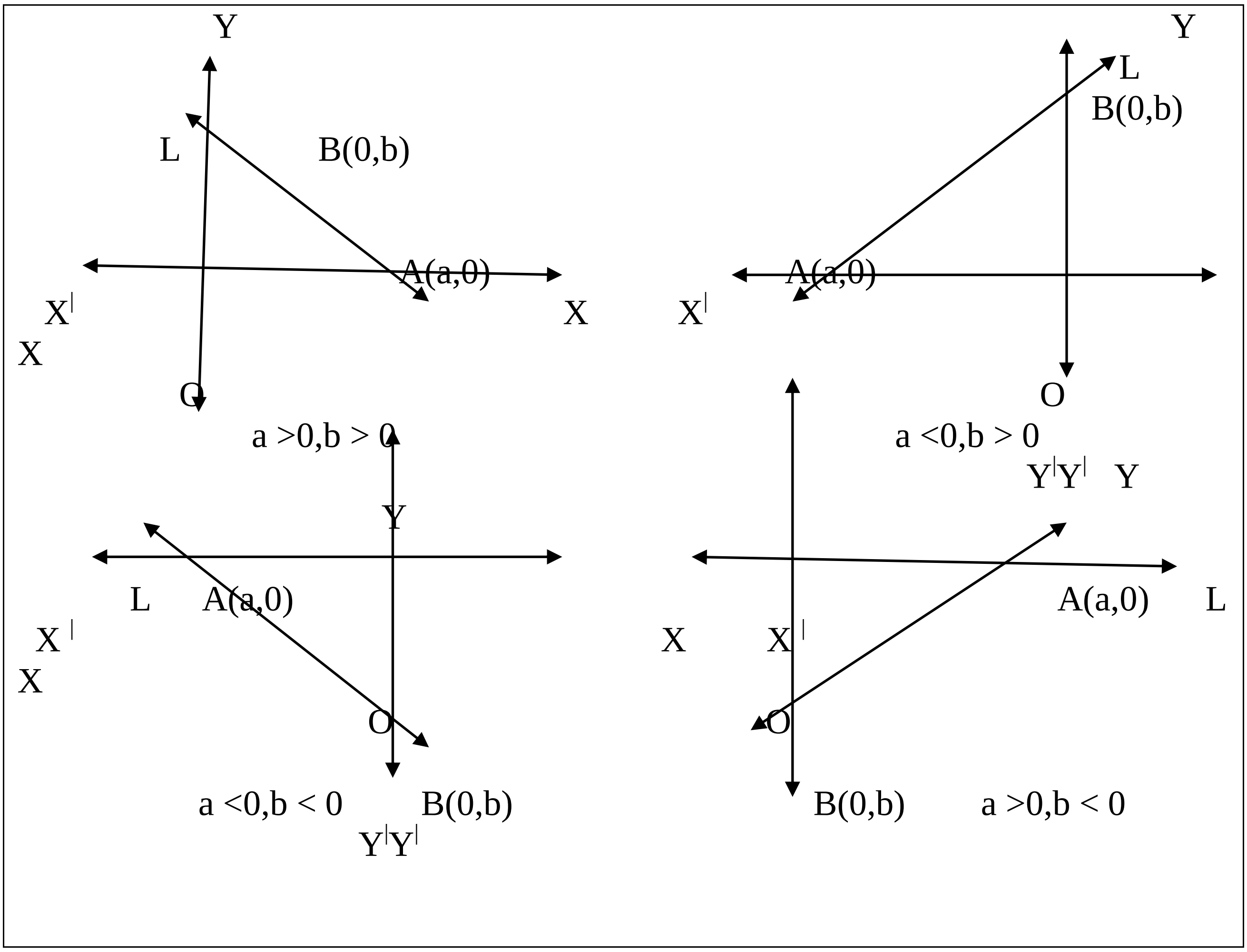
$$\Leftrightarrow m_1 = m_2$$

And L_1 and are perpendicular L_2 then $m_1 m_2 = -1$

(vi) If anon vertical straight line passes through the points (x_1, y_1) *and* (x_2, y_2) then its slope is $\frac{y_1 - y_2}{x_1 - x_2}$.

14.1.4 Intercepts

Definition: If a starlight line L intersects X-axis at A(a, 0) AND y-axis at B(0, b) then a and b are respectively called X-intercept and Y-intercept of the line L. Depending on the values of a and b the position of the line $\overleftrightarrow{AB}$ ia as given Fig . 14.



14.1.5 Note

1. A straight line passes through origin if and only if the X-intercept and Y-intercept of the

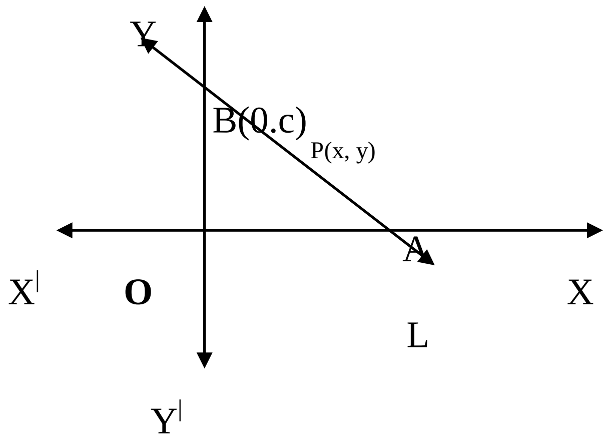
straight line are both equal to zero.

2. The X-intercept of a horizontal line is not defined.

3. The Y-intercept of a vertical line is not defined.

14.1.6 The equation of a straight line in slope-intercept form

Theorem: The equation of the straight line with slope m and cutting off Y-intercept c is $y = mx + c$..



14.1.7 Note: The straight line $y = mx + c.$ passes through the origin if c= 0. Thus the equation of the non vertical straight line passing through the origin and having slope m is y = m x.

14.1.8 Example: Find the equation of the straight line making an angle 60^0 with the positive direction of the X-axis and passing through the point (0, -2).

Solution : Slope of the line $m = tan 60^0 = \sqrt{3}$ and Y- intercept of the line c = -2. Hence the equation of the line using slope- intercept form is $y = \sqrt{3}x - 2.$

14.1.9 The equation of a straight line in intercept form

Theorem : The equation of the Straight line which cuts off non –zero intercepts a and b on the X- axis and Y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$

14.1.8 Example : Find the equation of the straight line which makes intercepts whose sum 5 and product 6.

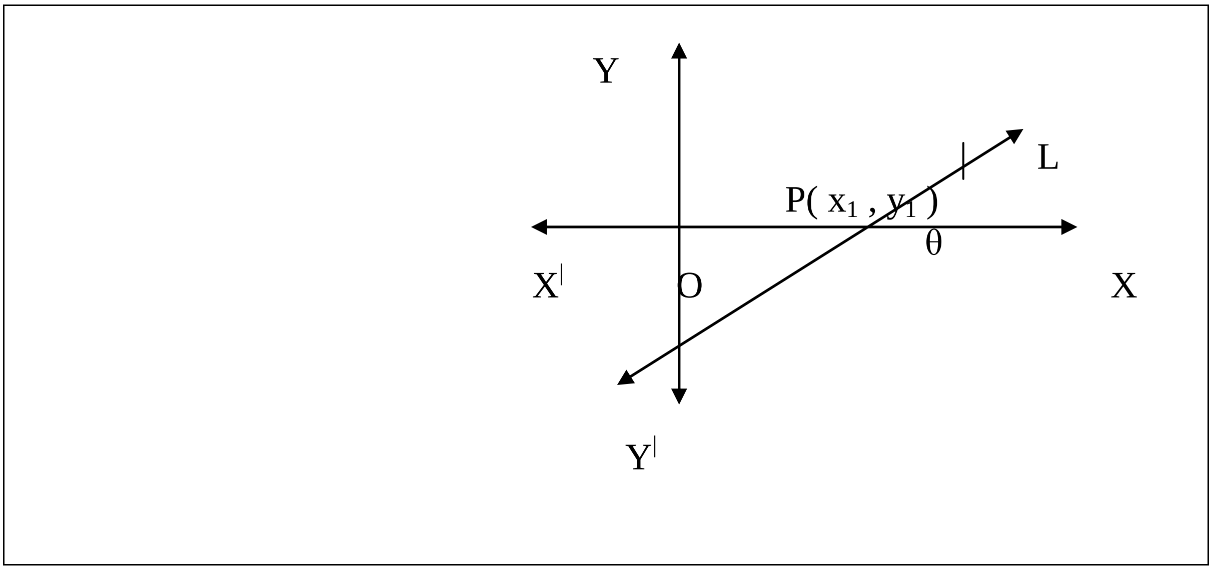
Solution : If a and b are the intercepts of the line on the axes of coordinated then a + b = 5 and ab = 6.Solving these we get a = 2, b =3 or a =3 , b = 2 .

If a = 2, b =3 ; the equation of the line is $\frac{x}{2} + \frac{y}{3} = 1.$

If a =3 , b = 2 ; the equation of the line is $\frac{x}{3} + \frac{y}{2} = 1.$

14.1.11 The equation of a straight line in point – slope form

Theorem: The equation of a straight line with slope m and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1).$

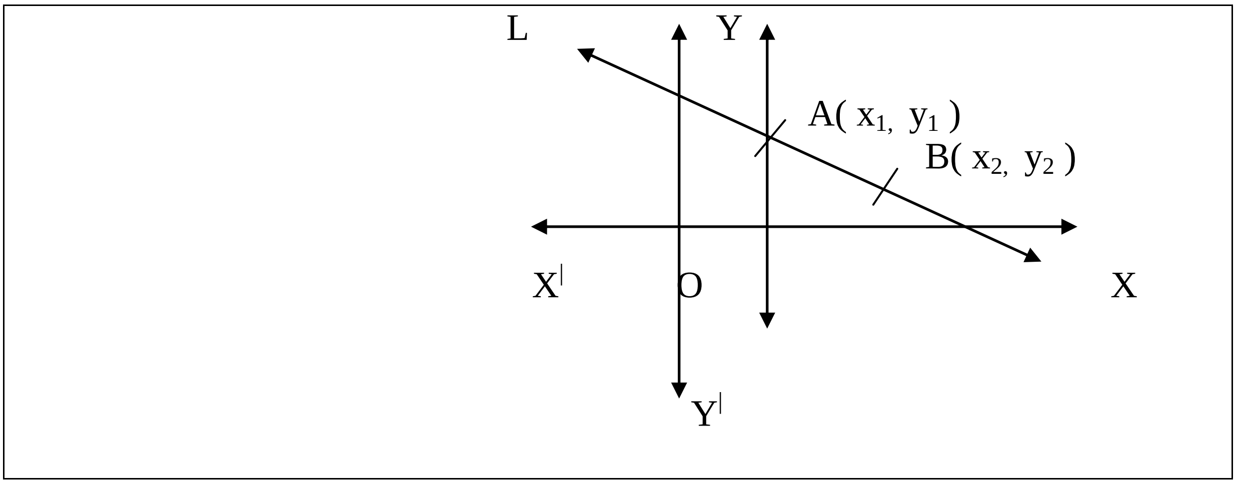


14.1.12 Example : Find the equation of the straight line making an angle 135^0 with the positive direction of the X-axis and passing through the point (-3, 2).

Solution: Slope of the line m= $\tan 135^0 = -1$ and the point is (-2, 3). Hence the equation of the line using the point- slope form is $y - 2 = (-1)(x + 3)$ (or) $x + y + 1 = 0.$

14.1.13. The equation of a straight line - Two point form

Theorem: The equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is $(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2)$.



14.1.14 Note

1. Three points A(x_1, y_1) ,B (x_2, y_2) and C (x_3, y_3) are collinear if and only if the point C lies on the line $\overleftrightarrow{AB}$. Hence $x_1 (y_1 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0$.

i.e., $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

2. The equation of a straight line containing (x_1, y_1) and (x_2, y_2) can also be written as

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

14.1.15 Example: Find the equation of the straight line passing through the points (1, 2) and (3, - 5).

Solution: The straight line passing through the points (1, 2) and (3, - 5) is

(x – 1) (2 + 5) = (y – 2) (1 – 3) (or) 7x + 2y - 3 = 0

14.1.16 Solved Problems

1. Find the equation of the straight line passing through the point (2, 3) and making non-zero intercepts n the coordinate axes whose sum is zero.

Solution: Let the intercepts made by the straight line on the coordinate axes be a, -a (a not equal to 0). Then the equation of the straight line is $\frac{x}{a}+\frac{x}{-a}=1$ (i.e.) x – y = a .

If this line passes through (2, 3) then a = 2 – 3 = -1.

Hence, equation of the required line is $x-y+1=0$.

2. Find the equation of the straight line passing through the points $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$.

Solution: The equation of the straight line containing the points $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$ is

$$(x-at_1^2)(2at_1-2at_2)=(y-2at_1)(at_1^2-at_2^2)$$

i.e.., $2(x-at_1^2)=(y-2at_1)(t_1+t_2)$

i.e..., $2x-(t_1+t_2)y+2at_1t_2=0$

Exercise 14 (a)

Short Answer Questions : -

1. Find the slopes of the lines $x+y=0$ and $x-y=0$
2. Find the equation of the line containing the points (1,2) and 1,-2)
3. . Find the angle which the straight line $y=\sqrt{3}x-4$ makes with the y-axis.
4. Write the equations of the straight lines parallel to X-axis and (i) at a distance of 3

units above the X-axis and (ii) at a distance of 4 units below the X-axis.

5. Write the equations of the straight lines parallel to Y-axis and (i) at a distance of 2 units from the Y-axis to the right of it, (ii) at a distance of 5 units from the Y-axis to the left of it.
6. Find the slopes of the straight lines passing through the following pair of points.
   $(i)\ (-3,8)\ ,\ (10,5)$ $(ii)\ (8,1),\ (-1,7)$
7. . Find the value of y if the line joining the points $(3,\ y)$ *and* $(2,\ 7)$ is parallel to the line joining the points $(-1,\ 4)$ *and* $(0,\ 6)$..
8. Find the slopes of the lines (i) parallel and (ii) perpendicular to the line passing through $(6,\ 3)$ *and* $(-4,\ 5)$.
9. Find the equations of the staright lines which makes the following angles with the positive X –axis in the positive direction and which pass through the the points given below.
   (i) $\frac{\pi}{4}$ and $(0,\ 0\ )$ (ii) $\frac{\pi}{3}$ and $(1,\ 2\ )$
10. Find the equations of the straight lines passing through the origin and making equal angles with the coordinate axes.
11. The angle made by a straight line with the positive X –axis in the positive direction and the Y-intercept cut off by it are given below. Find the equation of the straight lines. $(i)\ 150^0,\ 2\ (ii) Tan^{-1}\left(\frac{2}{3}\right),\ 3.$
12. Find the equation of the straight line passing through $(-4,\ 5)$ and cutting at equal non zero intercepts in the coordinate axes.
13. Find the equation of the straight line passing through $(-2,\ 4)$ and making non-zero intercepts whose sum is zero.
14. Find the sum of the squares of the intercepts of the line $4x\ -\ 3y\ =\ 12$ on the axes of coordinates.
15. Find the angle made by the straight line $y=\sqrt{3}x+3$ with positive direction of the X-axis measured in the counter- clock wise direction.
16. The intercepts of a straight line on the axes of coordinates are a and b. If P is the length of the perpendicular drawn from the origin to this line, write the value of P in terms of a and b.
17. Find the equations of the straight lines in the symmetric form, given the slope and a point on the line in each pat of the equation.
   $(i)\sqrt{3}\ ,\ (2,\ 3)$ $(ii)\ -\frac{1}{\sqrt{3}},\ (-2,\ 0)$ $(iii)\ -1,\ (1,\ 1)$

18. Transform the following equations into (a) slope – intercept form (b) intercept form
   (c) normal form.
   (i) $3x+\ 4y\ =5$ (ii) $4x-3y+12=0$
   (iii) $\sqrt{3}x+\ y\ =4$ (iv) $x+y-2=0.$

19. A line L has intercepts a and b on the coordinate axes. When the axes are rotated to through a given angle keeping the origin fixed, the same line L has intercepts p and q on the transformed equations. Prove that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

20. Transform the equation x/a + y/b = 1 into normal form where a > 0 and b > 0. If the perpendicular distance of the straight line from the origin is P then deduce that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Essay Type Questions

1. Find the equation of the straight line passing through the points (4, -3) and perpendicular to the line passing through the points (1, 1) and 2,3).
2. Show that the following sets of points are collinear and find the equation of the line L containing them.
   (i) (-5, 1), (5, 5), (10, 7) (ii) (1, 3), (-2,-6), (2,6)
3. A(10,4), B(-4, 9) and C(-2, -1) are the vertices of a triangle. Find the equation of (i) $\overline{AB}$ (ii) the median through to A (iii) the altitude through b (iv) the perpendicular bisector of the side $\overline{AB}$
4. Find the points on the line $3x-4y-1=0$ which are at a distance 5 units from the point (3,2)

14.2 Intersection of two straight lines

In this section, we find the point of intersection of two intersecting lines and we also discuss the two half – planes partitioned by a straight line in the coordinate plane.

14.2.1 Theorem:

If $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$ represents two intersecting lines, then their point of intersection is $\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$.

Proof: Consider the straight lines $L_1 \equiv a_1x+b_1y+c_1=0$ (1)

$$L_2 \equiv a_2x+b_2y+c_2=0$$ (2)

Since the lines intersect, we must have $a_1b_2 \neq a_2b_1$

If $P(x_0, y_0)$ is the point of intersection of lines (1) and (2), then P satisfies both the equations (1) and (2) and so,

$$a_1x_0+b_1y_0+c_1=0$$ (3)

$$a_2x_0+b_2y_0+c_2=0$$ (4)

By applying the rule of cross multiplication to (3) and (4) we obtain

$$x_0 : y_0 : 1 = (b_1c_2-b_2c_1):(c_1a_2-c_2a_1):(a_1b_2-a_2b_1)$$

Therefore $x_0 = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y_0 = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

And the point of intersection of the lines (1) and (2) is $P = \left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \right)$

Note that, if the straight lines (1) and (2) are parallel, then $a_1b_2 = a_2b_1$ and in this case, the equations (3) and (4) cannot be solved for x_0 & y_0. As such, the point of the lines doesn't exist.

14.2.2 Example:

Find the point of intersection of the straight lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1 (a \neq \pm b)$.

Solution: Let $P(x_0, y_0)$ be the point of intersection of the straight lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$.

Then, $\dfrac{x_0}{a} + \dfrac{y_0}{b} = 1$ and $\dfrac{x_0}{b} + \dfrac{y_0}{a} = 1$. From this

We obtain $\left(\dfrac{1}{a} - \dfrac{1}{b} \right) x_0 + \left(\dfrac{1}{b} - \dfrac{1}{a} \right) y_0 = 0$ (i.e.., $x_0 = y_0$)

But $\dfrac{x_0}{a} + \dfrac{y_0}{b} = 1$ and $x_0 = y_0 \Rightarrow x_0 = \dfrac{ab}{a+b} = y_0$

Therefore $P\left(\dfrac{ab}{a+b}, \dfrac{ab}{a+b} \right)$ is the point of intersection of the given lines.

14.2.3 Half – Planes:

A straight line divides the coordinate plane into three mutually disjoint sets of points,

namely

(i) The set of points on the straight line

(ii) The set of points on one side of the straight line

(iii) The set of points on the other side of the straight line.

Notation:

(i) The linear expression $ax + by + c$ is denoted by L. then the general form of the equation of a straight line is $ax + by + c = 0$ or, briefly, L = 0

(ii) We denote $ax_1 + by_1 + c$ by L_{11} and $ax_2 + by_2 + c$ by L_{22}. If the point $A(x_1, y_1)$ lies on the straight line L = 0, then the expression L_{11} equals zero. If the point A does not lie on the line L = 0, then L_{11} does not equal to equal and hence, L_{11} is either positive or negative. As such, the points of the plane are divided into three parts as

(a) The set of points for which L = 0

(b) The set of points for which L > 0

(c) The set of points for which L < 0.

In what follows, we find that the classification of points (x_1, y_1) on either side of a given straight line is based on whether L_{11} is positive or negative

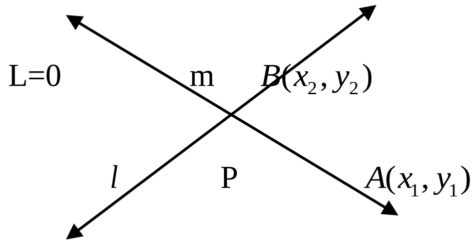
14.2.4 Theorem:

The ratio in which the straight line $L \equiv ax + by + c = 0$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $-L_{11} : L_{22}$.

Proof: Let the straight line divide the line segment $\overline{AB}$ in the ratio $l : m$ at P. Then

$P = \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m} \right)$ is a point on the straight line L = 0 and therefore,

$$a\left(\frac{lx_2 + mx_1}{l + m} \right) + b\left(\frac{ly_2 + my_1}{l + m} \right) + c = 0$$



(i.e.,) $a(lx_2 + mx_1) + b(ly_2 + my_1) + c(l + m) = 0$ or $l(ax_2 + by_2 + c) + m(ax_1 + by_1 + c) = 0$

Hence $l : m = -(ax_1 + by_1 + c) : (ax_2 + by_2 + c) = -L_{11} : L_{22}$

14.2.5 Note:

1. The points A, B are on opposite sides of the line L = 0

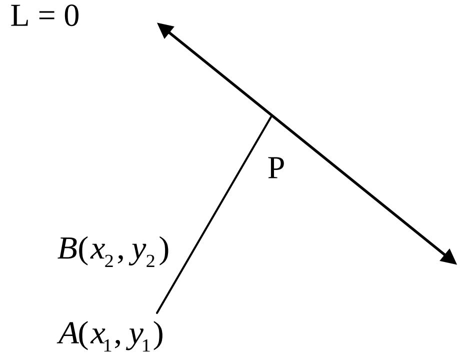
$\Leftrightarrow$ P divides $\overline{AB}$ internally (see fig)

$\Leftrightarrow l : m = -L_{11} : L_{22} > 0$

$\Leftrightarrow$ L_{11} & L_{22} Have opposite signs

2. The points A, B lie on the same side of the line L = 0

$\Leftrightarrow$ P divides $\overline{AB}$ externally (see fig)



$\Leftrightarrow l : m = -L_{11} : L_{22} < 0$

$\Leftrightarrow$ L_{11} & L_{22} Have the same sign.

3. X –axis divides $\overline{AB}$ in the ratio $-y_1 : y_2$ (since the equation of the X – axis is y = 0 and $L_{11} = y_1, L_{22} = y_2$). Similarly the Y – axis divides $\overline{AB}$ in the ratio $-x_1 : x_2$.

14.2.6 Examples:

1. Find the ratio in which the straight line $2x+3y-20=0$ divides the join of the points (2, 3) and (2, 10).

Solution: Here $L \equiv 2x+3y-20, L_{11}=2(2)+3(3)-20=-7$ and $L_{22}=2(2)+3(10)-20=14$

So the straight line L = 0 divides the given line segment in the ratio $-L_{11}:L_{22}=7:14=1:2$ and the division is internal.

2. State whether (3, 2) and (-4, -3) are on the same side or on opposite sides of the straight line $2x-3y+4=0$.

Solution: If $L \equiv 2x-3y+4$, then $L_{11}=2(3)-3(2)+4=4$ and $L_{22}=2(-4)-3(-3)+4=5$

As L_{11} & L_{22} have the same sign, the two points, lie on the same side of the given line L = 0.

3. Find the ratios in which (i) the X – axis and (ii) the Y – axis divide the line segment $\overline{AB}$ joining A (2, -3) and B (3, -6).

Solution:

(i) X – axis divides $\overline{AB}$ in the ratio $-y_1:y_2=-3:6=-1:2$

(ii) Y – axis divides $\overline{AB}$ in the ratio $-x_1:x_2=-2:3$

So both the axes of coordinates divide the line segment $\overline{AB}$ externally.

14.3 Family of straight lines - Concurrent lines

A set of straight lines having a common property is also known as a family of straight lines. In this section, we discuss (i) the family of straight lines parallel to a given line and (ii) the family of straight lines concurrent with two given intersecting lines.

14.3.1 Theorem:

Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$ represent a pair of parallel straight lines. Then the straight line represented by $\lambda_1L_1+\lambda_2L_2=0$ is parallel to each of the straight lines $L_1=0$ & $L_2=0$.

Proof: The straight lines $L_1=0$ & $L_2=0$ are parallel only if $a_1b_2=a_2b_1$

But then, $\lambda_1L_1+\lambda_2L_2 \equiv \lambda_1(a_1x+b_1y+c_1)+\lambda_2(a_2x+b_2y+c_2)$

$$\equiv (\lambda_1a_1+\lambda_2a_2)x+(\lambda_1b_1+\lambda_2b_2)y+(\lambda_1c_1+\lambda_2c_2)$$

And $a_1(\lambda_1 b_1 + \lambda_2 b_2) - b_1(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_2(a_1 b_2 - a_2 b_1) = 0$. So the straight line represented by $\lambda_1 L_1 + \lambda_2 L_2 = 0$ is parallel to the straight line $L_1 = 0$ and hence, also to the line $L_2 = 0$.

14.3.2 Theorem:

Let $L_1 \equiv a_1 x + b_1 y + c_1 = 0$ and $L_2 \equiv a_2 x + b_2 y + c_2 = 0$ represent two intersecting lines. Then

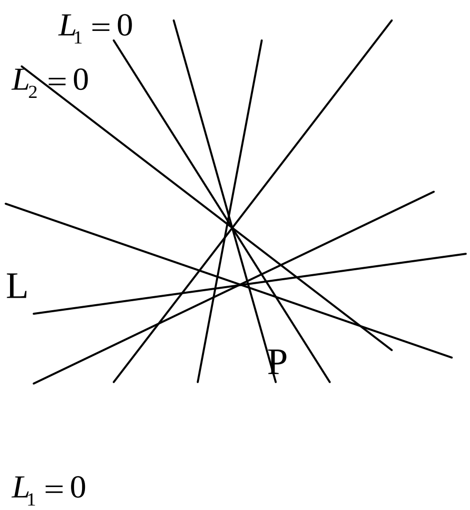
(i) The equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for parametric values of $\lambda_1 \& \lambda_2$ with $\lambda_1^2 + \lambda_2^2 \neq 0$, represents a family of straight lines passing through the point of intersection of the lines $L_1 = 0 \& L_2 = 0$.

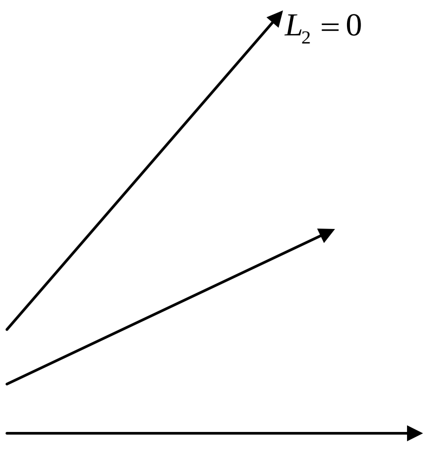
(ii) Conversely the equation of any straight line passing through the point of intersection of the given straight lines is of the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for some real $\lambda_1 \& \lambda_2$ such that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Proof: Let $P(x_1, y_1)$ be the point of intersection of the given pair of intersecting lines $L_1 = 0 \& L_2 = 0$. Then $a_1 x_2 + b_1 y_1 + c_1 = 0$ and $a_2 x_1 + b_2 y_1 + c_2 = 0$. Observe that $a_1 b_2 \neq a_2 b_1$, since $L_1 \& L_2$ intersect.

(i) If $\lambda_1^2 + \lambda_2^2 \neq 0$, then at least of $\lambda_1 \& \lambda_2$ is different from zero and since $a_1 b_2 \neq a_2 b_1$, it follows that the two numbers $\lambda_1 a_1 + \lambda_2 a_2 \& \lambda_1 b_1 + \lambda_2 b_2$ cannot be both equal to zero.
Hence the equation $\lambda_1 L_1 + \lambda_2 L_2 \equiv \lambda_1(a_1 x + b_1 y + c_1) + \lambda_2(a_2 x + b_2 y + c_2) = 0$ (i.e.) $(\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + (\lambda_1 c_1 + \lambda_2 c_2) = 0$ represents a straight line. Also $\lambda_1(a_1 x_1 + b_1 y_1 + c_1) + \lambda_2(a_2 x_1 + b_2 y_1 + c_2) = 0$. Therefore the above line passes through $P(x_1, y_1)$.

Hence, for parametric values of $\lambda_1 \& \lambda_2$ with $\lambda_1^2 + \lambda_2^2 \neq 0$, the equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ represents a family of straight lines passing through $P(x_1, y_1)$ (see fig).





(ii) Let $L \equiv px + qy + r = 0$ be a straight line passing through $P(x_1, y_1)$ (see fig). Then $px_1 + qy_1 + r = 0$. Since (p, q) not equal to (0, 0) and $a_1b_2 \neq a_2b_1$

The equations

$$\lambda_1 a_1 + \lambda_2 a_2 = p$$

$$\lambda_1 b_1 + \lambda_2 b_2 = q$$

Have unique solution for λ_1 & λ_2 such that $(\lambda_1, \lambda_2) \neq (0,0)$.

From (1), $r = -px_1 - qy_1 = -(\lambda_1 a_1 + \lambda_2 a_2)x_1 - (\lambda_1 b_1 + \lambda_2 b_2)y_1$

$$= -\lambda_1(a_1x_1 + b_1y_1) - \lambda_2(a_2x_1 + b_2y_1) = \lambda_1 c_1 + \lambda_2 c_2$$

$$\therefore px + qy + r \equiv (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + (\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 L_1 + \lambda_2 L_2$$

Thus, the equation of any straight line passing through the point of intersection of the lines $L_1 = 0 \,\&\, L_2 = 0$ can be expressed in the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for some real numbers λ_1 & λ_2 with $\lambda_1^2 + \lambda_2^2 \neq 0$.

14.3.3 Note:

1. The equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ represents L_1 if $\lambda_2 = 0 (\lambda_1 \neq 0)$ and L_2 if $\lambda_1 = 0 (\lambda_2 \neq 0)$. The equation of any straight line different from L_1 and L_2 and passing through the point of intersection of these two lines can hence be written in the form either $L_1 + \lambda L_2 = 0$ or $L_2 + \lambda L_1 = 0$ for some $\lambda \neq 0 \,\&\, \mu \neq 0$.

2. Suppose $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_1 \equiv a_2x + b_2y + c_2 = 0$ represent a pair of lines intersecting at P.

If L is a straight line in the plane of $L_1 = 0 \,\&\, L_2 = 0$ is a straight line passing through P and parallel to L, then by the above theorem, the equation of L' is of the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for $(\lambda_1, \lambda_2) \neq (0,0)$ and hence, the equation of L is of the form $\lambda_1 L_1 + \lambda_2 L_2 = \lambda_3$ for constant λ_3.

14.3.4 Example:

Find the equation of the straight line passing through the point of intersection of the lines $x + y + 1 = 0 \,\&\, 2x - y + 5 = 0$ and containing the point (5, -2).

Solution: Clearly the line $2x - y + 5 = 0$ does not contain the point (5, -2). So the equation of any straight line (other than the above line) passing through the point o intersection of the given lines is of the form $(x + y + 1) + \lambda(2x - y + 5) = 0$.

This line passing through (5, -2) only if $4 + \lambda(17) = 0$ or if $\lambda = -\frac{4}{17}$.

Therefore, the equation of the required line is $17(x + y + 1) - 4(2x - y + 5) = 0$

(i.e.) $9x + 21y - 3 = 0$ or $3x + 7y - 1 = 0$.

14.4 Condition for Concurrent lines

Given three straight lines in the XY – plane, we first obtain a necessary and sufficient condition for concurrency of these lines. This is followed by a sufficient condition for concurrency of three lines.

14.4.1 Theorem:

Let $L_1 \equiv a_1x + b_1y + c_1 = 0, L_2 \equiv a_2x + b_2y + c_2 = 0$ and $L_3 \equiv a_3x + b_3y + c_3 = 0$ be three straight lines, no two of which are parallel. Then these lines are concurrent if and only if $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

Proof: By theorem 3.5.1, the point of intersection of the lines $L_1 = 0 \,\&\, L_2 = 0$ is

$$p\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

The given straight lines are concurrent

$\Leftrightarrow$ the point P lies on the line $L_3 = 0$

$$\Leftrightarrow a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Leftrightarrow a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Leftrightarrow \sum a_1(b_2c_3 - b_3c_2) = 0$$

14.4.2 Note:

The above necessary and sufficient condition for concurrency of three straight lines can also be expressed in the determination form as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

14.4.3 Theorem (A sufficient condition for concurrency of three straight lines)

If $L_1 \equiv a_1x+b_1y+c_1=0, L_2 \equiv a_2x+b_2y+c_2=0, L_3 \equiv a_3x+b_3y+c_3=0$ are three straight lines, no two of which are parallel and if non zero real numbers $\lambda_1, \lambda_2 \,\&\, \lambda_3$ exist such that $\lambda_1L_1+\lambda_2L_2+\lambda_3L_3 \equiv 0$, then the straight lines $L_1=0, L_2=0 \,\&\, L_3=0$ are concurrent.

Proof: If $P(x_0, y_0)$ is the point intersection of the lines $L_1=0, L_2=0$, then

$$a_1x_0+b_1y_0+c_1=0 \,\&\, a_2x_0+b_2y_0+c_2=0$$

Since $$L_3 \equiv \left(\frac{-\lambda_1}{\lambda_3}\right)L_1+\left(\frac{-\lambda_2}{\lambda_3}\right)L_2$$, we have

$$a_3x_0+b_3y_0+c_3=\left(\frac{-\lambda_1}{\lambda_3}\right)(a_1x_0+b_1y_0+c_1)+\left(\frac{-\lambda_2}{\lambda_3}\right)(a_2x_0+b_2y_0+c_2)=0$$

$\therefore P(x_0, y_0)$ lies on the straight line $L_3=0$ and accordingly, the lines $L_1=0, L_2=0 \,\&\, L_3=0$ are concurrent at P.

14.4.4 Solved Examples:

1. Find the value of k, if the lines $2x-3y+k=0, 3x-4y-13=0 \,\&\, 8x-11y-33=0$ are concurrent.

Solution: Let L_1, L_2, L_2 be the straight lines whose equations are respectively

$$2x-3y+k=0 \qquad (1)$$

$$3x-4y-13=0 \qquad (2)$$

$$8x-11y-33=0 \qquad (3)$$

Solving (2) and (3) for x and y, we obtain (by applying the rule of cross – multiplication)

$$\frac{x}{132-142}=\frac{y}{-104+99}=\frac{1}{-33+32}$$

And this gives x = 1 and y = 5.

Therefore, point of intersection of the lines (2) and (3) is (11, 5)

Since L_1, L_2, L_3 are concurrent, L_1 contains (11, 5) and therefore, 2 (11) – 3 (5) + k = 0 (i.e.) k = -7.

2. If the straight lines $ax+by+c=0, bx+cy+a=0, cx+ay+b=0$ are concurrent, then prove that $a^3+b^3+c^3=3abc$.

Solution: Let L_1, L_2, L_3 be the straight lines whose equation are respectively

$$ax+by+c=0 \qquad (1)$$

$$bx+cy+a=0 \qquad (2)$$

$$cx+ay+b=0 \qquad (3)$$

Solving the equations (1) and (2), we obtain

$$\frac{x}{ab-c^2}=\frac{y}{bc-a^2}=\frac{1}{ca-b^2}$$

Therefore, the point of intersection of L_1 and L_2 is $\left(\frac{ab-c^2}{ca-b^2}, \frac{bc-a^2}{ca-b^2}\right)$

If the lines L_1, L_2, L_3 are concurrent, L_3 contains the above point of intersection of L_1 and L_2.

Hence, $$c\left(\frac{ab-c^2}{ca-b^2}\right)+a\left(\frac{bc-a^2}{ca-b^2}\right)+b=0$$

i.e.., $$c(ab-c^2)+a(bc-a^2)+b(ca-b^2)=0$$

i.e.., $$a^3+b^3+c^3=3abc$$

3. A variable straight line drawn through the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the coordinate axes at A and B. Show that the locus of the midpoint of $\overline{AB}$ is $2(a+b)xy=ab(x+y)$.

Solution: The straight lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ intersect at P whose coordinates are $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$ (see example)

$\therefore Q(x_0, y_0)$ is a point on the given locus

$\Leftrightarrow$ The straight line with x – intercept $2x_0$ and y – intercept $2y_0$ passes through P.

$\Leftrightarrow$ P lies on the straight line $\frac{x}{2x_0}+\frac{y}{2y_0}=1$.

$$\Leftrightarrow \left(\frac{1}{2x_0}+\frac{1}{2y_0}\right)\left(\frac{ab}{a+b}\right)=1$$

$$\Leftrightarrow 2(a+b)x_0y_0 = ab(x_0+y_0)$$

$\Leftrightarrow Q(x_0, y_0)$ lies on the locus $2(a+b)xy = ab(x+y)$

5. If a, b, c are in arithmetic progression, then show that the equation $ax+by+c=0$ represents a family of concurrent lines and find the point of concurrency.

Solution: If a, b, c are in arithmetic progression, then $2b=a+c$ or $a-2b+c=0$. Therefore, each member of the family of straight lines given by $ax+by+c=0$ passes through the fixed point (1, -2). Hence, the set of lines $ax+by+c=0$ for parametric values of a, b, c is a family of concurrent lines and the point of concurrency is (1, -2).

Exercise 14 (b)

Short Answer Questions:

1. Find the ratio in which the following straight lines divides the line segment joining the given points. State whether the points lie on the same side or on either side of the straight-line.
   (i) $3x-4y = 7: (2, -7)$ *and* $(-1, 3)$ (ii). $3x+4y = 6: (2, -1)$ *and* $(1, 1)$
2. Find the point of intersection of the following lines.
   (i) $4x + 8y - 1 = 0$, $2x - y + 1 = 0$ (ii) $7x + y + 3 = 0$, $x + y = 0$
3. Find the value of p if the following straight lines are concurrent.
   (i) $x + p = 0$, $y + 2 = 0$ *and* $3x + 2y + 5 = 0$
   (ii) $3x + 4y = 0$, $2x + 3y = 0$, $px + 4y = 6$
   (iii) $4x - 3y - 7 = 0$, $2x + py + 2 = 0$, $6x + 5y - 1 = 0$
4. Find the area of the triangles formed by the following straight lines and the coordinate axes.
   (i) $x - 4y + 2 = 0$ (ii) $3x - 4y + 12 = 0.$
5. A straight line meets the coordinate axes in A and B . Find the the equation of the straight line, when
   (i) $\overline{AB}$ is divided in the ratio $2 : 3$ *at* $(-5, 2$
   (ii) $\overline{AB}$ is divided in the ratio $1: 2$ *at* $(-5, 4)$..
6. A straight line forms a triangle of area 24 sq.units with the coordinate axes in the first quadrant find the equation of the line if it passes through $(3, 4)$.
7. Find the equation of the straight line passing through the points $(-1, 2)$. and $(5, -1)$.and also find the area of the triangle formed by it with the axes of coordinates.
8. A straight line with slope 1 passes through Q (-3 , 5) and meets the straight line

$x + y - 6 = 0$ at P. Find the distance PQ.

9. Show that the lines $2x + y - 3 = 0$, $3x + 2y - 2 = 0$ *and* $2x - 3y - 23 = 0$ are concurrent also find the point of concurrence.
10. If the straight lines $ax + by + c = 0$,$bx + cy + a = 0$,$cx + ay + b = 0$ are concurrent, then prove that $a^3 + b^3 + c^3 = 3abc$.
11. Determine whether or not the four straight lines with the equations $x + 2y - 3 = 0$,
    $3x + 4y - 7 = 0$ $2x + 3y - 4 = 0$ *and* $4x + 5y - 6 = 0$0 are concurrent.
12. If $3a + 2b + 4c = 0$ then show that the equation $ax + by + c = 0$ represents a family of concurrent starlight lines and fin the point of concurrency.

Essay Type questions:

1. Find the point on the starlight line $3x + y + 4 = 0$ which is equidistant from the points (-5, 6) and (3, 2).
2. Find the area of the triangle formed by the starlight lines
   $2x - y - 5 = 0$, $x - 5y + 11 = 0$ =0 and $x + y - 1 = 0$
3. A straight line through Q ($\sqrt{3}$, 2) makes an angle of ∏/6 with x-axis in positive direction if the straight line intersects the line $\sqrt{3}x - 4y + 8 = 0$ at P. Find the distance PQ.
4. Show that the straight lines
   $x + y = 0$, $3x + y - 4 = 0$ *and* $x + 3y - 4 = 0$ forms an isosceles triangle.

14.5 Angle between two lines

In this section, we first obtain a formula for the angle between two straight lines and then deduce the conditions for two lines to be parallel and perpendicular.

14.5.1 Theorem:

The angle between the straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ is $\cos^{-1}\left(\dfrac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}\right)$

Proof: Let $\overleftrightarrow{OA}$ & $\overleftrightarrow{OB}$ be the straight lines passing through the origin and parallel to the given lines $L_1 = 0$ & $L_2 = 0$ (see fig). Then the equations $\overleftrightarrow{OA}$ & $\overleftrightarrow{OB}$ are $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ respectively.

If $\angle XOB = \theta_2$, then the measure of $|\theta_1 - \theta_2|$ that lies in the interval of $\left[0, \dfrac{\pi}{2}\right]$ is the angle between the lines $L_1 = 0$ & $L_2 = 0$. Clearly, $P(b_1, -a_1)$ and $Q(b_2, -a_2)$ are points on the lines $\overleftrightarrow{OA}$ & $\overleftrightarrow{OB}$ respectively. Therefore,

$$\cos\theta_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2}}$$

$$\sin\theta_1 = \frac{-a_1}{\sqrt{a_1^2 + b_1^2}}$$

$$\cos\theta_2 = \frac{b_2}{\sqrt{a_2^2+b_2^2}}$$

And $$\sin\theta_2 = \frac{-a_2}{\sqrt{a_2^2+b_2^2}}$$

Hence, $\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$

$$= \frac{a_1a_2+b_1b_2}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}} = \cos(\theta_2-\theta_1)$$

Thus, if θ is the angle between $L_1=0\,\&\,L_2=0$, then $\theta\in\left[0,\frac{\pi}{2}\right]$ and so, $\cos\theta\geq 0$.

$$\therefore \cos\theta = \frac{|a_1a_2+b_1b_2|}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}}$$

Or $$\theta = \cos^{-1}\frac{|a_1a_2+b_1b_2|}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}}$$

14.5.2 Note:

1. A necessary and sufficient condition for the lines $L_1\,\&\,L_2$ with equations $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ to be perpendicular is that $a_1a_2+b_1b_2=0$ (since $\theta=90^\circ$). Hence the equation of a straight line perpendicular to the straight line $ax+by+c=0$ is of the form $bx-ay=k$.
2. By theorem, the straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel iff $a_1b_2=a_2b_1$. Therefore the equation of any straight line parallel to the straight line $ax+by+c=0$ is of the form $ax+by=k$.
3. The straight line containing the points $A(x_1,y_1), B(x_2,y_2)$ is $(x-x_1)(y-y_2)=(y-y_1)(x-x_2)$. Similarly the straight line containing the points $C(x_3,y_3)\,\&\,D(x_4,y_4)$ is $(x-x_3)(y_3-y_4)=(y-y_3)(x_3-x_4)$.

   Therefore, by the above note (1), the lines $\overleftrightarrow{AB}\,\&\,\overleftrightarrow{CD}$ are perpendicular if and only if $(x_1-x_2)(x_3-x_4)+(y_1-y_2)(y_3-y_4)=0$.

14.5.3 Corollary:

If $L_1\,\&\,L_2$ are non – vertical straight lines with slopes $m_1\,\&\,m_2$ respectively, then the angle between them is $Tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$ if $m_1m_2\neq -1$ and $\frac{\pi}{2}$ if $m_1m_2=-1$.

Proof: Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ be the equations of $L_1\,\&\,L_2$ respectively. Then $m_1=\frac{-a_1}{b_1}$ and $m_2=\frac{-a_2}{b_2}$

Now $$L_1\perp L_2 \Leftrightarrow a_1a_2+b_1b_2=0$$

$$\Leftrightarrow \frac{a_1 a_2}{b_1 b_2} + 1 = 0 \text{ (since } b_1 b_2 \neq 0\text{)}$$

$$\Leftrightarrow m_1 m_2 + 1 = 0$$

$$\Leftrightarrow m_1 m_2 = -1$$

Therefore, angle between $L_1 \& L_2$ is $\frac{\pi}{2}$ if $m_1 m_2 = -1$. However if $m_1 m_2 \neq -1$, then the angel

between $L_1 \& L_2$

$$= \cos^{-1}\left|\frac{a_1 a_2 + b_1 b_2}{\sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)}}\right|$$

$$= \cos^{-1}\frac{\left|\frac{a_1 a_2}{b_1 b_2} + 1\right|}{\sqrt{\left(\frac{a_1^2}{b_1^2} + 1\right)\left(\frac{a_2^2}{b_2^2} + 1\right)}}$$

$$= \cos^{-1}\left|\frac{m_1 m_2 + 1}{\sqrt{(1 + m_1^2)(1 + m_2^2)}}\right| = Tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Thus, the angle between two non perpendicular, non vertical lines with slopes $m_1 \& m_2$ is $Tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

14.5.4 Example: Find the angle between the lines $2x + y + 4 = 0$ and $y - 3x = 7$

Solution: The angle between the given lines = $\cos^{-1}\frac{|-6+1|}{\sqrt{5 \times 10}}$

$$= \cos^{-1}\left(\frac{5}{5\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

14.5.5 Example: Find the angle between the lines $\sqrt{3}x + y + 1 = 0$ $\sqrt{3}x + y + 1 = 0$ and $x + 1 = 0$.

Solution: The slope of the straight line $\sqrt{3}x + y + 1 = 0$ is $\sqrt{3}$. Therefore this line makes an angle 60° with the X – axis and 30° with the Y – axis.

But the equation $x+1=0$ represents a vertical line.

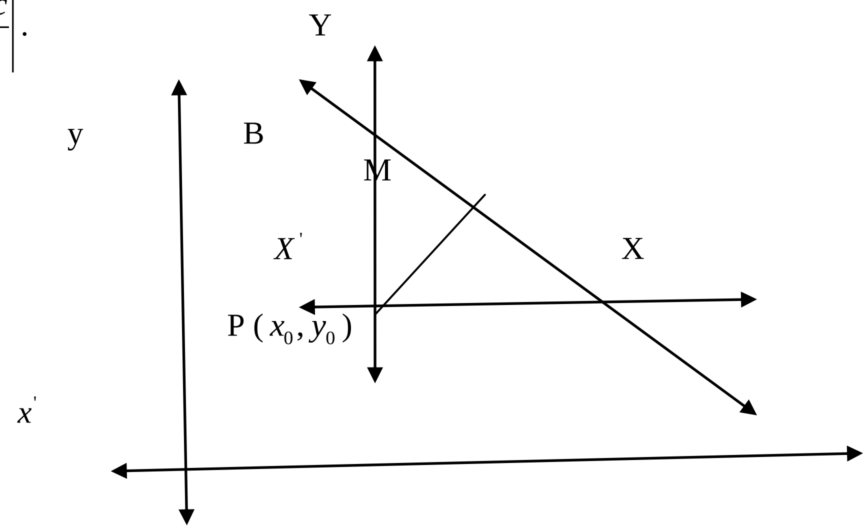
Hence, the angle between the given lines = 30°

14.6 Length of the perpendicular from a point to a line

In this section, we obtain formulas for the perpendicular distance of a point from a given straight line.

14.6.1 Theorem:

The length of the perpendicular from the point $P(x_0, y_0)$ to the straight line $ax+by+c=0$ is $\left|\dfrac{ax_0+by_0+c}{\sqrt{a^2+b^2}}\right|$.



Proof: Let $\overleftrightarrow{AB}$ be the straight line $ax+by+c=0$

If the axes of coordinates are translated to the new origin $P(x_0, y_0)$ then the coordinates of a point (x, y) will be changed to (X, Y) where $x = X + x_0$ and $y = Y + y_0$ (See fig).

Then the equation of $\overleftrightarrow{AB}$ w.r.t P as the origin is

$$a(X+x_0)+b(Y+y_0)+c=0$$

$$aX+bY+(ax_0+by_0+c)=0$$

Therefore, the perpendicular distance of $\overleftrightarrow{AB}$ from the origin P w.r.t the new axes is

$$PM = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$ (See note 2)

14.6.2 Example:

Find the perpendicular distances from the point (-3, 4) to the straight line $5x-12y=2$.

Solution: The perpendicular distance of the point (-3, 4) from the line $5x-12y=2$ is equal to $\dfrac{|5(-3)-12(4)-2|}{\sqrt{5^2+12^2}} = \dfrac{65}{13} = 5$.

14.7 Distance between two parallel lines

In this section, we obtain formulas for the distance between two parallel lines.

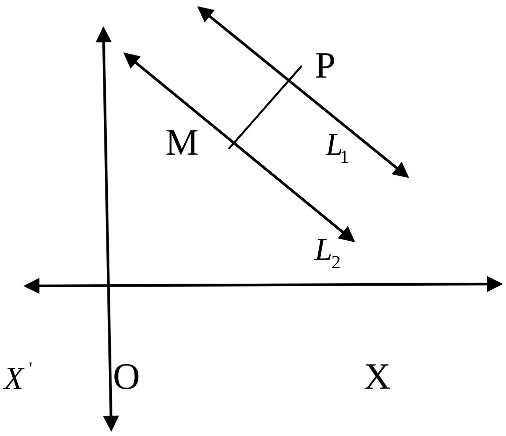
14.7.1 Theorem:

The distance between the parallel straight lines $ax+by+c_1=0$ and

$$ax + by + c_2 = 0 \text{ is } \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.$$

Proof: Let $P(x_0, y_0)$ be a point on the straight line $L_1 : ax + by + c_1 = 0$. Let L_2 be the other line. Then $ax_0 + by_0 = -c_1$ (1)

Now the distance between the parallel lines $L_1 \& L_2$ is equal to PM where PM is perpendicular distance of P from L_2 (See fig)



Therefore, $PM = \frac{|ax_0 + by_0 + c_2|}{\sqrt{a^2 + b^2}}$ (by theorem 14.9.1)

$$\frac{|(ax_0 + by_0 + c_1) + (c_2 - c_1)|}{\sqrt{a^2 + b^2}} = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \text{ (From 1)}$$

14.7.2 Example

Find the distance between the parallel straight lines $3x + 4y - 3 = 0$ and $6x + 8y - 1 = 0$.

Solution: The equations of the given straight lines can be taken as $6x + 8y - 6 = 0$ and $6x + 8y - 1 = 0$.

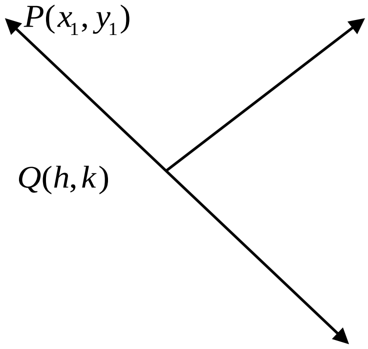
Hence by theorem 14.3.1, the perpendicular distance between these parallel lines

$$\frac{|-6 + 1|}{\sqrt{6^2 + 8^2}} = \frac{5}{10} = \frac{1}{2}$$

14.7.3 Theorem:

If $Q(h, k)$ is the foot of the perpendicular from $P(x_1, y_1)$ on the straight line $ax + by + c = 0$, then $(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$.

Proof: Equation $\overleftrightarrow{PQ}$ which is normal to the straight line $L : ax + by + c = 0$ (see fig) is $bx - ay = bx_1 - ay_1$, since $Q \in \overleftrightarrow{PQ}$, we have



$$bh - ak = bx_1 - ay_1$$

(i.e.) $b(h - x_1) = a(k - y_1)$

Or $\qquad (h - x_1) : a = (k - y_1) : b$

But, this implies that $h = a\lambda + x_1$ and $k = b\lambda + y_1$ for some $\lambda \in R$.

Since Q (h, k) is a point on L, we have $a(a\lambda + x_1) + b(b\lambda + y_1) + c = 0$

i.e.., $\qquad \lambda = -\dfrac{(ax_1 + by_1 + c)}{(a^2 + b^2)}$

Therefore, $\qquad (h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$.

14.7.4 Example:

Find the root of the perpendicular from (-1, 3) on the straight line $5x - y - 18 = 0$.

Solution: (h, k) is the foot of the perpendicular from (-1, 3) on the line $5x - y - 18 = 0$

$$\Rightarrow \frac{h-(-1)}{5} = \frac{k-3}{-1} = -\frac{(-5-3-18)}{5^2+1^2} = 1$$

$$\Rightarrow h+1 = 5 \,\&\, k-3 = -1$$

$$(h,k) = (4,2)$$

14.7.5 Theorem:

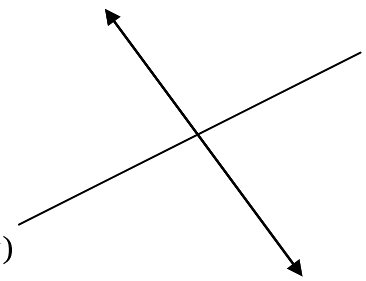
If $Q(h,k)$ is the image of the point $P(x_1, x_2)$ w.r.t the straight line $ax + by + c = 0$, then $(h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)$

Proof: Q (h , k) is the image of the point $P(x_1, y_1)$ w.r.t the line $L : ax + by + c = 0$ (see fig).

$$\Rightarrow \left(\frac{x_1+h}{2}, \frac{y_1+k}{2}\right)$$ Is the foot of the perpendicular from P on the line L

$$\Rightarrow \left(\frac{x_1+h}{2} - x_1\right) : a = \left(\frac{y_1+k}{2} - y_1\right) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$$ (from theorem 3.10.3)

$P(x_1, y_1)$



$$\Rightarrow (h - x_1) : 2a = (k - y_1) : 2b = -(ax_1 + by_1 + c) : (a^2 + b^2)$$

$$\Rightarrow (h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)$$ $\qquad Q(h,k)$

L

14.5.6Example: find the image of (1,-2) w.r.t the line $2x - 3y + 5 = 0$

$$\Rightarrow \frac{h-1}{2} = \frac{k+3}{-3} = \frac{-2(2+6+5)}{4+9} = -2$$

$$\Rightarrow h = -3, k = 4$$

Therefore (-3, 4) is the image of (1, -2) in the line $2x-3y+5=0$.

14.5.7 Solved problems:

1. Find the value of k, if the angle between the straight line $4x-y+7=0 \,\&\, kx-5y-9=0$ is 45°.

Solution: $\cos^{-1}\left(\frac{|a_1a_2+b_1b_2|}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}}\right) = \cos^{-1}\frac{|4k+5|}{\sqrt{17\left(k^2+25\right)}} = \frac{\pi}{4}$

$$\Leftrightarrow \frac{|4k+5|}{\sqrt{17\left(k^2+25\right)}} = \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow 2(4k+5)^2 = 17(k^2+25)$$

$$\Leftrightarrow 15k^2+80k-375=0$$

$$\Leftrightarrow (k-3)(3k+25)=0$$

$$\Leftrightarrow k=3 \text{ or } \frac{-25}{3}.$$

2. Find the equations of the straight lines passing through (x_0, y_0) and

(i) Parallel

(ii) Perpendicular to the straight line $ax+by+c=0$.

Solution:

(i) The equation of the straight line parallel to the line $ax+by+c=0$ and passing through (x_0, y_0) is $ax+by=k$ where $k=ax_0+by_0$ (i.e.) $a(x-x_0)+b(y-y_0)=0$.

(ii) The equation of the straight line perpendicular to the line $ax+by+c=0$ and containing the point (x_0, y_0) is $bx-ay=k$ where $k=bx_0-ay_0$ (i.e.) $b(x-x_0)-a(y-y_0)=0$.

3. Find the equation of the straight line perpendicular to the line $5x-2y=7$ and passing through the point of intersection of the lines $2x+3y=1 \,\&\, 3x+4y=6$.

Solution: Clearly neither of the straight lines $2x+3y=1 \,\&\, 3x+4y=6$ is perpendicular to the straight line $5x-2y=7$. Therefore the equation of the required line is of the form

$(2x+3y-1)+\lambda(3x+4y-6)=0$ for some $\lambda(\neq 0)\in R$. This line is perpendicular to the line $5x-2y=7$ if and only if $(2+3\lambda)5+(3+4\lambda)(-2)=0$

$$\text{(i.e.) iff } \lambda=\frac{-4}{7}$$

So the equation of the required line is $7(2x+3y-1)-4(3x+4y-6)=0$ i.e.., $2x+5y+17=0$.

4. If $2x-3y-5=0$ is the perpendicular bisector of the line segment joining (3, -4) and (α,β). Find $\alpha+\beta$.

Solution: (α,β) is the reflection of (3, -4) in the line $2x-3y-5=0$ and therefore,
$$\frac{\alpha-3}{2}=\frac{\beta+4}{-3}=\frac{-2(6+12-5)}{13}=-2$$

So $\alpha=-1,\beta=2$ and $\alpha+\beta=1$.

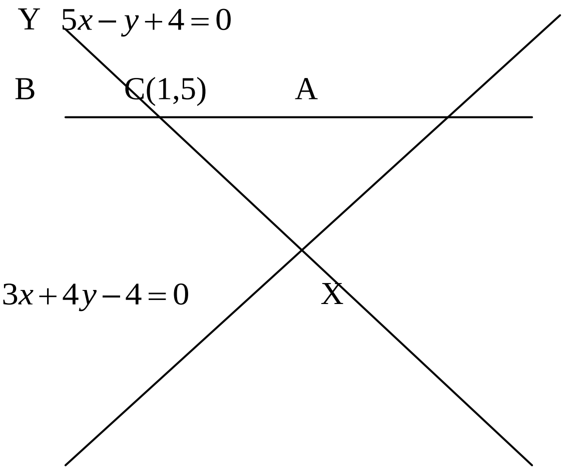
5. A line is such that its segment between the lines $5x-y+4=0$ and $3x+4y-4=0$ is bisected at the point (1, 5). Obtain its equation.

Solution: Let the required line meet $3x+4y-4=0$ at A and $5x-y+4=0$ at B, so that AB is the segment between the given lines, with its mid point at C = (1, 5).

The equation $5x-y+4=0$ can be written as $y=5x+4$ so that any point on $\overleftrightarrow{BX}$ is $(t,5t+4)$ for all real t.

Therefore $B=(t,5t+4)$ for some t. Since (1, 5) is the mid – point of $\overline{AB}$,

$$A=[2-t,10-(5t+4)]=[2-t,6-5t]$$



Since A lies on $3x+4y-4=0$, $3(2-t)+4(6-5t)-4=0$

$$\Rightarrow -23t+26=0\Rightarrow t=\frac{26}{23}$$

Therefore, $A=\left[2-\frac{26}{23},6-5\left(\frac{26}{23}\right)\right]=\left(\frac{20}{23},\frac{8}{23}\right)$

Since slope of $\overleftrightarrow{AB}$ is $\dfrac{5-\frac{8}{23}}{1-\frac{20}{23}}=\dfrac{107}{3}$

Equation of $\overleftrightarrow{AB}$ is $y-5=\frac{107}{3}(x-1)\Rightarrow 3y-15=107x-107\Rightarrow 107x-3y-92=0$.

6. An equilateral triangle has its centre at the origin and one side as $x+y-2=0$. Find the vertex opposite to $x+y-2=0$.

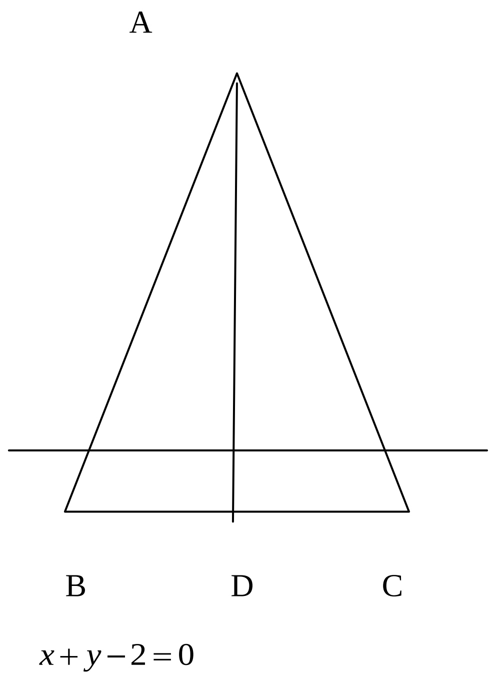
Solution: Let ABC be the equilateral triangle and $x+y-2=0$ represent the side $\overleftrightarrow{BC}$

Since O is the incentre of the triangle, $\overleftrightarrow{AD}$ is the bisector of $\underline{|BAC}$. Since the triangle is equilateral, $\overleftrightarrow{AD}$ is the perpendicular bisector of $\overleftrightarrow{BC}$.

Since O is also the centriod, $AO:OD = 2:1$. [The centriod, circumcentre incentre and orthocentre coincide]

Let D = (h ,k). Since D is the foot of the perpendicular from O onto $\overleftrightarrow{BC}$, D is given by

$$\frac{h-0}{1} = \frac{k-0}{1} = \frac{-(-2)}{2}$$



Therefore h = 1 and k = 1, D = (1, 1)

Let $A = (x_1, y_1)$

$$\therefore (0,0) = \left(\frac{2+x_1}{3}, \frac{2+y_1}{3}\right)$$

$$\therefore x_1 = -2, y_1 - 2$$

$\therefore A = (-2,-2)$, the required vertex.

Exercise 14(c)

Short Answer Questions:

1. Find the angle between the lines
   (i) $2x + y + 4 = 0$ *and* $y - 3x = 7$ (ii) $\sqrt{3}x + y + 1 = 0$ *and* $x + 1 = 0$.
2. Find the length of the perpendicular drawn from the point given against the following straight lines. (i) $5x - 2y = 4 + 0$; $(-2, -3)$ (ii) $3x - 4y + 10 + 0$:$(3, 4)$.
3. Find the equation of the straight line parallel to the line $2x + 3y + 7 = 0$ and passing through (5, 4).
4. Find the equation of the straight line perpendicular to the line $5x - 3y + 1 = 0$ and passing through (4, –3).
5. Find the value of K, if the straight lines $6x - 10y + 3 = 0$ *and* $kx - 5y + 8 = 0$ are parallel.
6. Find the value of P, if the straight lines $3x + 7y - 1 = 0$ *and* $7x - py + 3 = 0$ are mutually perpendicular.
7. Find the value of k if the straight lines

$y - 3kx + 4 = 0$ *and* $(2k-1)x - (8k-1)y - 6 = 0$ are perpendicular.

8. Find the equations of the straight lines passing through (1,3) and (i) parallel to (ii) perpendicular to the line passing through the points $(3, -5)$ *and* $(-6, 1)$..
9. Find the equation of the straight line perpendicular to the line $3x + 4y + 6 = 0$ and making an intercept – 4 on X – axis.
10. Find the foot of the perpendicular drawn from (4, 1) upon the straight line $3x - 4y + 12 = 0$
11. Find the foot of the perpendicular drawn from(3, 0) upon the straight line $5x + 12y - 41 = 0$
12. x $-3y - 5 = 0$ is the perpendicular bisector of the line segment joining the points A , B. If A = (-1, -3), find the coordinates of the point B.
13. Find the image of the point (1, 2) in the straight line $3x + 4y - 1 = 0$..
14. Show that the distance of the point (6, -2) from the line $4x + 3y = 12$ is half the distance of the point (3, 4) from the line 4x – 3y = 12.
15. Find the locus of the foot of the perpendicular from the origin to a variable straight line which always passes through a fixed point (a, b).

Essay Type questions

1. Show that the lines $x - 7y - 22 = 0,\ 3x + 4y + 9 = 0$ *and* $7x + y - 54 = 0$ form a right angled isosceles triangle.
2. Find the equations of the straight lines passing through the point (-3,2) and making tan angle of 45^0 with the straight line $3x - y + 4 = 0$..
3. Prove that the feet of the perpendiculars from origin to the lines $x + y = 4$, $x + 5y = 26$ and $15x - 27y = 424$ are collinear.
4. Find the equation of the straight line passing through the point of intersection of the lines $3x + 2y + 4 = 0$, $2x + 5y = 1$ *and whose distance from* $(2, -1)$ *is* 2..
5. Find the area of the parallelogram whose sides are $3x + 4y + 5 = 0,\ 3x + 4y - 2 = 0$, $2x + 3y + 1 = 0$ *and* $2x + 3y - 7 = 0$..
6. Find the angles of the triangle whose sides are $x + y - 4 = 0,\ 2x + y - 6 = 0$ and $5x + 3y - 15 = 0$.

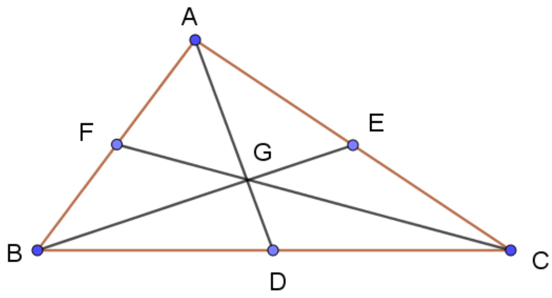
14.8 Concurrent lines - Properties related to a triangle.

There are various triads of concurrent straight lines associated with a triangle, viz. medians, altitudes, angular bisector, perpendicular bisectors of the sides etc. Geometric proofs for the concurrency of the each of the triads are already learnt in lower classes. In what follows, we give the analytical proofs for the concurrency of such triads of lines. Recall the vectorial proofs of these also.

Concurrency of the medians of a triangle

14.8.1 Theorem: The medians of a triangle are concurrent.

Proof: Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of the triangle ABC and



$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

$$L_2 \equiv a_2x + b_2y + c_2 = 0$$

$$L_3 \equiv a_3x + b_3y + c_3 = 0$$

be respectively the sides $\overleftrightarrow{BC}, \overleftrightarrow{CA} \& \overleftrightarrow{AB}$ (see fig).

Then $\lambda_r = a_rx_r + b_ry_r + c_r \neq 0$ for r = 1, 2, 3 and $a_rx_s + b_ry_s + c_s = 0$ for r, s = 1, 2,3 and $r \neq s$.

Suppose D, E, F are the mid points of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Then the equation of the median $\overleftrightarrow{AD}$ is $L_3 + \lambda L_2 = 0$ where $\lambda (\neq 0)$ is given by (1)

$$\left[a_3\left(\frac{x_2+x_3}{2}\right)+b_3\left(\frac{y_2+y_3}{2}\right)+c_3\right]+\lambda\left[a_2\left(\frac{x_2+x_3}{2}\right)+b_2\left(\frac{y_2+y_3}{2}\right)+c_2\right]=0$$

Or $\lambda_3 + \lambda\lambda_2 = 0$

Eliminating λ from (1) and (2), we obtain the equation of $\overleftrightarrow{AD}$ as $\lambda_2 L_3 - \lambda_3 L_2 = 0$

Similarly the equation of $\overleftrightarrow{BE}$ is $\lambda_3 L_1 - \lambda_1 L_3 = 0$

And the equation of $\overleftrightarrow{CF}$ is $\lambda_1 L_2 - \lambda_2 L_1 = 0$.

Since $\lambda_1(\lambda_2 L_3 - \lambda_3 L_2) + \lambda_2(\lambda_3 L_1 - \lambda_1 L_3) + \lambda_3(\lambda_1 L_2 - \lambda_2 L_1) = 0$ and $\lambda_1\lambda_2\lambda_3 \neq 0$, by theorem, it follows that the median $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent.

Note that G is the centroid of triangle ABC.

Concurrency of the altitudes of a triangle

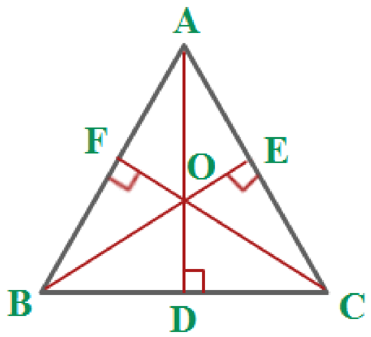
14.8.2 Theorem: The altitudes of a triangle are concurrent

Proof: Let $\overline{AD}, \overline{BE} \& \overline{CF}$ be the altitudes of triangle ABC drawn from the vertices A, B and C respectively. Let the altitudes $\overline{AD}, \overline{BE}$ intersect at 'O' (see fig). Choose 'O' as the origin of coordinates and a pair of perpendicular straight lines through O (not shown in fig) as the axes of coordinates w.r.t these axes, let $A=(x_1, y_1), B=(x_2, y_2) \& C(x_3, y_3)$.

Then $\overleftrightarrow{AD} \perp \overleftrightarrow{BC} \Rightarrow (x_1-0)(x_2-x_3)+(y_1-0)(y_2-y_3)=0$ (by note 3)

$$\Rightarrow x_1(x_2-x_3)+y_1(y_2-y_3)=0$$

Similarly $\overleftrightarrow{BE} \perp \overleftrightarrow{CA} \Rightarrow x_2(x_3-x_1)+y_2(y_3-y_1)=0$



From (1) and (2), we obtain $x_3(x_2-x_1)+y_3(y_2-y_1)=0$

(i.e.) $(x_3-0)(x_2-x_1)+(y_3-1)(y_2-y_1)=0$

This shows that $\overleftrightarrow{CO} \& \overleftrightarrow{AB}$ are perpendicular. But $\overleftrightarrow{CF}$ is the altitude drawn to $\overleftrightarrow{AB}$ from the vertex C.

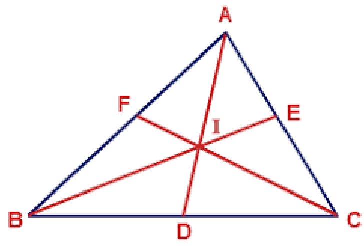
Hence $\overleftrightarrow{CF}$ passes through O. Accordingly, the altitudes $\overleftrightarrow{AD}, \overleftrightarrow{BE} \& \overleftrightarrow{CF}$ are concurrent at 'O'.

Note that 'O' is the orthocentre of triangle ABC.

Concurrency of the internal bisectors of the angles of a triangle

14.8.3 Theorem: The internal bisectors of the angle of a triangle are concurrent.

Proof: Let $A(x_1, y_1), B(x_2, y_2) \& C(x_3, y_3)$ be the vertices of the triangle ABC and $L_r \equiv a_r x + b_r y + c_r = 0$ (r = 1, 2, 3) be respectively the sides $\overleftrightarrow{BC}, \overleftrightarrow{CA} \& \overleftrightarrow{AB}$.



Then $\lambda_r = a_r x_r + b_r y_r + c_r \neq 0$ (r = 1, 2, 3)

And $\quad a_r x_s + b_r y_s + c_r = 0 (r, s = 1, 2, 3 \& r \neq s)$

Suppose $\overleftrightarrow{AD}, \overleftrightarrow{BE} \& \overleftrightarrow{CF}$ are the internal bisectors of the angles A, B, C respectively. With the usual notation in triangle ABC, we write $a = BC, b = CA, c = AB$ (see fig). Then D divides $\overline{BC}$ internally in the ratio $AB : AC = c : b$. And so $D = \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c} \right)$

Equation of the bisector $\overleftrightarrow{AD}$ is $L_3 + \lambda L_2 = 0$ (1)

Where $\lambda (\neq 0)$ is given by

$$\left[a_3 \left(\frac{bx_2 + cx_3}{b+c} \right) + b_3 \left(\frac{by_2 + cy_3}{b+c} \right) + c_3 \right] + \lambda \left[a_2 \left(\frac{bx_2 + cx_3}{b+c} \right) + b_2 \left(\frac{by_2 + cy_3}{b+c} \right) + c_2 \right] = 0$$

(i.e.) $c\lambda_3 + \lambda (b\lambda_2) = 0$ (2)

Eliminating λ from (1) and (2), we obtain the equation of the internal bisector $\overleftrightarrow{AD}$ of angle A as $u_1 \equiv (b\lambda_2) L_3 - (c\lambda_3) L_2 = 0$

Similarly the other bisectors $\overleftrightarrow{BE} \& \overleftrightarrow{CF}$ are given by

$u_2 \equiv (c\lambda_3) L_1 - (a\lambda_1) L_3 = 0$ And

$u_3 \equiv (a\lambda_1) L_2 - (b\lambda_2) L_1 = 0$ Respectively.

Writing $k_1 = a\lambda_1, k_2 = b\lambda_2, k_3 = c\lambda_3$, we observe that $k_1 k_2 k_3 \neq 0$ and $k_1 u_1 + k_2 u_2 + k_3 u_3 = 0$

Hence by theorem, it follows that the bisectors $\overleftrightarrow{AD}, \overleftrightarrow{BE} \& \overleftrightarrow{CF}$ are concurrent. The point of concurrency is called the incentre of triangle ABC, usually denoted by I (see fig)

Concurrency of the perpendicular bisectors of the sides of a triangle

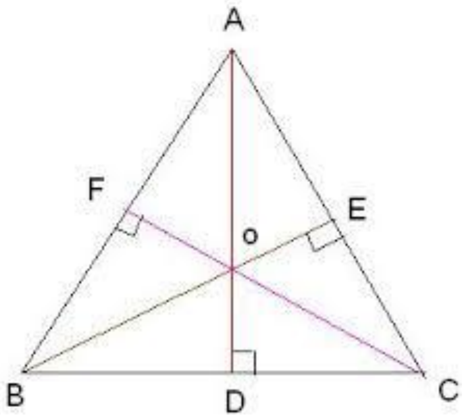
14.8.4 Theorem: The perpendicular bisectors of the sides of a triangle are concurrent.

Proof: Let D, E, F be the mid points of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively of triangle ABC; and let the perpendicular bisectors of the sides $\overline{BC}, \overline{CA}$ at O (See fig). Choose O as the origin of the coordinates and a pair of perpendicular lines through O as the axes of coordinates (not shown in the fig).

Let $(x_1, y_1), (x_2, y_2) \& (x_3, y_3)$ be the coordinates of the vertices A, B, C respectively w.r.t these axes of coordinates.

Then $D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$; $E = \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$ and $F = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Also



$$\overleftrightarrow{OD} \perp \overleftrightarrow{BC} \Rightarrow \left(\frac{x_2 + x_3}{2}\right)(x_2 - x_3) + \left(\frac{y_2 + y_3}{2}\right)(y_2 - y_3) = 0 \Rightarrow \left(x_2^2 - x_3^2\right) + \left(y_2^2 - y_3^2\right) = 0 \ldots (1)$$

$$\overleftrightarrow{OE} \perp \overleftrightarrow{CA} \Rightarrow \left(\frac{x_3 + x_1}{2}\right)(x_3 - x_1) + \left(\frac{y_3 + y_1}{2}\right)(y_3 - y_1) = 0 \Rightarrow \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right) = 0 \ldots. (2)$$

From (1) and (2) we obtain

$$\left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right) = 0 \text{ i.e.., } \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1) + \left(\frac{y_2 + y_1}{2}\right)(y_2 - y_1) = 0$$

But, this implies that $\overleftrightarrow{OF} \perp \overleftrightarrow{AB}$

Since F is the mid point of $\overleftrightarrow{OF} \& \overleftrightarrow{AB}$ is therefore the perpendicular bisector of $\overline{AB}$. Thus the perpendicular bisectors of the sides are concurrent. The point of concurrence O is the circumcentre of the triangle ABC.

Solved problems:

1. If the equation of the sides of the triangle are $7x + y - 10 = 0, x - 2y + 5 = 0, x + y + 2 = 0$. Find the orthocenter of the triangle.

Solution: Let the given triangle be ABC with the side $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{AC}$ represented by

$$7x + y - 10 = 0 \qquad (1)$$

$$x-2y+5=0 \qquad (2)$$

$$x+y+2=0 \text{ (see fig)} \qquad (3)$$

Let $\overleftrightarrow{AD} \& \overleftrightarrow{BE}$ be the altitudes drawn from A and B respectively to the sides $\overleftrightarrow{AD} \& \overleftrightarrow{BE}$.

Solving the equations (1) and (3) we obtain A = (-3, 1)

Since $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$ the equation of $\overleftrightarrow{AD}$ is $x-7y=-3-7=-10$ (4)

Solving the equation (1) and (2) we obtain B = (1, 3)

Since $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$ the equation of $\overleftrightarrow{BE}$ is $x-y=1-3=-2$ (5)

Point of intersection of the lines (4) and (5) is $H\left(\frac{-2}{3},\frac{4}{3}\right)$, which is the orthocentre of the triangle ABC

2. Find the circumcentre of the triangle whose vertices are (1, 3), (-3, 5) and (5, -1).

Solution: Let the vertices of the triangle be A (1, 3) , B(-3, 5) and C (5, -1) (see fig)

The midpoints of the sides $\overline{BC} \& \overline{CA}$ are respectively D (1, 2) and E (3, 1)

Let S be the point of intersection of the perpendicular bisectors of the sides $\overline{BC} \& \overline{CA}$.

Slope of $\overleftrightarrow{BC} = \frac{5+1}{-3-5} = \frac{-3}{4}$

Slope of $\overleftrightarrow{SD} = \frac{4}{3}$ and so the equation of $\overleftrightarrow{SD}$ is $4x-3y=4-6=-2$ (1)

Slope of $\overleftrightarrow{AC} = \frac{3+1}{1-5} = -1$

Slope of $\overleftrightarrow{SE}$ is 1 and so, we obtain of $\overleftrightarrow{SE}$ is $x-y=3-1=2$

Solving the equations (1) and (2) we obtain S = (-8, -10) which is the circumcentre of the triangle ABC.

3. Find the circumcentre of the triangle whose sides are $3x-y-5=0, x+2y-4=0, 5x+3y+1=0$.

Solution: Let the given equations represent the sides $\overleftrightarrow{BC}, \overleftrightarrow{CA} \& \overleftrightarrow{AB}$ repectively of triangle ABC (See fig).

Solving the above equations by taking two at a time, we obtain the vertices A (-2, 3), B(1, -2) and C (2, 1) of the given triangle.

The mid points of the sides $\overline{BC}$ & $\overline{CA}$ are respectively $D\left(\frac{3}{2},\frac{-1}{2}\right)$ & $E(0,2)$

Equations of $\overleftrightarrow{SD}$, the perpendicular bisector of $\overline{BC}$ is $x+3y=0$ and that of $\overleftrightarrow{SE}$, the perpendicular bisector of $\overline{CA}$ is $2x-y+2=0$.

Solving these two equations we obtain the point of intersection of the lines $\overleftrightarrow{SD}$ & $\overleftrightarrow{SE}$ which is therefore, $S=\left(\frac{-6}{7},\frac{2}{7}\right)$, the circumcentre of the triangle ABC.

4. Find the incentre of the triangle formed by the straight lines $y=\sqrt{3}x, y=-\sqrt{3}x$ & $y=3$.

Solution: The straight lines $y=\sqrt{3}x, y=-\sqrt{3}x$ makes angles 60° & 120° respectively, with OX in the anti – clock wise sense (See fig). Since y = 3 is a horizontal line, the triangle formed by the three given lines is equilateral. So its incentre is same as the centroid which will be at a distance of 2 units from the origin (the vertex of the triangle) on the positive Y – axis (which is a median).

Therefore Incentre of the triangle is I = (0, 2).

Exercise 14 (d)

1. Find the incentre of the triangle whose vertices are $(1,\sqrt{3}),(2,0)$ & $(0,0)$
2. Find the orthocenter of the triangle whose sides are given by $x+y+10=0, x-y-2=0, 2x+y-7=0$
3. Find the orthocenter of the triangle whose sides are given by $4x-7y+10=0, x+y=5$ & $7x+4y=15$.
4. Find the circumcentre of the triangle whose sides are $x=1, y=1$ & $x+y=1$.
5. Find the incentre of the triangle formed by the lines $x=1, y=1$ & $x+y=1$.
6. Find the circumcentre of the triangle whose vertices are $(1,0),(-1,2)$ & $(3,2)$
7. Find the values of k, if the angle between the straight lines $kx+y+9=0$ & $3x-y+4=0$ is $\frac{\pi}{4}$.
8. Find the equation of the straight line passing through the origin and also through the point of intersection of the lines $2x-y+5=0$ and $x+y+1=0$.
9. Find the equation of the straight line parallel to the line $3x+4y=7$ and passing through the point of intersection of the lines $x-2y-3=0$ & $x+3y-6=0$.
10. Find the equations of the straight line perpendicular to the line $2x+3y=0$ and passing through the point of intersection of the lines $x+3y-1=0$ & $x-2y+4=0$.
11. Find the equations of the straight line making non – zero equal intercepts on the coordinate axes and passing through the point of intersection of the lines $2x-5y+1=0$ & $x-3y-4=0$.
12. Find the length of the perpendicular drawn from the point of intersection of the

lines $3x+2y+4=0 \& 2x+5y-1=0$ to the straight line $7x+24y-15=0$

13. Find the value of 'a' if the distances of the points (2, 3) and (-4, a) from the straight line $3x+4y-8=0$ are equal.
14. Find the circumcentre of the triangle formed by the straight lines $x+y=0, 2x+y+5=0, x-y=2$.
15. If θ is the angle between the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$. Find the value of $\sin\theta$ when a > b.

Essays

1. Find the equations of the straight lines passing through the point (-10, 4) and making an angle θ with the line $x-2y=10$ such that $\tan\theta=2$.
2. Find the equations of the straight lines passing through the point (1, 2) and making an angle of 60° with the line $\sqrt{3}x+y+2=0$.
3. The base of an equilateral triangle is $x+y-2=0$ and the opposite vertex is (2, -1). Find the equations of the remaining sides.
4. Find the orthocentre of the triangle with the following vertices
   (i) $(-2,-1),(6,-1) \& (2,5)$
   (ii) $(5,-2),(-1,2) \& (1,4)$
5. Find the circumcentre of the triangle whose vertices are given below
   (i) $(-2,3),(2,-1) \& (4,0)$
   (ii) $(1,3),(0,-2),(-3,1)$
6. Let $\overline{PS}$ be the median of the triangle with the vertices $P(2,2), Q(6,-1), R(7,3)$. Find the equation of the straight line passing through (1, -1) and parallel to the median $\overleftrightarrow{PS}$.
7. Find the orthocentre of the triangle formed by the lines $x+2y=0, 4x+3y-5=0$ and $3x+y=0$.
8. Find the circumcentre of the triangle whose sides are given by $x+y+2=0, 5x-y-2=0, x-2y+5=0$.
9. Find the equations of the straight lines passing through (1, 1) and which are at a distance of 3 units from (-2, 3)
10. If 'p' and 'q' are the lengths of the perpendiculars from the origin to the straight lines $x\sec\alpha+y\operatorname{cosec}\alpha=a$ and $x\cos\alpha-y\sin\alpha=a\cos2\alpha$. Prove that $4p^2+q^2=a^2$.
11. Two adjacent sides of a parallelogram are given $4x+5y=0 \& 7x+2y=0$ and one diagonal is $11x+7y=9$. Find the equations of the remaining sides and the other diagonal.
12. Find the incentre of the triangle formed by the following straight lines
    (i) $x+1=0, 3x-4y=5 \& 5x+12y=27$
    (ii) $x+y-7=0, x-y+1=0, x-3y+5=0$
13. A triangle is formed by the lines $ax+by+c=0, lx+my+n=0, px+qy+r=0$. Given that the triangle is not right angled, show that the straight line $\frac{ax+by+c}{ap+bq}=\frac{lx+my+n}{lp+mq}$ passes through the orthocentre of the triangle.
14. The Cartesian equations of the sides $\overleftrightarrow{BC}, \overleftrightarrow{CA}, \overleftrightarrow{AB}$ of a triangle are respectively

$u_r \equiv a_r x + b_r y + c_r = 0, r = 1, 2, 3$. Show that the equation of the straight line passing through A and bisecting the side $\overline{BC}$ is $\frac{u_3}{a_3 b_1 - a_1 b_3} = \frac{u_2}{a_1 b_2 - a_2 b_1}$.

Key Concepts

1. Slope of a non vertical straight line passes through the points (x_1, y_1) and (x_2, y_2) is $\frac{y_1 - y_2}{x_1 - x_2}$ 2. A straight line passes through origin if and only if the X-intercept and Y-intercept of the

straight line are both equal to zero.

3 The X-intercept of a horizontal line is not defined.

4. The Y-intercept of a vertical line is not defined.

5. The equation of the straight line with slope m and cutting off Y-intercept c is y = mx + c

6.The equation of the non vertical straight line passing through the origin and having slope m is y = m x.

7.The equation of the Straight line which cuts off non –zero intercepts a and b on the X-axis and Y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$

8. The equation of a straight line with slope m and passing through the point $_1$, y_1) is $y - y_1 = m(x - x_1)$.

9. The equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is $(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2)$.

10. Three points A(x_1, y_1) ,B (x_2, y_2) and C (x_3, y_3) are collinear if and only if the point C lies on the line $\overleftrightarrow{AB}$. Hence $x_1(y_1 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.

i.e., $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

11. The equation of a straight line containing (x_1, y_1) and (x_2, y_2) can also be written as

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

12.Point of intersection of the lines $L_1 \equiv a_1 x + b_1 y + c_1 = 0$ and $L_2 \equiv a_2 x + b_2 y + c_2 = 0$ is

$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$.

13.

(i) The linear expression $ax+by+c$ is denoted by L. then the general form of the equation of a straight line is $ax+by+c=0$ or, briefly, L = 0

(ii) We denote ax_1+by_1+c by L_{11} and ax_2+by_2+c by L_{22}. If the point $A(x_1, y_1)$ lies on the straight line L = 0, then the expression L_{11} equals zero. If the point A does not lie on the line L = 0, then L_{11} does not equal to equal and hence, L_{11} is either positive or negative. As such, the points of the plane are divided into three parts as

(a) The set of points for which L = 0

(b) The set of points for which L > 0

(c) The set of points for which L < 0.

we can find that the classification of points (x_1, y_1) on either side of a given straight line is based on whether L_{11} is positive or negative

14. The ratio in which the straight line $L \equiv ax+by+c=0$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $-L_{11}:L_{22}$.

15.Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$ represent a pair of parallel straight lines. Then the straight line represented by $\lambda_1L_1+\lambda_2L_2=0$ is parallel to each of the straight lines $L_1=0$ & $L_2=0$.

16.Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$ represent two intersecting lines. Then

(i) The equation $\lambda_1L_1+\lambda_2L_2=0$ for parametric values of λ_1 & λ_2 with $\lambda_1^2+\lambda_2^2 \neq 0$, represents a family of straight lines passing through the point of intersection of the lines $L_1=0$ & $L_2=0$.

(ii) Conversely the equation of any straight line passing through the point of intersection of the given straight lines is of the form $\lambda_1L_1+\lambda_2L_2=0$ for some real λ_1 & λ_2 such that $\lambda_1^2+\lambda_2^2 \neq 0$.

17.. The equation $\lambda_1L_1+\lambda_2L_2=0$ represents L_1 if $\lambda_2=0(\lambda_1 \neq 0)$ and L_2 if $\lambda_1=0(\lambda_2 \neq 0)$. The equation of any straight line different from L_1 and L_2 and passing through the point of intersection of these two lines can hence be written in the form either $L_1+\lambda L_2=0$ or $L_2+\lambda L_1=0$ for some $\lambda \neq 0$ & $\mu \neq 0$.

18.. If $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_1 \equiv a_2x+b_2y+c_2=0$ represent a pair of lines intersecting at P.If L is a straight line in the plane of $L_1=0$ & $L_2=0$ is a straight line passing through P and parallel to L, then by the above theorem, the equation of L' is of the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for $(\lambda_1, \lambda_2) \neq (0,0)$ and hence, the equation of L is of the form $\lambda_1 L_1 + \lambda_2 L_2 = \lambda_3$ for constant λ_3.

18.Let $L_1 \equiv a_1x + b_1y + c_1 = 0, L_2 \equiv a_2x + b_2y + c_2 = 0$ and $L_3 \equiv a_3x + b_3y + c_3 = 0$ be three straight lines, no two of which are parallel. Then these lines are concurrent if and only if $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

19.The above necessary and sufficient condition for concurrency of three straight lines can also be expressed in the determination form as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

20.The angle between the straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ is $\cos^{-1}\left(\dfrac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}} \right)$

21. A necessary and sufficient condition for the lines $L_1 \& L_2$ with equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to be perpendicular is that $a_1a_2 + b_1b_2 = 0$ (since $\theta = 90^\circ$). Hence the equation of a straight line perpendicular to the straight line $ax + by + c = 0$ is of the form $bx - ay = k$.

22. By theorem, the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel iff $a_1b_2 = a_2b_1$. Therefore the equation of any straight line parallel to the straight line $ax + by + c = 0$ is of the form $ax + by = k$.

23. The straight line containing the points $A(x_1, y_1), B(x_2, y_2)$ is $(x - x_1)(y - y_2) = (y - y_1)(x - x_2)$. Similarly the straight line containing the points $C(x_3, y_3) \& D(x_4, y_4)$ is $(x - x_3)(y_3 - y_4) = (y - y_3)(x_3 - x_4)$.

Therefore, by the above note (1), the lines $\overleftrightarrow{AB} \& \overleftrightarrow{CD}$ are perpendicular if and only if $(x_1 - x_2)(x_3 - x_4) + (y_1 - y_2)(y_3 - y_4) = 0$.

24. If $L_1 \& L_2$ are non – vertical straight lines with slopes $m_1 \& m_2$ respectively, then the angle between them is $Tan^{-1}\left| \dfrac{m_1 - m_2}{1 + m_1m_2} \right|$ if $m_1m_2 \neq -1$ and $\dfrac{\pi}{2}$ if $m_1m_2 = -1$.

25.The length of the perpendicular from the point $P(x_0, y_0)$ to the straight line $ax + by + c = 0$ is $\left| \dfrac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$.

26. The distance between the parallel straight lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is $\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$.

27. If $Q(h,k)$ is the foot of the perpendicular from $P(x_1,y_1)$ on the straight line $ax+by+c=0$, then $(h-x_1):a=(k-y_1):b=-(ax_1+by_1+c):(a^2+b^2)$.

28. If $Q(h,k)$ is the image of the point $P(x_1,x_2)$ w.r.t the straight line $ax+by+c=0$, then $(h-x_1):a=(k-y_1):b=-2(ax_1+by_1+c):(a^2+b^2)$

29. The medians of a triangle are concurrent.

30. The altitudes of a triangle are concurrent

31. The internal bisectors of the angle of a triangle are concurrent.

32. The perpendicular bisectors of the sides of a triangle are concurrent.

Answers

Exercise 14 (a)

1. -1, 1
2. x – 1 = 0
3. $\frac{\pi}{6}$
4. (i) y – 3 = 0, (ii) y + 4 = 0
5. (i) x – 2 = 0, (ii) x + 5 =0
6. (i) $\frac{-3}{13}$ (ii) $\frac{-5}{2}$
7. y = 9
8. (i) – 1/5 (ii) 5
9. (i) y = x (ii) $\sqrt{3}x-y+(2-\sqrt{3})=0$
10. x = y , x = - y
11. (i) $x+\sqrt{3}y-2\sqrt{3}=0$ (ii) $2x-3y+9=0$
12. $x+y-1=0$
13. $x-y+6=0$
14. 25
15. $\frac{2\pi}{3}$
16. $\frac{|ab|}{\sqrt{a^2+b^2}}$
17. (i) $\frac{x-2}{\cos\frac{\pi}{3}}=\frac{y-3}{\sin\frac{\pi}{3}}$ (ii) $\frac{x+2}{\cos\frac{5\pi}{6}}=\frac{y}{\sin\frac{5\pi}{6}}$ (iii) $\frac{x-1}{\cos\frac{3\pi}{4}}=\frac{y-1}{\sin\frac{3\pi}{4}}$

18.(i) $y=\left(-\frac{3}{4}\right)x+\frac{5}{4}, \frac{x}{(5/3)}+\frac{y}{(5/4)}=1, x\cos\alpha+y\sin\alpha=1\left(\alpha=Tan^{-1}\frac{4}{3}\right)$

(ii) $y=\left(\frac{4}{3}\right)x+4, \frac{x}{-3}+\frac{y}{4}=1, x\cos\alpha+y\sin\alpha=\frac{12}{5}\left(\alpha=\pi-Tan^{-1}\frac{3}{4}\right)$

(iii) $y=-\sqrt{3}x+4, \frac{x}{(4/\sqrt{3})}+\frac{y}{4}=1, x\cos\left(\frac{\pi}{6}\right)+y\sin\left(\frac{\pi}{6}\right)=2$

(iv) $y=-x-2, \frac{x}{-2}+\frac{y}{-2}=1, x\cos\frac{\pi}{4}+y\sin\frac{\pi}{4}=\sqrt{2}$

20. $\left(\frac{b}{\sqrt{a^2+b^2}}\right)x+\left(\frac{a}{\sqrt{a^2+b^2}}\right)y=\frac{ab}{\sqrt{a^2+b^2}}$

Essays

1. $x+2y+2=0$
2. (i) $3x-y=0$ (ii) $x+y=a+b+c$
3. (i) $5x+14y-106=0$ (ii) $y=4$ (iii) $12x+5y+3=0$
   (iv) $28x-10y-19=0$
4. (7, 5) and (- 1, - 1)

Exercise 14 (b)

1. (i) 27 : 22 ; opposite sides (ii) 4 : 1 ; opposite sides
2. $\left(\frac{-7}{20},\frac{3}{10}\right)$ (ii) $\left(-\frac{1}{2},\frac{1}{2}\right)$
3. (i) $\frac{1}{3}$ (ii) 2 (iii) 4
4. (i) $\frac{1}{2}$ (ii) 6
5. (i) $3x-5y+25=0$ (ii) $8x-5y+60=0$
6. $4x+3y-24=0$
7. $x+2y-3=0, \frac{9}{4}$
8. $2\sqrt{2}$
9. (4, -5)
11. Not concurrent
12. $\left(\frac{3}{4},\frac{1}{2}\right)$

Essays

1. (-2, 2)
2. 9

3. 6

Exercise 14 (c)

1. (i) $\frac{\pi}{4}$ (ii) $\cos^{-1}\left(\frac{1}{\sqrt{170}}\right)$
2. (i) 0 (ii) 3/5
3. $2x+3y-22=0$
4. $3x+5y+3=0$
5. 3
6. 3
7. -1 or 1/6
8. (i) $2x+3y-11=0$ (ii) $3x-2y+3=0$
9. $4x-3y+16=0$
10. $\left(\frac{8}{5},\frac{21}{5}\right)$
11. $\left(\frac{49}{13},\frac{24}{13}\right)$
12. $\left(-\frac{8}{5},-\frac{6}{5}\right)$
13. $\left(-\frac{7}{5},-\frac{6}{5}\right)$
15. $x^2+y^2-ax-by=0$

Essays

2. $x-2y+7=0, 2x+y+4=0$

4. $y=1, 4x+3y+5=0$

5. 56

6. $\cos^{-1}\left(\frac{4}{\sqrt{17}}\right), \cos^{-1}\left(\frac{13}{\sqrt{170}}\right) \& \pi-\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

Exercise 14 (d)

1. $\left(1,\frac{1}{\sqrt{3}}\right)$
2. (- 4, - 6)
3. (1, 2)

4. $\left(\frac{1}{2}, \frac{1}{2}\right)$

5. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

6. (1, 2)

7. $2, -\frac{1}{2}$

8. $x + 2y = 0$

9. $3x + 4y - 15 = 0$

10. $3x - 2y + 8 = 0$

11. $x + y + 32 = 0$

12. $\frac{1}{5}$

13. $\frac{15}{2}$ or $\frac{5}{2}$

14. (-3, 1)

15. $\frac{a^2 - b^2}{a^2 + b^2}$

Essays

1. $3x + 4y + 14 = 0, x + 10 = 0$

2. $y = 2, y - 2 = \sqrt{3}(x - 1)$

3. $y + 1 = (2 \pm \sqrt{3})(x - 2)$

4. (i) $\left(2, \frac{5}{3}\right)$ (ii) $\left(\frac{1}{5}, \frac{14}{5}\right)$

5. (i) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (ii) $\left(-\frac{1}{3}, \frac{2}{3}\right)$

6. $2x + 9y + 7 = 0$

7. (-4, -3)

8. $\left(-\frac{1}{3}, \frac{2}{3}\right)$

9. $5x - 12y + 7 = 0, x = 1$

11. $7x + 2y = 9, 4x + 5y = 9, x - y = 0$

12. (i) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (ii) $(3, 1 + \sqrt{5})$

15. PAIR OF STRAIGHT LINES (18 HOURS)

Introduction

Given the equations of two straight lines, the methods of finding their point of intersection and the angle between them were discussed in chapter 3. In this chapter we shall find the conditions under which a second degree equation in x and y represents a pair of straight lines.

15.1Equations of a pair of lines passing through the origin, Angle between a pair of lines

In this section, we find the nature of the combined equation of a pair of straight lines passing through the origin.

15.1.1 Combined equation of a pair of straight lines:

Let $L_1 \& L_2$ denote two straight lines and let their equations be $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, i.e.., which are linear in x and y (i.e.., $a_1 \& b_1$ are not both zero and $a_2 \& b_2$ are not both zero.

Consider the equation $\left(a_1x+b_1y+c_1\right)\left(a_2x+b_2y+c_2\right)=0$. (1)

Now $P(\alpha,\beta)$ is a point on the locus represented by (1)

$$\Leftrightarrow \left(a_1\alpha+b_1\beta+c_1\right)\left(a_2\alpha+b_2\beta+c_2\right)=0$$

$$\Leftrightarrow a_1\alpha+b_1\beta+c_1=0 \text{ or } a_2\alpha+b_2\beta+c_2=0$$

That implies that P lies on L_1 or P lies on L_2.

We therefore, conclude that the locus or the graph of the equation (1) is the pair of straight lines $L_1 \& L_2$. We say that (1) is the combined equation or simply the equation of $L_1 \& L_2$.

15.1.2 Example:

The equation $6x^2+11xy-10y^2=0$ represents the pair of straight lines $3x-2y=0$ and $2x+5y=0$, since $(3x-2y)(2x+5y)\equiv 6x^2+11xy-10y^2$ (i)

Similarly, since $(3x+2y-1)(2x-3y+1)\equiv 6x^2-5xy-6y^2+x+5y-1$

(ii)

The equation $6x^2-5xy-6y^2+x+5y-1=0$ represents the pair of straight lines $3x+2y-1$ and $2x-3y+1$.

15.1.3 Definition:

If a, b, h are real numbers, not all zero, then $H \equiv ax^2+2hxy+by^2=0$ is called a homogeneous equation of second degree in x and y; and $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ is called a general equation of second degree in x and y.

The equation (1) in 4.1.1 and the combined equations (i) and (ii) of example 4.1.2 are second degree equations in x any y.

We shall now investigate the conditions under which the above two equations represent a pair of straight lines

15.1.4 Theorem:

If a, b, h are not all zero and the locus of the equation $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ contains a straight line L, then S can be written as the product of two linear factors in x and y (with real coefficient).

Proof: Let $P(x_0, y_0)$ be a point on the straight line L. By translating the origin to the point and the rotating the axes of coordinated through a suitable angle θ about the new origin, the equation of the straight line L can be transformed into the X-axis in the new coordinate system. Let a point (x, y) have coordinate (X, Y) w.r.t the new system of coordinate axes.

Then $x = x_0 + X\cos\theta - Y\sin\theta$ and $y = y_0 + X\sin\theta - Y\cos\theta$

From these equations, we obtain

$$X = (x-x_0)\cos\theta + (y-y_0)\sin\theta \text{ and } Y = (y-y_0)\cos\theta - (x-x_0)\sin\theta$$

Writing the given equation S = 0 in the new coordinates X and Y, we find the equation changes to the form $S \equiv AX^2+2HXY+BY^2+2GX+2FY+C=0$ which is a second degree equation in X and Y. (we note that A, B, H are not all zero, since $a+b=A+B, ab-h^2=AB-H^2$ and a, b, h are not all zero).

Since the locus of S = 0 contains the straight line L whose equation is Y = 0 in the new coordinate system, every point on the line Y = 0 satisfies the equation S = 0 and hence,

$$AX^2+2GX+C=0 \text{ for all real numbers X.}$$

Since the equation is satisfied by more than two values of X, we must have $A = G = C = 0$. Hence

$$S \equiv BY^2 + 2HXY + 2FY = Y(BY + 2HX + 2F)$$

That is S can be expressed as a product of two linear factors in X and Y. But X and Y are linear in x and y; and so using (i) and (ii), S can be factorised as a product o two real linear factors in x and y.

15.1.5 Note:

1. If the locus of a second degree equation in x and y contains a straight line, then the equation represents a pair of straight lines.
2. If the locus of a second degree equation S = 0 in the two variables x and y is a pair of straight lines, then we can write

   $$S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$$

   Where $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ are linear in x and y.

We now find the condition under which a homogeneous equation of second degree in x and y represents a pair of straight lines.

15.1.6 Theorem:

If a, b, h are not all zero, then the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines if and only if $h^2 \geq ab$.

Proof: Suppose that H = 0 represents a pair of straight lines. Then by note 4.1.5 (2), we can write

$$H \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$$

Here $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ are linear in x and y.

Since (0, 0) is a point on the locus of $ax^2 + 2hxy + by^2 = 0$, it follows that (0, 0) is a point on the line $l_1x + m_1y + n_1 = 0$ or on the line $l_2x + m_2y + n_2 = 0$.

Hence $n_1 = 0$ or $n_2 = 0$, say $n_1 = 0$. Then

$ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$ so that $l_1n_2 = 0 = m_1n_2$. Since l_1 & m_1 are not both zero, we get $n_2 = 0$. Hence

$ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$. Therefore $l_1l_2 = a, m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$.

Hence $h^2 - ab = \left(\frac{l_1m_2 + l_2m_1}{2}\right)^2 - l_1l_2m_1m_2 = \left(\frac{l_1m_2 - l_2m_1}{2}\right)^2 \geq 0$ so that $h^2 \geq ab$

Conversely, suppose that $h^2 \geq ab$.

Case (i): Let $a \neq 0$. Then $ax^2 + 2hxy + by^2$

$$= \frac{1}{a}\left(a^2x^2 + 2ahxy + aby^2\right)$$

$$= \frac{1}{a}\left[(ax)^2 + 2(ax)(hy) + h^2y^2 + aby^2 - h^2y^2\right]$$

$$= \frac{1}{a}\left[(ax + hy)^2 - (h^2 - ab)y^2\right]$$

$$= \frac{1}{a}\left[(ax + hy + \sqrt{h^2 - ab}\,y)(ax + hy - \sqrt{h^2 - ab}\,y\right] \text{ since } h^2 \geq ab$$

$$= \frac{1}{a}\left[ax + (h + \sqrt{h^2 - ab})y\right]\left[ax + (h - \sqrt{h^2} - ab)y\right]$$

Therefore the equation H = 0 represents the pair of straight lines. $ax + (h + \sqrt{h^2 - ab})y = 0$ and $ax + (h - \sqrt{h^2 - ab})y = 0$

Observe that each of these lines passes through the origin.

Case (ii) Let a = 0. Then $H \equiv 2hxy + by^2 \equiv y(2hx + by)$ and so, in this case, the equation H =0 represents the straight lines y = 0 and $2hx + by = 0$ (since h and b are not both zero), each of which passes through the origin.

15.1.7 Note:

If $h^2 = ab$, we observe that the lines represented by H = 0 are coincident

If $h^2 \geq ab$, then we can write $H \equiv (l_1x + m_1y)(l_2x + m_2y)$ so that $l_1l_2 = a, m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$. Also $l_1x + m_1y = 0 \,\&\, l_2x + m_2y = 0$ are the straight lines represented by H = 0. If H = 0 represents a pair of a straight lines and $b \neq 0$, then these lines are non – vertical (prove). If $m_1 \,\&\, m_2$ are the slopes of these lines, then

$$ax^2 + 2hxy + by^2 \equiv b(y - m_1x)(y - m_2x)$$

So that $$m_1 + m_2 = \frac{-2h}{b} \,\&\, m_1m_2 = \frac{a}{b}$$

When the equations of two straight lines are given separately, finding the angle between them was discussed in 3.8. The following theorem aims at finding the angle between a pair of straight lines when their combined equation is given.

15.1.8 Theorem:

Let the equation $ax^2+2hxy+by^2=0$ represent a pair of straight lines. Then the angle θ between the lines is given b y $\cos\theta = \dfrac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$

Proof: It is obvious that $(a-b)^2+4h^2>0$

Let $H \equiv ax^2+2hxy+by^2 = (l_1x+m_1y)(l_2x+m_2y)$. Then the lines represented by the given equations are $l_1x+m_1y=0 \,\&\, l_2x+m_2y=0$. Further $l_1l_2=a, m_1m_2=b \,\&\, l_1m_2+l_2m_1=2h$. Therefore the angle θ between these lines is given by

$$\cos\theta = \frac{|l_1l_2+m_1m_2|}{\sqrt{(l_1^2+m_1^2)(l_1^2+m_1^2)}}$$

$$= \frac{|l_1l_2+m_1m_2|}{\sqrt{(l_1l_2-m_1m_2)^2+(l_1m_2-m_1l_2)^2}}$$

Therefore $$\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$$

15.2 Condition for perpendicular and coincident lines, bisectors of angles

It is already observed in 4.1.7 that the equation H = =0 represents a pair of coincident lines if $h^2=ab$.

Now the lines given by H = 0 are perpendicular

$$\Leftrightarrow \cos\theta = 0$$

$$\Leftrightarrow a+b=0$$

Sum of the coefficient of $x^2 \,\&\, y^2$ in H = 0 is zero.

If $a+b\neq 0$, then the lines represented by H = 0 are not perpendicular and in such a situation, the angle θ between the lines is also given by the formula

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{|a+b|} \text{ because } \cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}} \text{ gives}$$

$$\sin\theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}$$

15.2.1 Example :

Let us find the angle between the straight lines represented by the equation $2x^2 - 3xy - 6y^2 = 0$.

Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we find a = 2, b = -6 and $h = \frac{-3}{2}$.

Therefore angle θ between the given pair of lines is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|} = \frac{2\sqrt{\frac{9}{4} + 12}}{|2-6|} = \frac{\sqrt{57}}{4}$$

Hence the angle between the lines is $\tan^{-1}\left(\frac{\sqrt{57}}{4}\right)$

15.2.2 Theorem:

Let the equations of two intersecting lines be $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$. Then the equations of the angles (angle and its supplement) between $L_1 = 0 \,\&\, L_2 = 0$ are $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

Proof: The locus of the points equidistant from $L_1 \,\&\, L_2$ is the pair of lines bisecting the angles between $L_1 \,\&\, L_2$. Let PM, PN be the perpendicular distances of a point $P(x_1, y_1)$ from the lines $L_1 \,\&\, L_2$ respectively (see fig)

Then P is a point on the given locus $\Leftrightarrow PM = PM$

$$\Leftrightarrow \left|\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$$\Leftrightarrow \frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Note: The equations of the lines bisecting the angles between $L_1 \,\&\, L_2$ are also written as

$$\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}} = 0$$

15.2.3 Examples:

1) Let us find the equations of the straight lines bisecting the angles between the lines $7x + y + 3 = 0$ & $x - y + 1 = 0$. By theorem 4.2.2 the equations of the straight lines bisecting the angles between the given lines are

$$\left(\frac{7x + y + 3}{\sqrt{50}}\right) \pm \left(\frac{x - y + 1}{\sqrt{2}}\right) = 0$$

That is $(7x + y + 3) \pm 5(x - y + 1) = 0$ or $x + 3y - 1 = 0$ & $3x - y + 2 = 0$

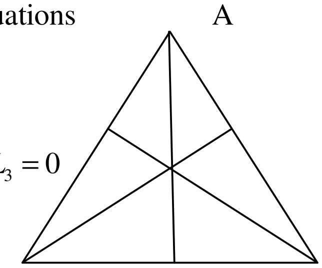
2) Let us prove that the internal bisectors of the angles of a triangle are concurrent.

Proof: Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of a given triangle ABC (see fig) whose sides are $\overleftrightarrow{BC}, \overleftrightarrow{CA}, \overleftrightarrow{AB}$ are represented by the equations

$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

$$L_2 \equiv a_2x + b_2y + c_2 = 0$$

$$L_3 \equiv a_3x + b_3y + c_3 = 0$$



Without loss of generality, we can assume that the non zero numbers $a_rx_r + b_ry_r + c_r$ (r= 1, 2, 3) are positive (that is, if necessary, we write the equation so that these numbers are positive).

Also $a_rx_s + b_ry_s + c_r = 0$ for $r \neq s$ and r, s=1, 2, 3.

Now the equation $u_1 \equiv \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} - \dfrac{a_3x + b_3y + c_3}{\sqrt{a_3^2 + b_3^2}}$

Or $$u_1 \equiv \frac{L_1}{\sqrt{a_2^2 + b_2^2}} - \frac{L_3}{\sqrt{a_3^2 + b_3^2}} = 0$$

Represents one of the bisectors of the angle BAC.

Since $a_3x_2 + b_3y_2 + c_3 = a_2x_3 + b_2y_3 + c_2 = 0$ we have

$$\frac{a_2x_2 + b_2y_2 + c_2}{\sqrt{a_2^2 + b_2^2}} - \frac{a_3x_2 + b_3y_2 + c_3}{\sqrt{a_3^2 + b_3^2}} > 0 \text{ and}$$

$$\frac{a_2x_3+b_2y_3+c_2}{\sqrt{a_2^2+b_2^2}}-\frac{a_3x_3+b_3y_3+c_3}{\sqrt{a_3^2+b_3^2}}<0$$

Hence the vertices $B(x_2, y_2)$ & $C(x_3, y_3)$ lie on either side of the bisector $u_1 = 0$ and accordingly, it is the internal bisector o angle A of triangle ABC.

Similarly the internal bisectors of the angles B and C of the triangle are respectively

$$u_2 \equiv \frac{L_3}{\sqrt{a_3^2+b_3^3}}-\frac{L_1}{\sqrt{a_1^2+b_1^2}}=0$$

And $$u_3 \equiv \frac{L_1}{\sqrt{a_1^2+b_1^3}}-\frac{L_2}{\sqrt{a_2^2+b_2^2}}=0$$

Now letting $k_1 = k_2 = k_3 = 1$, we observe that $k_1u_1 + k_2u_2 + k_3u_3 \equiv 0$ and therefore, by theorem 3.7.3, the bisectors $u_1 = 0, u_2 = 0 \,\&\, u_3 = 0$ are concurrent.

15.3 Pair of bisectors of angles

15.3.1 Theorem:

If the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of intersecting lines the combined equation of the pair of bisection of the angles between these lines $h(x^2 - y^2) = (a-b)xy$.

15.3.2 Note:

The sum of the coordinates of x^2 & y^2 in the bisectors equation is zero, which verified perpendicularly.

15.3.3. Example:

Let us find the combined equation of the pair of bisectors of the angles between that of straight lines represented by $6x^2 + 11xy + 3y^2 = 0$.

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we observe that a = 6, b = 3 and h.

Therefore the equation of the pair of bisectors of the angles between the given pair of linear $h(x^2 - y^2) = (a-b)xy$.

That is $\frac{11}{2}(x^2 - y^2) = (6-3)xy$ or $11(x^2 - y^2) - 6xy = 0$

15.3.4 Solved problem:

1. Does the equation $x^2 + xy + y^2 = 0$ represent a pair of lines?

Solution: No. For, a = b = 1, $h = \frac{1}{2}$ and $h^2 - ab = \frac{1}{4} - 1 < 0$, that is, $h^2 < ab$.

2. Find the nature of the triangle formed by the lines $x^2 - 3y^2 = 0$ and x = 2.

Solution: The lines $x^2 - 3y^2 = 0$, that is $y = \frac{1}{\sqrt{3}}x, y = -\frac{1}{\sqrt{3}}x$ are equally inclined to the x - axis, the inclination being 30°

Further $\angle OAB = \angle OBA = 60^\circ$

Hence the triangle is equilateral.

3. Find the centriod of the triangle formed by the lines $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.

Solution: The pair of straight lines $12x^2 - 20xy + 7y^2 = 0$ intersects the straight line $2x - 3y + 4 = 0$ in the points A and B whose coordinates are given by the equation $3(3y-4)^2 - 10y(3y-4) + 7y^2 = 0$ (eliminate of x from the above equations)

That is $y^2 - 8y + 12 = 0$ or $(y-2)(y-6) = 0$ and so, y = 2 or 6 and correspondingly x = 1 or 7.

Therefore, the points of intersection are A (1, 2) and B (7, 6). Accordingly, the triangle OAB formed by the given triad of lines has its centriod at $\left(\frac{8}{3}, \frac{8}{3}\right)$.

4. Prove that the lines represented by the equations $x^2 - 4xy + y^2 = 0$ and $x + y = 3$ form an equilateral triangle.

Solution: The slope of the line $L \equiv x + y - 3 = 0$ is -1 and hence it makes an angle of 45° with the negative direction of the x- axis. Therefore, no straight line which makes an angle of 60° with L is vertical. Let the equation of a line passing through the origin and making an angle of 60° with L be y = mx. Then Y

$$\sqrt{3} = \tan 60^\circ = \left|\frac{m+1}{m-1}\right|$$

$y = m_2 x \quad L_2$

So that $(m+1)^2 = 3(m-1)^2$ or $m^2 - 4m + 1 = 0$

$y = m_1 x$

Whose roots are $m_1 \& m_2$ are real and distinct. Therefore, there are two lines $L_1 \& L_2$ passing through the origin each making an angle of 60° with L. Their equations are $y = m_1 x \& y = m_2 x$ where $m_1 + m_2 = 4, m_1 m_2 = 1$.

The combined equation of $L_1 \& L_2$ is $(y - m_1 x)(y - m_2 x) = 0$

i.e.., $y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$

i.e.., $y^2 - 4xy + x^2 = 0$, which is same as the given pair of lines

Hence L, $L_1 \& L_2$ form an equilateral triangle.

5. Show that the product of the perpendicular distances from a point (α, β) to the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + 4h^2}}$.

Solution: Let $ax^2 + 2hxy + by^2 \equiv (l_1 x + m_1 y)(l_2 x + m_2 y)$

Then the lines represented by the equation are $l_1 x + m_1 y = 0$ and $l_2 x + m_2 y = 0$

Further $l_1 l_2 = a; m_1 m_2 = b \& l_1 m_2 + l_2 m_1 = 2h$.

d_1 = length of the perpendicular from (α, β) to $l_1 x + m_1 y = 0$

$$= \frac{|l_1\alpha + m_1\beta|}{\sqrt{l_1^2 + m_1^2}}$$

d_2 = length of the perpendicular from (α, β) to $l_2 x + m_2 y = 0$

$$= \frac{|l_2\alpha + m_2\beta|}{\sqrt{l_2^2 + m_2^2}}$$

Then, the product of the lengths of the perpendiculars from (α, β) to the given pair of lines = $d_1 d_2 = \dfrac{|(l_1\alpha + m_1\beta)(l_2\alpha + m_2\beta)|}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}} = \dfrac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + 4h^2}}$

6. Let $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then show that the equation of the pair of straight lines.

(i) Passing through (x_0, y_0) and parallel to the given pair of lines is $a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0$ and

(ii) Passing through (x_0, y_0) and perpendicular to the given pair of lines is $b(x-x_0)^2-2h(x-x_0)(y-y_0)+a(y-y_0)^2=0$.

Solution: Let $ax^2+2hxy+by^2=0 \equiv (l_1x+m_1y)(l_2x+m_2y)$

Then the equations of the lines are $L_1 \equiv l_1x+m_1y=0, L_2 \equiv l_2x+m_2y=0$

Further $l_1l_2=a, m_1m_2=b \,\&\, l_1m_2+l_2m_1=2h$

(i) Now the equations of the straight lines passing through (x_0, y_0) and parallel to $L_1 \,\&\, L_2$ respectively are

$$l_1(x-x_0)+m_1(y-y_0)=0 \text{ And } l_2(x-x_0)+m_2(y-y_0)=0$$

Therefore their combined equation $[l_1(x-x_0)+m_1(y-y_0)][l_2(x-x_0)+m_2(y-y_0)]=0$ or $a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0$

(ii) The straight lines passing through (x_0, y_0) and perpendicular to the pair $L_1 \,\&\, L_2$ are respective $m_1x-l_1y=m_1x_0-l_1y_0$ or $m_1(x-x_0)-l_1(y-y_0)=0$ and $m_2(x-x_0)-l_2(y-y_0)=0$.

Hence their combined equation is $[m_1(x-x_0)-l_1(y-y_0)][m_2(x-x_0)-l_2(y-y_0)]=0$.

That is $b(x-x_0)^2-2h(x-x_0)(y-y_0)+a(y-y_0)^2=0$

Note: The pair of lines passing through the origin and perpendicular to the pair of lines given by $ax^2+2hxy+by^2=0$ is $bx^2+2hxy+ay^2=0$

7. Show that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$, $lx+my+n=0$ is $\left|\dfrac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$

Solution: Let $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ be the pair of straight lines represented by the equation $ax^2+2hxy+by^2$ (see fig) and $\overleftrightarrow{AB}$ be the line $lx+my+n=0$.

Let $ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y)$ and $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ be the lines $l_1x+m_1y=0 \,\&\, l_2x+m_2y=0$ respectively.

Let $A=(x_1,y_1), B=(x_2,y_2)$. Then since A lies on $\overleftrightarrow{OA}$ and $\overleftrightarrow{AB}$,
$l_1x+m_1y=0 \,\&\, l_2x+m_2y=0$

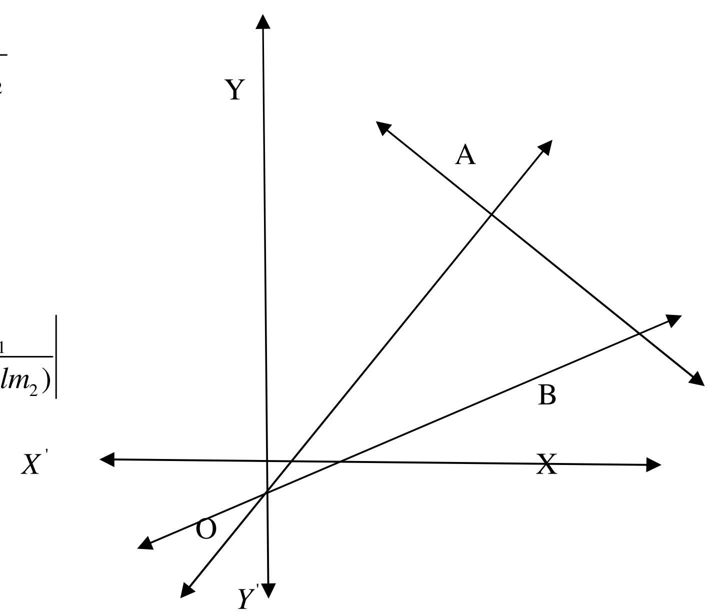
Since, by hypothesis the given three lines form a triangle, $l_1m-lm_1 \neq 0 \,\&\, l_2m-lm_2 \neq 0$

So, by the rule of cross-multiplication, we obtain

$$\frac{x_1}{m_1n}=\frac{y_1}{-nl_1}=\frac{1}{l_1m-lm_1} \text{ and hence}$$

$$x_1=\frac{m_1n}{l_1m-lm_1};\ y_1=\frac{-nl_1}{l_1m-lm_1}$$

Similarly $$x_2=\frac{m_2n}{l_2m-lm_2};\ y_1=\frac{-nl_2}{l_2m-lm_2}$$



Therefore area of triangle OAB

$$=\frac{1}{2}\left|x_1y_2-x_2y_1\right|$$

$$=\frac{1}{2}\left|\frac{n^2(l_1m_2-l_2m_1}{(l_1m-lm_1)(l_2m-lm_2)}\right|$$

$$\frac{1}{2}\left|\frac{n^2\sqrt{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}}{l_1l_2m^2-lm(l_1m_2+l_2m_1)+m_1m_2l^2}\right|=\frac{1}{2}\left|\frac{n^2\sqrt{4h^2-4ab}}{am^2-2hlm+bl^2}\right|=\left|\frac{n\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$$

(since $l_1l_2=a, m_1m_2=b$ and $l_1m_2+l_2m_1=2h$)

Exercise 15 (a)

1. Find the acute angle between the pair of lines represented by the following equations.
   (i) $x^2-7xy+12y^2=0$
   (ii) $y^2-xy-6x^2=0$
   (iii) $(x\cos\alpha-y\sin\alpha)^2=(x^2+y^2)\sin^2\alpha$
   (iv) $x^2+2xy\cot\alpha-y^2=0$

2. Show that the following pairs of straight lines have the same set of angular bisectors (that is they are equally inclined to each other).

(i) $2x^2+6xy+y^2=0, 4x^2+18xy+y^2=0$

(ii) $a^2x^2+2h(a+b)xy+b^2y^2; ax^2+2hxy+by^2=0; a+b\neq 0$

(iii) $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0; (\lambda\in R)$
$ax^2+2hxy+by^2=0$

3. Find the value of 'h' if the slope of the lines represented by $6x^2+2hxy+y^2=0$ are in the ratio 1 : 2.

4. If $ax^2 2hxy+by^2=0$ represents two straight lines such that the slope of one line is twice the slope of the other, prove that $8h^2=9ab$

5. Show that the equation of the pair of straight lines passing through the origin and making an angle of 30° with the line $3x-y-1=0$ is $13x^2+12xy-3y^2=0$

6. Find the equation of the pair of straight lines passing through the origin and making an acute angle α with the straight line $x+y+5=0$

7. Show that the straight lines represented by $(x+2a)^2-3y^2=0$ and x = a form an equilateral triangle.

8. Show that the pair of bisectors of the angles between the straight lines $(ax+by)^2=c(bx-ay)^2, c>0$ are parallel and perpendicular to the line $ax+by+k=0$.

9. The adjacent sides of a parallelogram are $2x^2-5xy+3y^2=0$ and one diagonal is $x+y+2=0$. Find the other vertices and the other diagonal.

10. Find the centriod and the area of the triangle formed by the following lines

(i) $2y^2-xy-6x^2=0, x+y+4=0$

(ii) $3x^2-4xy+y^2=0, 2x-y=6$

11. Find the equations of the pair of lines intersecting at (-2, -1) and

(i) Perpendicular to the pair of $6x^2-13xy-5y^2=0$ and

(ii) Parallel to the pair $6x^2-13xy-5y^2=0$

12. Find the equation of the bisector of the acute angle between the lines $3x-4y+7=0 \& 12x+5y-2=0$.

13. Find the equation of the bisector of the obtuse angle between the lines $x+y-5=0 \& x-7y+7=0$

14. Show that the lines represented by $(lx+my)^2-3(mx-ly)^2=0$ and $lx+my+n=0$ form an equilateral triangle with area $\dfrac{n^2}{\sqrt{3}(l^2+m^2)}$

15. Show that the straight lines represented by $3x^2+48xy+23y^2=0$ and $3x-2y+13=0$ form equilateral triangle of area $\dfrac{13}{\sqrt{3}}$ sq.units.

16. Show that the equation of the pair of lines bisecting the angles between the pair of bisectors of angles between the pair of lines $ax^2+2hxy+by^2=0$ is $(a-b)(x^2-y^2)+4hxy=0$

17. If one line of the pair of lines $ax^2+2hxy+by^2=0$ bisects the angle between the coordinate axis. Prove that $(a+b)^2=4h^2$

18. If (α,β) is the centriod of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$. Prove that $\dfrac{\alpha}{bl-hm}=\dfrac{\beta}{am-hl}=\dfrac{2}{3(bl^2-2hlm+am^2)}$

19. Prove that the distance from the origin to the orthocentre of the triangle formed by the line $\dfrac{x}{\alpha}+\dfrac{y}{\beta}=1$ and $ax^2+2hxy+by^2=0$ is $\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}\left|\dfrac{(a+b)\alpha\beta}{a\alpha^2-2h\alpha\beta+b\beta^2}\right|$

20. The straight line $lx+my+n=0$ bisects an angle between the pair of lines o which one is $px+qy+r=0$. Show that the other side is $(px+qy+r)(l^2+m^2)-2(lp+mq)(lx+my+n)=0$

15.4 Pair of lines – Second degree general equation

We now obtain conditions for a general equation of second degree in x and y to represent a pair of lines.

15.4.1Theorem:

If the second degree equation $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ in the two variables x and y represents a pair of straight lines, then

(i) $abc+2fgh-af^2-bg^2-ch^2=0$ and

(ii) $h^2\geq ab, g^2\geq ac, f^2\geq bc$

15.4.2 Note:

Both the above sets of conditions are necessary for the equation S = 0 to represent a pair of straight lines. That is S = 0 cannot represent a pair of lines if any of the above conditions fails.

For example in the equation $x^2+y^2+2xy+1=0$, we have a = b = c = h = 1; f = g = 0 and therefore $abc+2fgh-af^2-bg^2-ch^2=0$. But this equation which is the same as $(x+y)^2+1=0$ does not represent a pair of straight lines. Infact the locus of this equation is the empty set.

Similarly in the equation $x^2+y^2=0$, we have a = b = 1; c = f = g = h = 0 and so $abc+2fgh-af^2-bg^2-ch^2=0; f^2=bc, g^2=ac$. But $h^2<ab$, and once again this

equation does not represent a pair of straight lines. The locus of the equation $x^2 + y^2 = 0$ is the origin.

The equation of second degree in x and y, $S \equiv ax^2 + 2hxy + by^2 - af^2 - bg^2 - ch^2 = 0$ represents a pair of straight lines if $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$, $g^2 \geq ac$ & $f^2 \geq bc$.

(This proof of this is beyond the scope of the book)

From 4.15.1theorem and 4.4.2 Note (2) we have

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ Represents a pair of straight lines

$\Leftrightarrow \Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0, h^2 \geq ab, g^2 \geq ac$ & $f^2 \geq bc$

15.4.3 Theorem:

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation $ax^2 + 2hxy + by^2$ represents a pair of lines passing through the origin and parallel to the former pair of lines.

15.5 Conditions for parallel lines – Distance between them, Point of intersection of pair of lines:

In this section we find the condition for two lines to be parallel and to find the distance between two parallel lines. We will also find the intersection of two lines when their combined equation is given.

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ Represents a pair of straight lines. The angle between this pair of lines is the same as the angle between the pair of lines represented by $H \equiv ax^2 + 2hxy + by^2 = 0$. Hence the angle between the pair of lines S = 0 is $\cos^{-1}\left(\frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}\right)$.

$= \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$ if (a + b) >0

$= \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{-(a+b)}\right)$ if (a – b) <0

$= \frac{\pi}{2}$ if a + b = 0

Therefore, the lines represented by S = 0 are parallel if $h^2 = ab$, perpendicular if a + b = 0 and intersecting if $h^2 > ab$.

15.5.1 Theorem:

If the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2hy + c = 0$ represents a pair of parallel straight lines, then

(i) $h^2 = ab$

(ii) $af^2 = bg^2$ and

(iii) The distance between the parallel lines = $2\sqrt{\frac{g^2 - ac}{a(a+b)}} = 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$

15.5.2 Theorem:

If the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines intersecting at the origin, then g = f = c = 0.

15.5.3 Theorem:

If the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting straight lines, then their point of intersection is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

15.5.4 Note:

If $h^2 > ab$, then the point of intersection of the pair of lines S = 0 satisfies the three equations $ax + hy + h = 0, hx + by + f = 0 \;\&\; gx + fy + c = 0$.

Observe that the eliminant of x_0, y_0 from the equations (1), (2) and (3) above is $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} = 0$ which is the same as one of the necessary conditions given in the theorem 4.15.1for the equations S = 0 to represent a pair of straight lines.

15.5.5 Example:

Let us find the point of intersection of the pair of straight lines represented by $x^2 + 4xy + 3y^2 - 4x - 10y + 3 = 0$.

Comparing this equation with the general equation of second degree in x and y, we get
a = 1; f = - 5

b = 3; g = -2

$c = 3;$ $h = 2$

Therefore the point of intersection of the lines is

$$\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)=\left(\frac{-10+6}{3-4},\frac{-4+5}{3-4}\right)=(4,-1)$$

The point of intersection can also be obtained by solving the equations

$ax+hy+g=0$ That is, $x+2y-2=0$

$hx+by+f=0$ That is, $2x+3y-5=0$

15.5.6 Solved Problems:

1. Find the angle between the straight lines represented by $2x^2+5xy+2y^2-5x-7y+3=0$.

Solution: Here a = 2, 2h = 5 and b = 2 and

$$\theta=\cos^{-1}\frac{a+b}{\sqrt{(a-b)^2+(2h)^2}}=\cos^{-1}\frac{4}{\sqrt{0+5^2}}=\cos^{-1}\left(\frac{4}{5}\right)$$

2. Find the equation of the pair of lines passing through the origin and parallel to the pair of lines $2x^2+3xy-2y^2-5x+5y-3=0$

Solution: Equation of the pair of lines passing through the origin and parallel to the lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $ax^2+2hxy+by^2=0$. Hence the required equation is $2x^2+3xy-2y^2=0$.

3. Find the equation of the pair of lines passing through the origin and perpendicular to the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$

Solution: The equation of the lines passing through the origin and parallel to the given pair of lines is $ax^2+2hxy+by^2=0$. Hence their equation is $bx^2-2hxy+ay^2=0$

4. If $x^2+xy-2y^2+4x-y+k=0$ represents a pair of straight lines, find k.

Solution: Since the given equation represents a pair of lines, $abc+2fgh-af^2-bg^2-ch^2=0$

Here, $a=1, b=-2, c=k, f=-\frac{1}{2}, g=2, h=\frac{1}{2}$. Hence k = 3.

5. Prove that the equation $2x^2+xy-6y^2+7y-2=0$ represents a pair of straight lines.

Solution: Here a = 2, b = -6, c = -2, f = 7/2, g = 0,h = ½.

Hence $abc+2fgh-af^2-bg^2-ch^2=2(-6)(-2)+2.\frac{7}{2}.0.\frac{1}{2}-2\left(\frac{7}{2}\right)^2-(-6).0-(-2)\left(\frac{1}{2}\right)^2$

$$=24-\frac{49}{2}+\frac{1}{2}=\frac{1}{2}(48-49+1)=0$$

$$h^2-ab=\frac{1}{4}+12>0, g^2-ac=0-(2)(-2)=4>0$$

$$f^2-bc=\frac{49}{4}-(-6)(-2)=\frac{49}{4}-12=\frac{1}{4}>0$$

Therefore, $h^2>ab, g^2>ac, f^2>bc$. Hence the given equation represents a pair of straight lines.

6. Prove that the equation $2x^2+3xy-2y^2-x+3y-1=0$ represents a pair of perpendicular straight lines.

Solution: a = 2, b = -2, c = -1, h = 3/2, g = -1/2, f = 3/2

Hence $abc+2fgh-af^2-bg^2-ch^2=4+2.\frac{3}{2}\left(-\frac{1}{2}\right)\frac{3}{2}-2.\frac{9}{4}-(-2)\frac{1}{4}-(-1).\frac{9}{4}$

$$= \quad 4-\frac{9}{4}-\frac{9}{2}+\frac{1}{2}+\frac{9}{4}=0$$

$$h^2-ab=\frac{9}{4}+4>0, g^2-ac=\frac{1}{4}+2>0, f^2-bc=\frac{9}{4}-2=\frac{1}{4}>0$$

a + b = 2 – 2 = 0. Hence the given equation represents a pair of perpendicular lines.

7. Show that the equation $2x^2-13xy-7y^2+x+23y-6=0$ represents a pair of straight lines. Also find the angle between them and the coordinates of the point of intersection of the lines.

Solution: Let $S\equiv 2x^3-13xy-7y^2+x+23y-6$

Now $2x^3-13xy-7y^2=(x-7y)(2x+y)$

Let us see whether we can find $C_1 \& C_2$ such that $2C_1+C_2=1, C_1-7C_2=23, C_1C_2=-6$. From the first two, we get $C_1=2, C_2=-3$. These values satisfy $C_1C_2=-6$. Hence there exist $C_1 \& C_2$ such that

$$S\equiv(x-7y+C_1)(2x+y+C_2)=(x-7y+2)(2x+y-3)$$

Therefore the given equation represents the straight lines $2x+y-3=0 \& x-7y+2=0$

Angle between the lines = $\tan^{-1}\left|\dfrac{2+\frac{1}{7}}{1-\frac{2}{7}}\right| = \tan^{-1} 3$

Solving the equations $2x+y-3=0$ & $x-7y+2=0$, we obtain the point of intersection of the given pair of lines which is $\left(\dfrac{19}{15}, \dfrac{7}{15}\right)$.

8. Find the value of λ for which the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ which represents a pair of straight lines.

Solution: A necessary condition for the given equation to represent a pair of lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ where $a=\lambda, b=12, c=-3, h=-5, g=\dfrac{5}{2}, f=-8$

Therefore, $-36\lambda + 2\times -8\times\dfrac{5}{2}\times 5 - \lambda(-8)^2 - 12\left[\dfrac{5}{2}\right]^2 - (-3)(-5)^2 = 0$

This gives $\lambda = 2 = a$

Now $h^2 - ab = 25 - 24 = 1 > 0$

$g^2 - ac = \dfrac{25}{4} + 6 = \dfrac{49}{4} > 0$

$f^2 - bc = 64 + 36 = 100 > 0$

That is, $h^2 > ab, g^2 > ac, f^2 > bc$

Therefore, the given equation represents a pair of lines for $\lambda = 2$

8. Show that the pairs of straight lines $6x^2 - 5xy - 6y^2 = 0$ and $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ form a square.

Solution: $H \equiv 6x^2 - 5xy - 6y^2 = (3x+2y)(2x-3y)$ and

$$S \equiv 6x^2 - 5xy - 6y^2 + x + 5y - 1 = (3x+2y-1)(2x-3y+1)$$

Clearly, H = 0 represents a pair of perpendicular lines and S = 0 also represents a pair of perpendicular lines. Further the lines represented by H = 0 are parallel to the lines represented by S= 0. Therefore, the four lines form a triangle.

But the distance of each of the lines $3x+2y-1=0, 2x-3y+1=0$ from the origin is $\dfrac{1}{\sqrt{13}}$. Hence the rectangle is a square.

Aliter

Since the second degree terms in both the equations are identical, they form a parallelogram. Also, since the sum of the coefficients of x^2 & y^2 is zero, the parallelogram becomes the rectangle.

If OABC represents the rectangle, then $6x^2-5xy-6y^2+x+5y-1=0$...... (1) represents the combined equation of $\overleftrightarrow{AB}$ & $\overleftrightarrow{BC}$.

Comparing the equation (1) with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get $a=6, b=-6, h=-\frac{5}{2}, g=\frac{1}{2}, f=\frac{5}{2}$

Since $\quad B=\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)=\left(\frac{1}{13}, \frac{5}{13}\right)$

Therefore, slope of $\overleftrightarrow{OB}$ is 5.

Since the equation $\overleftrightarrow{AC}$ is $x+5y-1=0$, its slope is $-\frac{1}{5}$.

Clearly, $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{OB}$. OABC is thus a square.

9. Show that the equation $8x^2-24xy+18y^2-6x+9y-5=0$ represents a pair of parallel straight lines and find the distance between them.

Solution: $\quad S\equiv 8x^2-24xy+18y^2-6x+9y-5$

$=2(2x-3y)^2-3(2x-3y)-5$

$=[2(2x-3y)-5][(2x-3y)+1]$

$=(4x-6y-5)(2x-3y+1)$

Therefore, the equation S = 0 represents the straight lines $4x-6y-5=0$ and $2x-3y+1=0$ which are clearly a pair of parallel lines.

Distance between them $\frac{2+5}{\sqrt{4^2+6^2}}=\frac{7}{\sqrt{52}}$.

Note: This problem can also be solved by using the result of 4.5.1 theorem.

10. If the pairs of lines represented by $ax^2+2hxy+by^2=0$ and $ax^2+2hxy+by^2+2gx+2fy+c=0$ form a rhombus, prove that $(a-b)fg+(f^2-g^2)=0$.

Solution: Let $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ be the pair of straight lines given by $H \equiv ax^2 + 2hxy + by^2 = 0$ and $\overleftrightarrow{AC}, \overleftrightarrow{BC}$ be the pair of lines given by $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. We know that the lines represented by H = 0 are parallel to the lines represented by S = 0 that is, figure OACB (see fig) is a parallelogram C, the point of intersection of lines S = 0 $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Since O and C are distinct points, hf – bg and gh – af are not both zero. Now the equation of the diagonal OC is $(gh - af)x - (hf - bg)y = 0$.

Since A is a point on the locus H = 0 as well as on the locus S = 0, coordinates of A satisfy the equation S – H = 0. Similarly the coordinates of B also satisfy the equation. Now $S - H \equiv 2gx + 2fy + c = 0$, being linear in x and y represents a straight line, (note that g and f are not both zero). Hence S – H = 0 is the equation of the diagonal AB. Since by hypothesis, figure OACB is a rhombus, the diagonals OC and AB are perpendicular to each other. Hence $(gf - af)2g - (hf - bg)2f = 0$ that is $(a - b)fg + h(f^2 - g^2) = 0$.

11. If two of the sides of a parallelogram are represented by $ax^2 + 2hxy + by^2 = 0$ and $px + qy = 1$ is one of its diagonals, prove that the other diagonal is $y(bp - hq) = x(aq - hp)$.

Solution: Let OACB be the parallelogram two of whose sides $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ are represented by the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ (see fig). Since the other pair of sides AC and BC are respectively parallel to $\overleftrightarrow{OB}$ & $\overleftrightarrow{OA}$ their combined equation will be of the form $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Then the equation of the diagonal AB is $2gx + 2fy + c = 0$ (see solved problem11). But this line is given to be

$$px + qy = 1 \text{ or} -pcx - qcy + c = 0 \text{, since } c \neq 0.$$

Therefore, $2g = -pc, 2f = -qc$

The vertex C of the parallelogram $= \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$. Therefore, the equation of the diagonal $\overleftrightarrow{OC}$ is $(gh - af)x = (hf - bg)y$ that is, $c(-ph + aq)x = c(-hq + bp)y$ (using (1))

$$\text{Or } (aq - hp)x = (bp - hq)y \text{, since } c \neq 0.$$

Exercise 15(b)

1. Find the angle between the lines represented by $2x^2+xy-6y^2+7y-2=0$
2. Prove that the equation $2x^2+3xy-2y^2+3x+y+1=0$ represents a pair of perpendicular lines.
3. Prove that the equation $3x^2+7xy+2y^2+5x+5y+2=0$ represents a pair of straight lines and find the coordinates of the point of intersection.
4. Find the value of k, if the equation $2x^2+kxy-6y^2+3x+y+1=0$ represents a pair of straight lines. Find the point of intersection of the lines and the angle between the straight lines for this value of k.
5. Show that the equation $x^2-y^2-x+3y-2=0$ represents a pair of perpendicular lines, and find their equations.
6. Show that the lines $x^2+2xy-35y^2-4x+44y-12=0$ and $5x+2y-8=0$ are concurrent.
7. Find the distances between the following pairs of parallel straight lines :
   (i) $9x^2-6xy+y^2+18x-6y+8=0$
   (ii) $x^2+2\sqrt{3}xy+3y^2-3x-3\sqrt{3}y-4=0$
8. Show that the two pairs of lines $3x^2+8xy-3y^2=0$ and $3x^2+8xy-3y^2+2x-4y-1=0$ form a square.
9. Find the product of the lengths of the perpendiculars drawn from (2, 1) upon the lines $12x^2+25xy+12y^2+10x+11y+2=0$
10. Show that the straight lines $y^2-4y+3=0$ and $x^2+4xy+4y^2+5x+10y+4=0$ form a parallelogram and find the lengths of its sides.
11. Show that the product of the perpendicular distances from the origin to the pair of straight lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $\dfrac{|c|}{\sqrt{(a-b)^2+4h^2}}$
12. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of intersecting lines, then show that the square of the distance of their point of intersection from the origin is $\dfrac{c(a+b)-f^2-g^2}{ab-h^2}$. Also show that the square of this distance is $\dfrac{f^2+g^2}{h^2+b^2}$ if the given lines are perpendicular.

Key Concepts

1.If a, b, h are not all zero, then the equation $H \equiv ax^2+2hxy+by^2=0$ represents a pair of straight lines if and only if $h^2 \geq ab$.

2. If $h^2 = ab$, the lines represented by H = 0 are coincident

3.If $h^2 \geq ab$, then we can write $H \equiv (l_1x+m_1y)(l_2x+m_2y)$ so that $l_1l_2 = a, m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$. Also $l_1x+m_1y=0 \,\&\, l_2x+m_2y=0$ are the straight lines represented by H = 0. If H = 0 represents a pair of a straight lines and $b \neq 0$, then these lines are non – vertical . If $m_1 \,\&\, m_2$ are the slopes of these lines, then

$$ax^2+2hxy+by^2 \equiv b(y-m_1x)(y-m_2x)$$

So that $m_1+m_2 = \frac{-2h}{b} \,\&\, m_1m_2 = \frac{a}{b}$

4. Let the equation $ax^2+2hxy+by^2=0$ represent a pair of straight lines. Then the angle θ between the lines is given b y $\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$

5. The equation H = =0 represents a pair of coincident lines if $h^2 = ab$.

6. The lines given by H = 0 are perpendicular

$$\Leftrightarrow \cos\theta = 0$$

$$\Leftrightarrow a+b=0$$

Sum of the coefficient of $x^2 \,\&\, y^2$ in H = 0 is zero.

7.If $a+b \neq 0$, then the lines represented by H = 0 are not perpendicular and in such a situation, the angle θ between the lines is also given by the formula

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{|a+b|} \text{ because } \cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}} \text{ gives } \sin\theta = \frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$$

8.Let the equations of two intersecting lines be $L_1 \equiv a_1x+b_1y+c_1=0$ and

$L_2 \equiv a_2x+b_2y+c_2=0$. Then the equations of the angles (angle and its supplement) between $L_1=0 \,\&\, L_2=0$ are $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

9.If the equation $ax^2+2hxy+by^2=0$ represents a pair of intersecting lines the combined equation of the pair of bisection of the angles between these lines $h(x^2-y^2)=(a-b)xy$

10.Let $ax^2+2hxy+by^2=0$ represent a pair of straight lines. Then the equation of the pair of straight lines.

(i) Passing through (x_0, y_0) and parallel to the given pair of lines is $a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0$ and

(ii) Passing through (x_0, y_0) and perpendicular to the given pair of lines is $b(x-x_0)^2-2h(x-x_0)(y-y_0)+a(y-y_0)^2=0$.

11. The pair of lines passing through the origin and perpendicular to the pair of lines is $ax^2+2hxy+by^2=0$ is $bx^2+2hxy+ay^2=0$

12. The area of the triangle formed by the lines $ax^2+2hxy+by^2=0$, $lx+my+n=0$ is $\left|\dfrac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$

13.If the second degree equation $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ in the two variables x and y represents a pair of straight lines, then

(i) $abc+2fgh-af^2-bg^2-ch^2=0$ and

(ii) $h^2 \geq ab, g^2 \geq ac, f^2 \geq bc$

14. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, then the equation $ax^2+2hxy+by^2=0$ represents a pair of lines passing through the origin and parallel to the former pair of lines.

15.If the equation $S \equiv ax^2+2hxy+by^2+2gx+2hy+c=0$ represents a pair of parallel straight lines, then

(i) $h^2=ab$

(ii) $af^2=bg^2$ and

(iii) The distance between the parallel lines = $2\sqrt{\dfrac{g^2-ac}{a(a+b)}}=2\sqrt{\dfrac{f^2-bc}{b(a+b)}}$

16. If the equation $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines intersecting at the origin, then g = f = c = 0.

17.If the equation $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of intersecting straight lines, then their point of intersection is $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$

18. Show that the product of the perpendicular distances from the origin to the pair of straight lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $\dfrac{|c|}{\sqrt{(a-b)^2+4h^2}}$

ANSWERS

Exercise 15 (a)

1. (i) $\tan^{-1}\left(\frac{1}{13}\right)$ (ii) $\frac{\pi}{4}$ (iii) 2α (iv) $\frac{\pi}{2}$
2. .
3. $\pm\frac{3\sqrt{3}}{2}$
4. .
5. .
6. $x^2+2xy\sec 2\alpha$ if $\alpha\neq\frac{\pi}{4}$ and $xy=0$ if $\alpha=\frac{\pi}{4}$
7. Proof
8. $(0,0),\left(-\frac{6}{5},-\frac{4}{5}\right),\left(-\frac{11}{5},-\frac{9}{5}\right),(-1,-1);9x-11y=0$
9. (i) $\left(\frac{20}{9},-\frac{44}{9}\right),\frac{56}{3}$ (ii) $(0,-4),36$
10. (i) $5x^2-13xy-6y^2-33x+14y+40=0$
    (ii) $6x^2-13xy-5y^2-37x+16y+45=0$
11. $11x-3y+9=0$
12. $3x-y-9=0$

Exercise 15 (b)

1. $\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)$
2. $\left(-\frac{3}{5},-\frac{1}{5}\right)$
3. k = 4, $\left(-\frac{5}{8},-\frac{1}{8}\right)$ and $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ or k = -1, $\left(-\frac{5}{7},\frac{1}{7}\right)$ and $\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)$
4. $x+y-2=0, x-y+1=0$
5. .
6. (i) $\sqrt{\frac{2}{5}}$ (ii) $\frac{5}{2}$
7. Proof
8. $\frac{143}{25}$
9. $2\sqrt{5},3$

16 THREE DIMENSIONAL COORDINATES

Introduction

Geometric shapes like spheres, cubes and cones do not exist in a single plane. These shapes require third dimension to describe their location in shape. To create this third dimension, a third axis is added to the co – ordinate system. Consequently, the location of each point in space is defined by three real numbers. Three dimensional geometry deals with geometry of solids like cone, sphere and also plane, lines using algebraic equations. The study of analytical geometry is important because of its major applications.

In this chapter we learn how to determine the position of a point in space and the distance between two points. We derive a formula to find the coordinates of a point dividing a line segment in a certain ratio. As an application of this, we determine the coordinates of the centriod of a triangle and tetrahedron.

16.1 Coordinates:

Let $\overleftrightarrow{X'OX}, \overleftrightarrow{Y'OY}$ be two mutually perpendicular straight lines passing through a fixed point 'O'. These two lines determine the XOY – plane or briefly XY – plane. Draw the line $\overleftrightarrow{Z'OZ}$ perpendicular to XY – plane and passing through O (this is unique). The fixed point O is called the origin and these three mutually perpendicular lines $\overleftrightarrow{X'OX}, \overleftrightarrow{Y'OY}, \overleftrightarrow{Z'OZ}$ are called Rectangular Coordinate axes. $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ are the positive directions of coordinate axes. In fig, the positive directions of these axes are represented by arrow – heads.

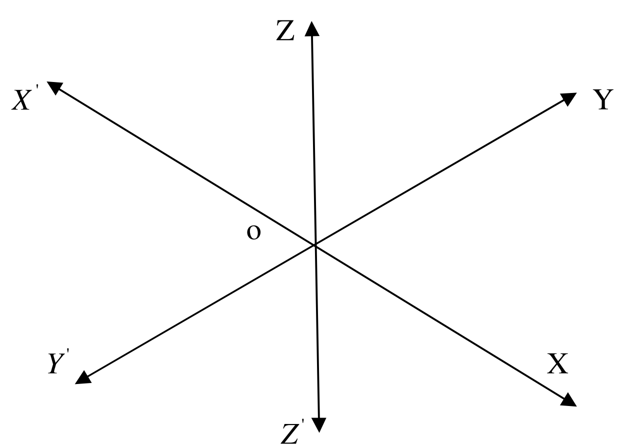


Fig.16.1

The three coordinate axes are taken two at a time determine three planes namely, XOY – plane, YOZ – plane and ZOX – plane or briefly XY, YZ, ZX – planes respectively. These planes are mutually perpendicular and they are called Rectangular coordinate planes. The triple coordinate axes $\overleftrightarrow{X^{'}OX}, \overleftrightarrow{Y^{'}OY}, \overleftrightarrow{Z^{'}OZ}$ are called the rectangular frame of reference and is written as OXYZ.

This frame of reference is said to be aright handed system if a right threaded screw advances in the direction of $\overrightarrow{OZ}$, when it is rotated from $\overrightarrow{OX}$ to $\overrightarrow{OY}$. Otherwise it is said to be a left handed system.

Given point P in space other than O, through P, we can exactly draw three planes parallel to the coordinate planes so that they meet the axes. $\overleftrightarrow{X^{'}OX}, \overleftrightarrow{Y^{'}OY}, \overleftrightarrow{Z^{'}OZ}$ in the points A, B, C respectively. Let x, y, z be real numbers such that $OA = |x|, OB = |y|, OZ = |z|$ and the signs of x, y, z are positive or negative according as A, B, C lie on the positive or negative according as A, B, C lie on the positive or negative directions of the axes. Then the real numbers x, y, z taken in this order are called the coordinates of P with respect to OXYZ. We write the coordinates of P as the ordered triad (x, y, z). The co – ordinates of the origin are (0, 0, 0).

Conversely, given an ordered triad of real numbers (x, y, z), we choose points A, B, C on the X, Y, Z – axes respectively so that $OA = |x|, OB = |y|, OZ = |z|$. The positions of A, B, C on the positive or negative side of the axes are determined according as x, y, z are positive or negative respectively. Through A, B, C draw planes parallel to YZ, ZX, XY planes respectively. These planes intersect at a unique point P in space. We observe that the coordinates of P are nothing but (x, y, z).

Thus, for every point P in space, we can associate an ordered triad (x, y, z) of real numbers formed by its coordinates and conversely, every ordered triad (x, y, z). So we often refer to the triad (x, y, z) as the point P itself. The set of all points in space is referred to as 3 – dimensional space or R^3 space.

If P (x, y, z) is a point in space

x is called the X – coordinate of P

y is called the Y – coordinate of P

z is called the Z – coordinate of P.

16.1.1 Remark:

Given a point P (x, y, z) other than O in space, draw three planes PLCM, PLAN, PMBN parallel to XY, YZ, ZX planes respectively (fig). These three planes along with three coordinate planes constitute a rectangular parallelepiped. From the fig, we have

$|x| = OA = CL = BN = MP$ = perpendicular distance of P from YZ - plane .

$|x| = OB = AN = CM = LP$ = perpendicular distance of P from ZX – plane.

$|z| = O = BM = AL = NP$ = perpendicular distance of P from XY – plane.

Therefore the coordinates of P are (x, y, z), then its perpendicular distances from YZ, ZX, XY planes are $|x|, |y|, |z|$ respectively.

16.1.2 Remark:

From fig, $\overline{OA}$ is perpendicular to the plane PLAN. So it is perpendicular to every line on the plane and in particular to $\overline{PA}$, that is $\overline{OA} \perp \overline{PA}$. Similarly, $\overline{OB} \perp \overline{PB}$ & $\overline{OC} \perp \overline{PC}$. Thus if the coordinates of P are (x, y, z) then $|x|, |y|, |z|$ are the perpendicular distances from the origin of the feet of the perpendiculars A, B, C from P to X, Y, Z – axes respectively.

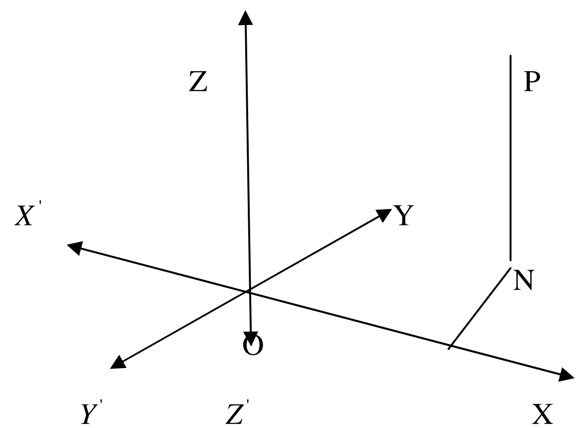


Figure 16.2

16.1.3 Remark:

From the fig, $NP = AL = OC = |z|, AN = OB = |y|, OA = |x|$. Thus if P (x, y, z) is a given point in space, from P draw PN perpendicular to the XY – plane meeting it at N. Draw parallel to Y – axis meeting OX at A (see fig)

Then $PN = |z|, NA = |y|, OA = |x|$

16.1.4 Note:

1. If a point P (x, y, z) lies in the XY – plane then from Remark 5.1.1, $|z|$ = perpendicular distance of P from XY – plane = 0 i.e., z = 0. Therefore P is of the form (x, y, 0).
   Similarly, the coordinates of points in YZ and ZX planes may be taken as (0, y, z) and (x, 0, z) respectively.
2. If P (x, y, z) lies on the X – axis, then its perpendicular distances from ZX and XY planes are zero. So from Remark 5.1.1, y = 0, z = 0. Thus any point on X – axis is of the form (x, 0, 0).
   Similarly points on Y and Z axes are of the form (0, y, 0), (0, 0, z) respectively.

16.1.5 Octants:

The three coordinate planes divide the space into eight parts called Octants. The octant formed by the edges $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ is called the first octant. We write it as OXYZ. The octant whose bounding edges are OX, $OY^{'}, OZ^{'}$ is denoted by $OXY^{'}Z^{'}$. In a similar fashion the remaining six octants can be found. The following table shows the octants and the sigh of coordinates in each octant.

Octant	$OXYZ$	$OX^{'}YZ$	$OX^{'}Y^{'}Z$	$OXY^{'}Z$	$OX^{'}YZ^{'}$	$OXYZ^{'}$	$OXY^{'}Z^{'}$	$OX^{'}Y^{'}Z^{'}$
x – coordinate	+	-	-	+	-	+	+	-
y – coordinate	+	+	-	-	+	+	-	-
z- coordinate	+	+	+	+	-	-	-	-

16.1.6 Distance between two points in space

First we find the distance between the origin and any point in space. Using this we find the distance between any two points in space.

16.1.7 Theorem:

The distance between the origin 'O' and any point P (x, y, z) in space is $OP = \sqrt{x^2 + y^2 + z^2}$.

Proof: We may assume $P \neq O$. Let the planes through P parallel to the coordinate planes intersect the X, Y, Z axes respectively at A, B and C (see fig)

Since $\overline{PA} \perp \overline{OX}$

In $\Delta OAP, OP^2 = OA^2 + AP^2$

Since $\overline{AP}$ is the diagonal of the rectangle PDAF, $AP^2 = AF^2 + FP^2$

From rectangle OAFC, AF = OC

From rectangle PDAF and OBEC, FP = AD = OB

$$OP^2 = OA^2 + AP^2 + OA^2 + AF^2 + FP^2 = OA^2 + OC^2 + OB^2$$

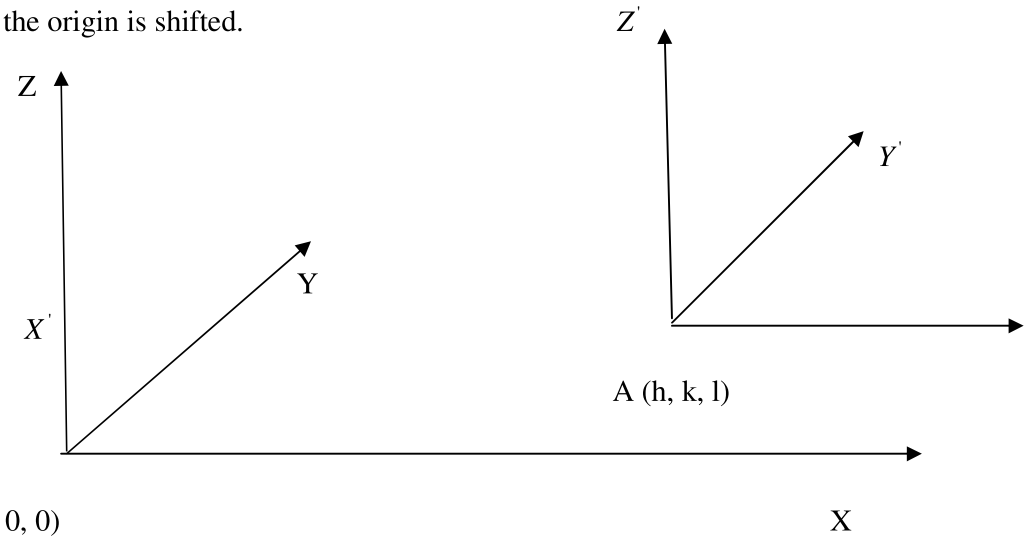
Since the coordinate of P are (x, y, z), $OA = |x|, OB = |y|, OC = |z|$

Therefore $OP^2 = x^2 + y^2 + z^2$. Hence $OP = \sqrt{x^2 + y^2 + z^2}$

16.1.8 Note: Distance is a non negative number. The distance of the point $\left(\sqrt{3}, 0, -1\right)$ from the origin is $\sqrt{3+0+1} = 2$.

16.1.9 Translation of axes:

If we keep the direction of coordinate axes unchanged and shift the origin to some other point, the change is called translation of axes. The coordinates of the point in space change when the origin is shifted.



Let P (x, y, z) and A (h, k, l) be two points in space with respect to the frame of reference OXYZ. Now treating A as origin, let $\overrightarrow{AX'}, \overrightarrow{AY'}, \overrightarrow{AZ'}$ be the new axes parallel to $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively (see fig)

If (x', y', z') are coordinates of P with respect to $AX'Y'Z'$, then $x' = x-h, y' = y-k, z' = z-l$ respectively, with reference to OXYZ frame. Shifting the origin to P, the new coordinates of Q are $(1-1, 0-2, -1+3) = (0, -2, 2)$

16.1.10 Theorem (Distance formula):

Distance between the points $P(x_1, y_1, z_1)\,\&\,Q(x_2, y_2, z_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

Proof: Shifting the origin to P, the new coordinates of Q are $(x_2-x_1, y_2-y_1, z_2-z_1)$

Using 5.1.7, Distance between P and Q = PQ

= Distance of Q from the new origin P

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Using the above formula, the distance between the points (1, -1, 1) and (3, -3, 2) is $PQ = \sqrt{(3-1)^2+(-3+1)^2+(2-1)^2} = \sqrt{4+4+1} = 3$.

16.1.11 Note:

1. Clearly QP = PQ

2. The foot of the perpendicular from P (x, y, z) to X – axis is A (x, 0, 0). Using 5.1.10 perpendicular distance of P from X – axis is

$$PA = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2} = \sqrt{y^2 + z^2}$$

Similarly perpendicular distances of P from X – axis are $\sqrt{x^2 + z^2}$ & $\sqrt{x^2 + y^2}$ respectively.

16.1.12 Solved problems:

1. Show that the points $A(-4,9,6), B(-1,6,6)$ & $C(0,7,10)$ form a right angled isosceles triangle.

Solution: Using distance formula

$$AB = \sqrt{(-1+4)^2 + (6-9)^2 + (6-6)^2} = \sqrt{9+9} = 3\sqrt{2}$$
$$BC = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2} = \sqrt{1+1+16} = 3\sqrt{2}$$
$$AC = \sqrt{(0+4)^2 + (7-9)^2 + (10-6)^2} = \sqrt{16+4+16} = 6$$

$$AB = BC = 3\sqrt{2}$$

Therefore the triangle is isosceles.

Also $AB^2 + BC^2 = 18 + 18 = 36 = AC^2$. Therefore $\underline{|B} = 90^\circ$.

Therefore the triangle ABC is right angled isosceles triangle.

2. Show that locus of the point whose distance from Y – axis is thrice its distance from (1, 2 , -1) is $8x^2 + 9y^2 + 8z^2 - 18x - 36y + 18z + 54 = 0$.

Solution: Let P (x, y, z) be any point on locus.

Distance of P from Y – axis = $\sqrt{x^2 + z^2}$.

Distance of P from (1, 2, -1) = $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$

Given that, $\sqrt{x^2 + z^2} = 3\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$

$\Rightarrow x^2 + z^2 = 9(x^2 - 2x + y^2 - 4y + z^2 + 2z + 6) \Rightarrow 8x^2 + 9y^2 + 8z^2 - 18x - 36y + 18z + 54 = 0$ which is the required equation.

3. A, B, C are three points on $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively at distances a, b, c ($a \neq 0, b \neq 0, c \neq 0$) from the origin 'O'. Find the coordinates of the point which is equidistant from A, B, C and O.

Solution: Let P (x, y, z) be the required point. The coordinates of A, B, C and O are (a, 0, 0), (0, b, 0), (0, 0, c) and (0, 0, 0) respectively.

Given that PA = PB = PC = PO

$$PA = PO \Rightarrow PA^2 = PO^2 \Rightarrow (x-a)^2 + y^2 + z^2 = x^2 + y^2 + z^2$$

$$\Rightarrow a^2 - 2ax = 0 \Rightarrow x = \frac{a}{2} (\because a \neq 0)$$

Similarly we get $y = \frac{b}{2}, z = \frac{c}{2}$

Therefore P = $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ is the point equidistant from A, B, C and O.

4. Show that the points A (3, -2, 4), B (1, 1, 1) and C (-1, 4, -2) are collinear. (points are said to be collinear if they lie on the same line. See definition 5.2.1)

Solution: By the distance formula

$$AB = \sqrt{(1-3)^2 + (1+2)^2 + (1-4)^2} = \sqrt{4+9+9} = \sqrt{22}$$

$$BC = \sqrt{(-1-1)^2 + (4-1)^2 + (-2-1)^2} = \sqrt{4+9+9} = \sqrt{22}$$

$$AC = \sqrt{(-1-3)^2 + (4+1)^2 + (-2-4)^2} = \sqrt{16+36+36} = \sqrt{88}$$

Now $AB + BC = 2\sqrt{22} = \sqrt{88} = AC$

Therefore A, B, C are collinear.

Exercise 16 (a)

1. Find the distance of P (3, -2, 4) from the origin.
2. Find the distance between the points (3, 4, -2) and (1, 0, 7).
3. Find 'x' if the distance between (5, -1, 7) and (x, 5, 1) is 9 units.
4. Show that the points (2, 3, 5), (-1, 5, -1) and (4, -3, 2) form a right angled isosceles triangle.
5. Show that the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral triangle.
6. P is a variable point which moves such that 3PA = 2PB. If A = (-2, 2, 3) and B = (13, -3, 13), prove that P satisfies the equation $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

7. Show that the points (1, 2, 3), (7, 0, 1) and (-2, 3, 4) are collinear.
8. Show that ABCD is a square where A, B, C, D are the points (0, 4, 1), (2, 3, -1), (4, 5, 0) and (2, 6, 2) respectively.

16.2 Section Formula

Section formula gives the coordinates of a point that divides the line segment joining two given points in a given ratio. Using this we derive the coordinates of the centriod of a triangle and tetrahedron.

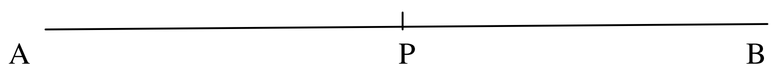
16.2.1 Definition:

If three or more points lie on the same line, they are said to be collinear points.

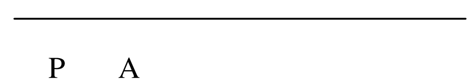
16.2.2 Division of a line segment in space:

Suppose A, B, P are three collinear points in space.

If P lies on the segment AB, we say P divides $\overline{AB}$ in the ratio AP : PB or P divides $\overline{AB}$ internally in the ratio AP : PB (see fig).



If P lies on the line $\overleftrightarrow{AB}$ and outside the segment $\overline{AB}$, we say that P divides $\overline{AB}$ in the ratio - AP : PB (or AP : - PB) or P divides $\overline{AB}$ externally in the ratio AP : PB.



16.2.3 Theorem (Section formula):

The point dividing the line segment joining the distinct points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ in the ratio m : n (m + n not equal to 0) is given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Proof: Suppose P (x, y, z) divides $\overline{AB}$ in the ratio m : n. Draw planes through A, P, B parallel to the YZ – plane so as to meet $\overrightarrow{OX}$ in A', P', B'. Then A', P', B' are the feet of the perpendiculars of A, P, B on the X – axis (see fig).

Therefore $A' = (x_1, 0, 0), P' = (x, 0, 0), B' = (x_2, 0, 0)$

Since parallel planes divide any two straight lines proportionally $\frac{\overline{A'P'}}{\overline{P'B'}} = \frac{\overline{AP}}{\overline{PB}} = \frac{m}{n}$

$$\Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} \text{ and so, } x = \frac{mx_2 + nx_1}{m+n}$$

Similarly, $y = \frac{my_2 + ny_1}{m+n}, z = \frac{mz_2 + nz_1}{m+n}$

Therefore the coordinates of P are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$.

16.2.4 Corollary:

The midpoint of the segment AB where $A = (x_1, y_1, z_1)$ & $B = (x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Proof: Since the midpoint divides the line segment AB in the ratio 1 : 1 taking m = n =1 in theorem 5.2.3, we get $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

16.2.5 Note:

1. If $k \neq -1$ from 5.2.3, the point which divides $\overline{AB}$ in the ratio k : 1 is

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$$

2. Further $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$ and C are collinear iff there exist m, n, $m \neq -n$ such that $C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

16.2.6 Example:

By section formula, the point which divides the line joining the points A (2, -3, 1) and B (3, 4, -5) in the ratio 1 : 3 is $\left(\frac{1\times3+3\times2}{1+3}, \frac{1\times4+3\times-3}{1+3}, \frac{1\times-5+3\times1}{1+3}\right) = \left(\frac{9}{4}, \frac{-5}{4}, \frac{-1}{2}\right)$

16.2.7 Example:

We can use section formula to find the ratio in which the line joining two points is divided by a given point on it.

Consider the points A (7, 0, -1), B (1, 2, 3) and C (-2, 3, 5). Suppose B divides $\overline{AC}$ in the ratio k : 1.

Then, by note 5.2.5,

$$B=\left(\frac{k\times -2+1\times 7}{k+1},\frac{k\times 3+1\times 0}{k+1},\frac{k\times 5+1\times -1}{k+1}\right)=\left(\frac{7-2k}{k+1},\frac{3k}{k+1},\frac{5k-1}{k+1}\right)$$

But. B = (1, 2, 3)

Therefore equating the corresponding coordinates, $\frac{7-2k}{k+1}=1,\frac{3k}{k+1}=1,\frac{5k-1}{k+1}=3$

Solving for k, we get k = 2.

Therefore B divides $\overline{AC}$ in the ratio 2 : 1.

Note that B divides $\overline{AC}$ internally since the ratio is positive.

16.2.8 Example: Using section formula, we can verify whether the given points are collinear or not. Consider the points A (2, -4, 3), B (-4, 5, 6), C(4, -7, 2).

A, B, C are collinear iff C divides $\overline{AB}$ in some ratio say m : n. Then, the coordinates of C, according to section formula are $\left(\frac{-4m+2n}{m+n},\frac{5m-4n}{m+n},\frac{6m+3n}{m+n}\right)$

But C = (4, -7, 2)

Equating the corresponding coordinates we have

$$\frac{-4m+2n}{m+n}=4,\frac{5m-4n}{m+n}=-7,\frac{6m+3n}{m+n}=2$$

From the above three relations we get a unique value $\frac{m}{n}=-\frac{1}{4}$.

So, we conclude that 'C' divides $\overline{AB}$ externally in the ratio 1 : 4.

Therefore A, B, C are collinear.

16.2.9 Example: Let A, B, C be the points (5, 4, 6), (1, -1, 3) and (4, 3, 2) respectively. If these points are collinear, C must divide $\overline{AB}$ in some ratio say m : n. Then coordinates of C are $\left(\frac{m+5n}{m+n},\frac{-m+4n}{m+n},\frac{3m+6n}{m+n}\right)$. Since C is (4, 3, 2) equating the corresponding coordinates,

We get $\frac{m+5n}{m+n}=4,\frac{-m+4n}{m+n}=3,\frac{3m+6n}{m+n}=2$

These relations respectively give $\frac{m}{n}=\frac{1}{3},\frac{1}{4},\frac{-4}{1}$.

We can see that there are no values of m and n that satisfy all the three equations simultaneously. So, we conclude that A, B, C are not collinear.

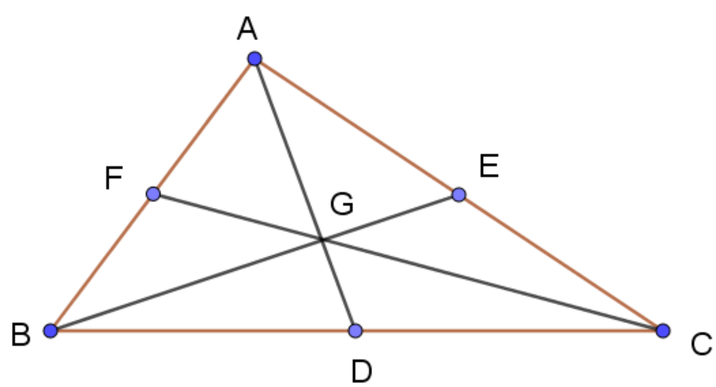
16.2.10 Definition:

Three or more straight lines passing through a single point P are called concurrent lines and the point P is called the point of concurrence.

We know that in a triangle, the medians are concurrent and the point of concurrence is called its centriod. The centriod of a triangle trisects each median.

16.2.11 Theorem: The centriod of a triangle whose vertices are $A(x_1, y_1, z_1), B(x_2, y_2, z_2).C(x_3, y_3, z_3)$ is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Proof: Let D, E, F are midpoints of the sides $\overline{BC}, \overline{AC}, \overline{AB}$ respectively. Then $\overline{AD}, \overline{BE}, \overline{CF}$ are medians of ΔABC (See fig).



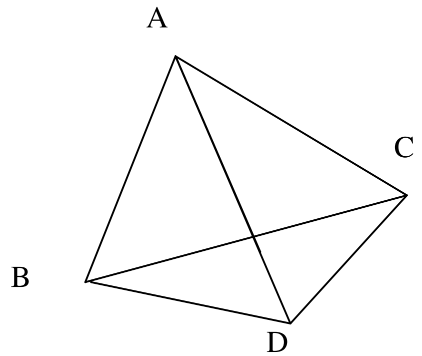
Since D is the midpoint of $\overline{BC}$, $D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$ by corollary 5.2.4.The centriod G divides $\overline{AD}$ in the ratio 2 : 1.

$$\therefore G = \left(\frac{x_1 + 2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{y_1 + 2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{z_1 + 2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

16.2.12 Example: The centriod of the triangle whose vertices are (5, 4, 6), (1, -1, 3) and (4, 3, 2) is $\left(\frac{5+1+4}{3}, \frac{4-1+3}{3}, \frac{6+3+2}{3}\right) = \left(\frac{10}{3}, 2, \frac{11}{3}\right)$.

16.2.13 Tetrahedron

A tetrahedron is a closed figure formed by four planes not all passing through the same point. It has four vertices and six edges. Each vertex is obtained as the point of intersection o three planes. Each edge arises as the line of intersection of two of the four planes. If all edges of a tetrahedron are equal in length, it is called a regular tetrahedron.



In the fig, A, B, C are three points and D is a point not lying in the plane of A, B, C, D. Now ABCD is a tetrahedron with vertices A, B, C, D. $\overline{AB}, \overline{AD}, \overline{AC}, \overline{BC}, \overline{BD}, \overline{CD}$ are its edges and $\Delta ABC, \Delta BCD, \Delta ACD \,\&\, \Delta ABD$ are its faces. $\overline{AB}, \overline{CD}; \overline{BC}, \overline{AD}; \overline{CA}, \overline{DB}$ are called three pairs of opposite edges.

It is known that the line segments joining the vertices to the centriod of opposite faces are concurrent. The point of concurrence is called the centriod of the tetrahedron. It divides each line segment in the ratio 3 : 1.

16.2.14 Theorem: The centriod of the tetrahedron whose vertices are

$$A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3), D(x_4, y_4, z_4) \text{ is}$$

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Proof: Let S be the centroid of the $\Delta ABCD$ .Then by theorem 16.2.11

$$S = \left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right).$$

Let G be the centriod of the tetrahedron ABCD. Then G divides $\overline{AS}$ in the ratio 3 : 1.

$$\therefore G = \left(\frac{\frac{3(x_2 + x_3 + x_4)}{3} + 1.x_1}{3+1}, \frac{\frac{3(y_2 + y_3 + y_4)}{3} + 1.y_1}{3+1}, \frac{\frac{3(z_2 + z_3 + z_4)}{3} + 1.z_1}{3+1}\right)$$

$$= \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right).$$

16.2.15 Example: The centriod of the tetrahedron whose vertices are (2, 3, -4), (-3, 3, -2), (-1, 4, 2), (3, 5, 1) is $\left(\frac{2-3-1+3}{4}, \frac{3+3+4+5}{4}, \frac{-4-2+2+1}{4} \right) = \left(\frac{1}{4}, \frac{15}{4}, \frac{-3}{4} \right)$.

16.2.16 Vector Method

The study o analytical geometry is so far confined to Cartesian methods only. Though this gives a clear geometric and analytical picture of the situation, vector approach to 3D – geometry makes the study simpler and more elegant. Since the students are familiar with vector algebra, vector methods are suggested for derivation of some formulae. According to convenience, either the classical method or the vector method may be used to solve the problems.

We know that if i, j, k are mutually orthogonal unit vectors along OX, OY, OZ in a right handed co-ordinate system, the position vector of a point $P(x, y, z)$ in space with reference to the origin 'O' is given by $OP = xi + yj + zk$.

Conversely, for every vector $xi + yj + zk$, there is a unique point $P(x, y, z)$ in space whose position vector is $xi + yj + zk$. Thus there is a one – one correspondence between the set of points $\mathbf{R}^3$ and the set of position vectors. We identify the point (x, y, z) with its position vector $xi + yj + zk$.

Now it follows that, Distance of $P(x, y, z)$ from the O (0, 0, 0) is $OP = |OP|$ = magnitude of $OP = |xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$.

Distance between the points $A(x_1, y_1, z_1) \& B(x_2, y_2, z_2)$ is $AB = |AB|$ = magnitude of $AB = |OB - OA| = |(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula: Suppose P (x, y, z) divides $\overline{AB}$ in the ratio m : n where $A(x_1, y_1, z_1) \& B(x_2, y_2, z_2)$ are given points. Then A, P, B are collinear and $\frac{AP}{PB} = \frac{m}{n}$

Therefore $nAP = mPB$

$$\Rightarrow n\mathbf{AP} = m\mathbf{PB} \Rightarrow n\left[(x - x_1)i + (y - y_1)j + (z - z_1)k\right] = m\left[(x_2 - x)i + (y_2 - y)j + (z_2 - z)k\right]$$

$$\Rightarrow n(x - x_1) = m(x_2 - x), n(y - y_1) = m(y_2 - y), n(z - z_1) = m(z_2 - z)$$

$$\Rightarrow x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}, z = \frac{mz_2 + nz_1}{m+n}$$

$$\Rightarrow P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

16.3 Solved problems:

1. Find the ratio in which YZ – plane divides the line joining A (2, 4, 5) and B (3, 5, -4). Also find the point of intersection.

Solution: Suppose the line segment $\overline{AB}$ meets YZ – plane in P. Then A, P, B are collinear. If P divides $\overline{AB}$ in the ratio k : 1, then

$$P = \left(\frac{3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{-4k+5}{k+1} \right)$$

Since P lies on the YZ – plane its X – coordinate is zero.

Therefore $\frac{3k+2}{k+1} = 0 \Rightarrow k = \frac{-2}{3}$

Thus YZ – plane divides $\overline{AB}$ in the ratio $\frac{-2}{3}:1$ i.e.., in the ratio $-2:3$

Substituting the value of k, the point of intersection P = (0, 2, 23).

2. Show that the points A (3, -2, 4), B (1, 1, 1) and C (-1, 4, -2) are collinear.

Solution: Suppose the point P divides $\overline{AB}$ in the ratio k : 1, then

$$P = \left(\frac{k+3}{k+1}, \frac{k-2}{k+1}, \frac{k+4}{k+1} \right) \qquad (1)$$

If C lies on $\overline{AB}$, then for some value of 'k', the coordinate of P must be same as those of C.

Equating the X – coordinates of P and C, $\frac{k+3}{k+1} = -1 \Rightarrow k = -2$.

Substituting k = -2 in (1), we get P = (-1, 4, -2) = C.

Therefore A, B, C are collinear.

3. Find the fourth vertex of the parallelogram whose consecutive are (2, 4, -1), (3, 6, -1) and (4, 5, 1).

Solution: Let ABCD be the parallelogram where A = (2, 4, -1), B = (3, 6, -1), C = (4, 5, 1) and D = (a, b, c).

Then the midpoint of AC = midpoint of BD (see fig)

$$\Rightarrow\left(\frac{2+4}{2},\frac{4+5}{2},\frac{-1+1}{2}\right)=\left(\frac{3+a}{2},\frac{6+b}{2},\frac{-1+c}{2}\right)$$

$$\Rightarrow\frac{3+a}{2}=3,\frac{6+b}{2}=\frac{9}{2},\frac{-1+c}{2}=0$$

$$\Rightarrow a=3,b=3,c=1$$

Therefore fourth vertex D = (3, 3, 1).

4. $A(5,4,6),B(1,-1,3),C(4,3,2)$ are three points. Find the coordinates of the point in which the bisector of $\underline{|BAC}$ meets the side $\overline{BC}$.

Solution: We know that the bisector of $\underline{|BAC}$ divides $\overline{BC}$ in the ratio AB : AC (see fig)

$$AB=\sqrt{(5-1)^2+(4+1)^2+(6-3)^2}=5\sqrt{2}$$

$$Ac=\sqrt{(5-4)^2+(4-3)^2+(6-2)^2}=3\sqrt{2}$$

AB : AC = 5 : 3.

If D is the point where the bisector of $\underline{|BAC}$ meets $\overline{BC}$ then D divides $\overline{BC}$ in the ratio 5 : 3.

$$\therefore D=\left(\frac{5\times4+3\times1}{5+3},\frac{5\times3+3\times-1}{5+3},\frac{5\times2+3\times3}{5+3}\right)=\left(\frac{23}{8},\frac{3}{2},\frac{19}{8}\right)$$

5. If $M(\alpha,\beta,\gamma)$ is the midpoint of the line segment joining the points A (x_1,y_1,z_1) and B, then find B.

Solution: Let $B(h,k,s)$ be the point required.

It is given that M is the midpoint of AB.

Therefore we have $(\alpha,\beta,\gamma)=\left(\frac{x_1+h}{2},\frac{y_1+k}{2},\frac{z_1+s}{2}\right)$

$$\Rightarrow 2\alpha=x_1+h;2\beta=y_1+k;2\gamma=z_1+s$$

$$\Rightarrow h=2\alpha-x_1;k=2\beta-y_1;s=2\gamma-z_1$$

Therefore point B is $(2\alpha-x_1,2\beta-y_1,2\gamma-z_1)$.

6. If H, G, S and I respectively denote orthocentre, centriod, circumcentre and incentre of a triangle formed by the points (1, 2, 3), (2, 3, 1) and (3, 1, 2), then find H, G, S, I.

Solution: $AB = \sqrt{(2-1)^2 + (3-2)^2 + (1-3)^2} = \sqrt{1+1+4} = \sqrt{6}$

$$BC = \sqrt{(3-2)^2 + (1-3)^2 + (2-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$CA = \sqrt{(1-3)^2 + (2-1)^2 + (3-2)^2} = \sqrt{4+1+1} = \sqrt{6}$$

Since AB = BC = CA, ABC is an equilateral triangle.

We know that orthocentre, centriod, circumcentre and incentre of an equilateral triangle are the same (i.e.., all the four points coincide).

Now, centroid $G = \left(\frac{1+2+3}{3}, \frac{2+3+1}{3}, \frac{3+1+2}{3}\right) = (2,2,2)$

Therefore H = (2, 2, 2), S = (2, 2, 2) and I = (2, 2, 2).

7. Find the incentre of the triangle formed by the points (0, 0, 0), (3, 0, 0) and (0, 4, 0).

Solution:

If a, b, c are the sides of the triangle ABC, where $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), C = (x_3, y_3, z_3)$ are the vertices, then the incentre of the triangle is given by

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}, \frac{az_1 + bz_2 + cz_3}{a+b+c}\right)$$

Here A = (0, 0, 0), B = (3, 0, 0) and C = (0, 4, 0).

$$a = BC = \sqrt{9+16+0} = 5$$

$$b = CA = \sqrt{0+16+0} = 4$$

$$c = AB = \sqrt{9+0+0} = 3$$

Therefore $I = \left(\frac{5(0)+4(3)+3(0)}{5+4+3}, \frac{5(0)+4(0)+3(4)}{5+4+3}, \frac{5(0)+4(0)+3(0)}{5+4+3}\right) = (1,1,0)$

8. If the point (1, 2, 3) is changed to the point (2, 3, 1) through translation of axes, find the new origin.

Solution: Let (x, y, z) be the coordinates of any point P, w.r.t the coordinate frame oxyz and (X, Y, Z) be the coordinates of P w.r.t the new frame of reference $O^{'}XYZ$.

Let $O^{'}(h,k,s)$ be the new origin so that $x = X + h, y = Y + k, z = Z + s$

$$\Rightarrow (h,k,s) = (x-X, y-Y, z-Z)$$

$$\Rightarrow (h,k,s) = (1-2, 2-3, 3-1) = (-1,-1,2)$$

Therefore $O' = (-1,-1,2)$ is the new origin.

9. Find the ratio in which the point P (5, 4, -6) divides the line segment joining the points A (3, 2, -4) and B (9, 8, -10). Also find the harmonic conjugate of P.

Solution: Let P divide the line segment AB in the ratio $l:m$.

Therefore we have $(5,4,-6) = \left(\frac{9l+3m}{l+m}, \frac{8l+2m}{l+m}, \frac{-10l-4m}{l+m}\right)$

$\Rightarrow l:m = 1:2$ or $2l = m$.

Now let Q divide AB in the ratio $l:-m$

Then $Q = \left(\frac{9l-3m}{l-m}, \frac{8l-2m}{l-m}, \frac{-10l+4m}{l-m}\right) = \left(\frac{9l-6l}{l-2l}, \frac{8l-4l}{l-2l}, \frac{-10l+8l}{l-2l}\right) = (-3,-4,2)$

Therefore Q (-3, -4, 2) is the harmonic conjugate of P (5, 4, -6).

Exercise 16 (b)

1. Find the ratio in which XZ – plane divides the line joining A (-2, 3, 4) and B (1, 2, 3).
2. Find the coordinates of the vertex 'C' of triangle ABC if its centriod is the origin and the vertices A, B are (1, 1, 1) and (-2, 4, 1) respectively.
3. If (3, 2, -1), (4, 1, 1) and (6, 2, 5) are three vertices and (4, 2, 2) is the centriod of a tetrahedron, find the fourth vertex.
4. Find the distance between the midpoint of the line segment AB and the point (3, -1, 2) where A = (6, 3, -4) and B= (-2, -1, 2).
5. Show that the points (5, 4, 2), (6, 2, -1) and (8, -2, -7) are collinear.
6. Show that the points A (3, 2, -4), B (5, 4, -6) and C (9, 8, -10) are collinear and find the ratio in which B divides the line segment AC.
7. If A (4, 8, 12), B (2, 4, 6), C (3, 5, 4) and D (5, 8, 5) are four points, show that the lines $\overleftrightarrow{AB}$ & $\overleftrightarrow{CD}$ intersect.
8. Find the point of intersection of the lines $\overleftrightarrow{AB}$ & $\overleftrightarrow{CD}$ where A = (7, -6, 1), B (17, -18, -3), C (1, 4, -5) and D = (3, -4, 11).
9. A (3, 2, 0), B (5, 3, 2), C (-9, 6, -3) are the vertices of a triangle. $\overline{AD}$, the bisector of $\underline{|BAC}$ meets the line segment BC at D. Find the coordinates of D.
10. Show that the points O (0, 0, 0), A (2, -3, 3), B (-2, 3, -3) are collinear. Find the ratio in which each point divides the segment joining the other two.

Key Concepts

- ❖ Distance of $P(x, y, z)$ from the origin = $\sqrt{x^2 + y^2 + z^2}$.
- ❖ The distance of the point $P(x, y, z)$ from the x, y, z axes are respectively $\sqrt{y^2 + z^2}, \sqrt{z^2 + x^2}, \sqrt{x^2 + y^2}$
- ❖ Distance between the points (x_1, y_1, z_1) & (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- ❖ The point dividing the segment AB where $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$ in the ratio $m : n$ is $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$
- ❖ Point dividing the line segment AB in k : 1 is $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$.
- ❖ Mid point of $\overline{AB}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
- ❖ Centriod of a triangle with vertices $(x_i, y_i, z_i), i = 1, 2, 3$ is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$
- ❖ Centriod of a tetrahedron with vertices $(x_i, y_i, z_i), i = 1, 2, 3, 4$ is $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$.

ANSWERS

Exercise 16 (a)

1. $\sqrt{29}$
2. $\sqrt{101}$
3. 8, 2

Exercise 16 (b)

1. -3 :2
2. (1, -5, -2)
3. (3, 3, 3)
4. $\sqrt{14}$

6. 1 : 2
8. (2, 0, 3)
9. $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$
10. $\frac{OA}{AB} = \frac{-1}{2}, \frac{AB}{BO} = \frac{-2}{1}, \frac{OB}{OA} = 1$

17. Direction Cosines and Direction Ratios

Introduction

Any two lines lying on the same plane are either parallel or intersecting. When two non-parallel lines on a plane meet at a point , an angle is formed and we know how to measure that angle. Sometimes we come across lines in space in space which are neither parallel nor intersecting. For example, the diagonal of the rectangle formed by the floor and the opposite diagonal of the rectangle formed by the roof of a room are two such lines, called skew lines. Measuring angle between such lines is very important.

In Analytical geometry of two dimensions the orientation of a line is given by slope. Whereas 3-dimensional geometry it is measured in terms of direction cosines. In this chapter we learn about the direction cosines and direction ratios of a line and use them to derive a formula to find the angle between lines.

17.1 Definition of Direction Cosines – Simple Problems

Consider a ray $\overrightarrow{OA}$ passing through O and making angles α, β, γ respectively with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ (i.e.., positive directions of X,Y,Z axes).The numbers $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a ray $\overrightarrow{OA}$.Usually they are denoted by (l,m,n) where $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ (fig 17.1)

By reversing the direction, we observe that the ray $\overrightarrow{AO}$ makes angles $\prod - \alpha, \prod - \beta, \prod - \gamma$ respectively with positive directions of X, Y, Z axes.

So $\cos(\prod - \alpha) = -\cos\alpha = -l, \cos(\prod - \beta) = -\cos\beta = -m, \cos(\prod - \lambda) = -\cos\gamma = -n$ are d.c's of $\overrightarrow{AO}$ If L is a directed line in space, we draw a line parallel to L passing through origin so that the direction of the line is same as that of L. The direction cosines of L are defines as the direction cosines of this line through 'O'.

17.1.1 Note:

1. Since a line in space has two directions, it has two sets of direction cosines, one for each direction. If (l, m, n) is one set of d.c's then (-l, -m, -n) is the other set. So it is enough to mention any one set of d.c's of a line.

2. It is clear from the definition and note (1) that if(l,m,n) are the d.c's of a line the d.c's of its parallel line L are $\pm$ (l,m,n).

17.1.2 Example: Since $\overrightarrow{OX}$ males angles 0 ,$90^\circ, 90^\circ$ with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively, $\cos 0, \cos 90^\circ, \cos 90^\circ$ i.e.., (1, 0, 0) are the of X-axes. Similarly d.c's of Y, Z axes are (0, 1, 0),(0, 0, 1) respectively.

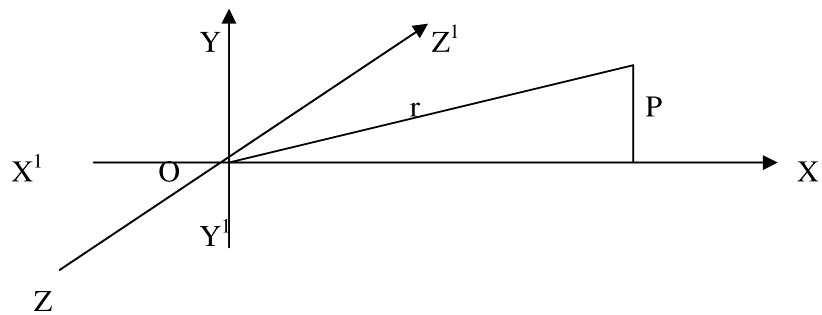
17.1.3 Theorem:

Suppose P(x, y, z) is any point in space such that OP=r.

If (l, m, n) are d.c's of $\overrightarrow{OP}$ then x=le, y=mr, z=nr.

From P draw $\overrightarrow{PA}$ perpendicular to the X-axis. Let A be the foot of the perpendicular. Suppose $\overrightarrow{OP}$ makes angles α, β, γ with the positive directions of X, Y, Z axes respectively.

In fig 17.2, ΔOAP is right angled.



Since A is the foot of the perpendicular from P to X-axes, A=(x, 0, 0).

If x>0 A is on the positive side of X-axes.

$\therefore OA = x$

$$\cos\alpha = \frac{OA}{OP} = \frac{x}{r}$$

If x<0,A is on the negative side of X-axes $\therefore OA = -x$

$$\therefore \cos(\Pi - \alpha) = \frac{OA}{OP} = \frac{-x}{r} \Rightarrow \cos\alpha = \frac{x}{r}$$

Similarly by dropping perpendiculars to Y and Z axes respectively we get y=mr, z=nr.

17.1.4 Note: If OP=r and d.c's of $\overrightarrow{OP}$ are (l,m,n) then the coordinates of P are (lr, mr ,nr)

17.1.5 Example: Suppose P is a point in the space such that $OP = \sqrt{3}$ and $\overrightarrow{OP}$ makes angles $\frac{\Pi}{3}, \frac{\Pi}{4}, \frac{\Pi}{3}$ with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$, respectively.

Then d.c's of $\overrightarrow{OP}$ are: $\cos\frac{\Pi}{3}, \cos\frac{\Pi}{4}, \cos\frac{\Pi}{3}$ *i.e..*, $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

By 17.1.4 coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)$.

17.1.6 Corollary: If P(x, y, z) is appoint in the space then the d.c's of $\overrightarrow{OP}$ are

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Proof: If P=(x, y, z) the OP=r=$\sqrt{x^2+y^2+z^2}$

By 17.1.3 d.c's of OP are $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r},\right)$

i.e.., $\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

17.1.7 Example:

Consider the point P(2,3,-1).By 17.1.6 direction cosines of $\overrightarrow{OP}$ are $\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}},\right)$

17.1.8 Corollary: If (l, m, n) are the direction cosines of a line L, then $l^2+m^2+n^2=1$

Proof: Draw a line parallel to the given line and passing through 'O'. Let P(x, y, z) be appoint on the line each that OP=r. Then $r=\sqrt{x^2+y^2+z^2}$.

By theorem 17.1.3

$x=\pm lr, y=\pm mr, z=\pm nr$ Where the sign should be taken positive or negative throughout, by note 17.1.1

Now $r^2=x^2+y^2+z^2=(l^2+m^2+n^2)r^2 \Rightarrow l^2+m^2+n^2=1$

17.1.9 Example: We cannot have a line direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ because

$$\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\frac{4}{3}\neq 1$$

17.1.10 Theorem:

The direction cosines of the directed line $\overrightarrow{PQ}$ joining the points $P(x_1+y_1+z_1)$ and $Q(x_2+y_2+z_2)$ are $\left(\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}\right)$.

17.1.11 Solved problems:

1. Problem: If P(2,3,-6),Q(3,-4,5) are two points, find the d.c's of $\overrightarrow{OP},\overrightarrow{QO},\overrightarrow{PQ}$ where O is the origin

Solution: $OP=\sqrt{4+9+36}=7; QO=\sqrt{9+16+25}=5\sqrt{2}$

$$PQ=\sqrt{1+49+121}=\sqrt{171}$$

$\therefore$ d.c's of $\overrightarrow{OP}$ are : $\left(\frac{2}{7},\frac{3}{7},\frac{-6}{7}\right)$

d.c's of $\overrightarrow{QO}$ are : $\left(\frac{0-3}{5\sqrt{2}},\frac{0-(-4)}{5\sqrt{2}},\frac{0-5}{5\sqrt{2}}\right)=\left(\frac{3}{5\sqrt{2}},\frac{4}{5\sqrt{2}},\frac{-1}{5\sqrt{2}},\right)$

d.c's of $\overrightarrow{PQ}$ are : $\left(\frac{3-2}{\sqrt{171}},\frac{-4-3}{\sqrt{171}},\frac{5+6}{\sqrt{171}}\right)=\left(\frac{1}{\sqrt{171}},\frac{-7}{\sqrt{171}},\frac{11}{\sqrt{171}}\right)$

2. Problem: Find the d.c's of a line that makes equal angles with the axes

Solution: Suppose the line makes an angle α with $\overrightarrow{OX}$ .since it makes equal angles with the axes, its d.c's are $(\cos\alpha,\cos\alpha,\cos\alpha)$

But
$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$
$$\Rightarrow 3\cos^2\alpha=1\Rightarrow\cos^2\alpha=\frac{1}{3}\Rightarrow\cos\alpha=\pm\frac{1}{\sqrt{3}}$$

Therefore the d.c's of the line are: $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$

3. Problem: If the d.c's of a line are $\left(\frac{1}{c},\frac{1}{c},\frac{1}{c}\right)$ find c

Solution: $\left(\frac{1}{c^2},\frac{1}{c^2},\frac{1}{c^2}\right)=1\Rightarrow\frac{3}{c^2}=1\Rightarrow c^2=3$

$$\Rightarrow c=\pm\sqrt{3}$$

4. Problem: Find the direction cosines of two lines which are connected by the relations l+m+n=0 and mn-2nl-2lm=0.

Solution:

Given that $l+m+n=0$ (1)

And $mn-2nl-2lm=0$ (2)

From (1) $l = -m - n$

Substituting in (2) $mn - 2n(-m-n) - 2m(-m-n) = 0$

$$mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$$

$$2m^2 + 5mn + 2n^2 = 0$$

$$(2m+n)(m+2n) = 0$$

$$2m + n = 0 or m + 2n = 0$$

$$\frac{m}{n} = \frac{-1}{2}$$
$$\frac{m}{n} = \frac{-2}{1} \qquad (3)$$

From 1 $\frac{l}{n} = \frac{-m}{n} = -1$ (4)

When $\frac{m}{n} = \frac{-1}{2}$

From 4 $\frac{l}{n} = \frac{1}{2} - 1 = \frac{-1}{2}$

$$\therefore \frac{m}{l} = \frac{l}{1} = \frac{n}{-2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt{6}}$$

$$\therefore l = \frac{1}{\sqrt{6}}, m = \frac{1}{\sqrt{6}}, n = \frac{-2}{\sqrt{6}}$$

Again from (3) and (4) $\frac{m}{n} = -2$ gives $\frac{l}{n} = +2 - 1 = 1$

$$\therefore \frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}$$

$$\therefore l = \frac{1}{\sqrt{6}}, m = \frac{-2}{\sqrt{6}}, n = \frac{1}{\sqrt{6}}$$

Thus the d.c's of the two lines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right); \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

5. Problem: A ray makes angles $\frac{\pi}{3}, \frac{\pi}{3}$ with $\overrightarrow{OX}, \overrightarrow{OY}$ respectively. Find the angle made by it with $\overrightarrow{OZ}$.

Solution: Let the angle made by the ray with $\overrightarrow{OZ}$ be γ

d.c's of the ray are: $\left(\cos\frac{\pi}{3}, \cos\frac{\pi}{3}, \cos\gamma\right) = \left(\frac{1}{2}, \frac{1}{2}, \cos\gamma\right)$

$$\frac{1}{4}+\frac{1}{4}+\cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}} \Rightarrow \gamma = \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \gamma = \frac{\pi}{4} or \frac{3\Pi}{4}$$

Exercise 17(a)

I. 1. A Line makes angles 90^0, 60^0, 30^0 with the positive directions of X,Y,Z axes respectively. Find its direction cosines.

2. If the line makes angles α, β, γ with the +*ve* directives of X, Y, Z axes, what is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$?

3. What are the direction cosines of the line joining the points (–4,1, 7) and (2, –3, 2)

II 1. Find the ratio in which the XZ-plnae divides the line joining $A(-2,3, 4)$ and $B(1, 2,3)$ **.**

2. Show that the lines PQ and RS are parallel, if $P = (2,3,4)$ $Q(4,7,8)$ $R = (-1,-2,1)$ $S = (1,2,5)$.

II. 1**.**If the direction cosines of two non-parallel lines are related by 2mn + 3n*l* – 5*l*m= 0 and *l* + m + n = 0, then show that these lines are perpendicular to each other.

17.2 Definition of Direction ratios – Simple Problems.

Any three real numbers which are proportional to the direction cosines of a line are called direction ratios (d.r's) of that line

If (a, b, c) are the direction ratios of a line then for every $\lambda \neq 0; (\lambda a, \lambda b, \lambda c)$ are also its direction ratios .Thus a line may have infinite number of direction ratios.

17.2.1 Determining the direction cosines with given direction ratios:

Let (a, b, c) be the direction ratios of a line whose direction cosines are (l, m, n).Then (a, b, c) are proportional to (l, m, n).

$$\therefore \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k(say)$$

$$\Rightarrow a^2+b^2+c^2 = k^2(l^2+m^2+n^2) = k^2$$

$$\Rightarrow k = \pm\sqrt{a^2+b^2+c^2}$$

Therefore the direction cosines of the line are:

$$\left(l,m,n\right)=\left(\frac{a}{k},\frac{b}{k},\frac{c}{k}\right)=\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

17.2.2 Note:

1. If (a, b, c) are direction ratios of a line $a^2+b^2+c^2\neq 1$ in general

2. The direction cosines of a line are its direction ratios but not vice versa.

17.2.3 Direction ratios of the line joining the points $(x_2, y_2, z_2)\,\&\,(x_2, y_2, z_2)$

By Theorem 17.1.10 the direction cosines of the line joining

$(x_2, y_2, z_2)\,\&\,(x_2, y_2, z_2)$ are $\left(\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}\right)$.

Since $\left(x_2-x_1, y_2-y_1, z_2-z_1\right)$ are proportional to direction cosines of the line, they are direction ratios of the line.

17.2.4 Note: If P(x, y, z) is a point in space , by corollary 17.1.6,direction cosines of $\overrightarrow{OP}$ $\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

Since x, y, z are proportional to these values, direction ratios of $\overrightarrow{OP}$ are (x, y, z).Thus the coordinates of any point on a line through the origin may be taken are direction ratios of the line.

17.2.5 Example: If P(-2,4,-5) and Q(1,2,3) are two points , direction ratios of the line $\overleftrightarrow{PQ}$ are(3,-2,8) Direction cosines of the line are

$$\left(\frac{3}{\sqrt{9+4+64}},\frac{-2}{\sqrt{9+4+64}},\frac{8}{\sqrt{9+4+64}}\right)$$
$$i.e..,\left(\frac{3}{\sqrt{77}},\frac{-2}{\sqrt{77}},\frac{8}{\sqrt{77}}\right)$$

17.2.6 Angle between two lines:

Let L_1, L_2 be two lines in space. Draw lines L_1', L_2' parallel to L_1, L_2 and passing through the origin. The angle between L_1', L_2' which lies in $\left[0,\frac{\pi}{2}\right]$ is defined as the angle between L_1, L_2.

17.2.7 Theorem: If $(l_1, m_1, n_1),(l_2, m_2, n_2)$ are direction cosines of two lines, and θ is the angle between them, then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

Proof: Let L_1, L_2 be the given lines with direction cosines $(l_1, m_1, n_1),(l_2, m_2, n_2)$ respectively

Case (1): If the lines L_1, L_2 are parallel then

$$\theta = 0$$
$$\cos\theta = 1$$

From 17.1.1 note (2),

$l_2 = kl_1, m_2 = km_1, n_2 = kn_1$ Where $k = \pm 1$

So that $|l_1 l_2 + m_1 m_2 + n_1 n_2| = |l_1^2 + m_1^2 + n_1^2| = 1$

Therefore result holds good in this case.

Case (2): Suppose L_1, L_2 are not parallel .Draw L_1', L_2' parallel to L_1, L_2 and passing through the origin. Let A, B be points on L_1', L_2' respectively at a distance of 1 unit from 'O'.

Then
$$A = \pm(l_1, m_1, n_1) and$$
$$B = \pm(l_2, m_2, n_2)$$

$$\therefore AB^2 = (l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2$$

$$= (l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2 = (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) \pm 2(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$= 1 + 1 \pm 2(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

Using cosine rule from ΔOAB,

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2OA.OB}$$

$$= \frac{1 + 1 - [1 + 1 \pm 2(l_1 l_2 + m_1 m_2 + n_1 n_2)]}{2} (\because OA = OB = 1)$$

$$= \pm(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

Since $\theta \in \left[0, \frac{\pi}{2}\right], \cos\theta$ is non-negative

$$\therefore \cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

17.2.8 Note: If the lines are perpendicular, $\theta = \frac{\pi}{2}$, so $\cos\theta = 0$

$\therefore$ From 17.2.7, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

17.2.9 Langange's identity:

For any two ordered triads of real numbers, $(a_1, b_1, c_1) and (a_1, b_1, c_1)$

Then $(a_1^2 + b_1^2 + c_1^2)(a_2^2 + a_2^2 + a_2^2) - (a_1 a_2 + b_1 b_2 + c_1 c_2)^2 = \Sigma(a_{12} - a_2 b_1)^2$.

Notice that simplification and rearrangement of terms on the left yields the right side.

17.2.10 Note:

1. If θ is the angle between two lines with direction cosines $(l_1, m_1, n_1) and (l_2, m_2, n_2)$ then,

$\sin^2\theta = 1 - \cos^2\theta = (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 = \Sigma(l_1 m_2 - m_1 l_2)^2$

(by langrange's identity)

Exercise 17(b)

1. If (6,10,10) (1,0,−5) (6,−10,0) are the vertices of a triangle, find the direction ratios of its sides. Also show that it is a right angle triangle.
2. If the direction cosines of two non-parallel lines are related by $2mn + 3nl - 5lm = 0$ $and l + m + n = 0$, then show that these lines are perpendicular to each other.
3. A line makes angles α,β,γ, δ, with the four diogonals of a cube. Show that $cos 2a + cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\alpha + cos^2\delta = 4/3$.

Key Concepts

1. Direction cosines of a ray $\overrightarrow{OA}$ are $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ where α, β, γ are the angles made by $\overrightarrow{OA}$ with positive directions of $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$.

2. By reversing the direction, the ray $\overrightarrow{AO}$ makes angles $\prod - \alpha, \prod - \beta, \prod - \gamma$ respectively with positive directions of X, Y, Z axes.

3. $\cos(\prod - \alpha) = -\cos\alpha = -l, \cos(\prod - \beta) = -\cos\beta = -m, \cos(\prod - \lambda) = -\cos\gamma = -n$ are d.c's of $\overrightarrow{AO}$

4. Since a line in space has two directions, it has two sets of direction cosines, one for each direction. If (l, m, n) is one set of d.c's then (-l, -m, -n) is the other set.

5. if(l,m,n) are the d.c's of a line then the d.c's of its parallel line L are ± (l,m,n).

6.If P(x, y, z) is a point in the space then the d.c's of $\overrightarrow{OP}$ are

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

7.If (l, m, n) are the direction cosines of a line L, then $l^2+m^2+n^2=1$

8.The direction cosines of the directed line $\overrightarrow{PQ}$ joining the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $\left(\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}\right)$.

.9. Direction cosines of a line whose the direction ratios are (a, b, c) are

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

10. Direction ratios of the line joining the points $(x_2,y_2,z_2)\,\&\,(x_2,y_2,z_2)$ are $(x_2-x_1,y_2-y_1,z_2-z_1)$ And the direction cosines of the line joining $(x_2,y_2,z_2)\,\&\,(x_2,y_2,z_2)$ are $\left(\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}},\frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}\right)$.

11. If P(x, y, z) is a point in space , by corollary 17.1.6,direction cosines of $\overrightarrow{OP}$

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

12.If $(l_1,m_1,n_1),(l_2,m_2,n_2)$ are direction cosines of two lines, and θ is the angle between them, then $\cos\theta=|l_1l_2+m_1m_2+n_1n_2|$

If the lines are perpendicular, $\theta=\frac{\pi}{2}$,so $\cos\theta=0$

$\therefore,\ l_1l_2+m_1m_2+n_1n_2=0$

13.Angle between the lines whose direction ratios are $(a_1,b_1,c_1),(a_2,b_2,c_2)$ is

$$\cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

The lines perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Answers

Exercise 17 (a)

I. 1. $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2. 2

3. $\left(\frac{6}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}\right)$

II. 1. $\left(\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right), \left(\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right), \left(\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right),$

III.1. $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Exercise 17 (b)

1. (-5, -10, -15), (5, -10, 5), (0, 20, 10), Triangle ABC is right angled triangle.
2. $\frac{\pi}{3}$
3. 4/3

References:

[1] ***Mathematics Textbook for Class XI© National Council of Educational Research and Training, 2006***

[2] ***Mathematics Textbook for Class XII© National Council of Educational Research and Training, 2006***

[3] ***Textbook for Intermediate First Year Mathematics IA© Telugu Akademi, Hyderabad, 2014***

[4] ***Textbook for Intermediate First Year Mathematics IB© Telugu Akademi, Hyderabad, 2014***

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VOCATIONAL BRIDGE COURSE

MATHEMATICS – First Year (w.e.f. 2018-2019)

MODEL QUESTION PAPER

Time: 3 Hours **Max.Marks: 75**

Section A **10x3=30**

Note:

*i) Answer **all** questions*

ii) *Each question carries **3** marks*

1. A function f : $A \to B$ is defined by $f(x) = x^2 + x + 1$. *If* $A = \{-2, -1, 0, 1, 2\}$, then find B.
2. If the vectors $-3i + 4j + pk$ and $qi + 8j + 6k$ are collinear, then find p and q.
3. $\lim_{x\to 2} \frac{2x^2 - 7x - 4}{2x - 1}$
4. Find $\frac{d}{dx}(\frac{Cosx}{Cosx + Sinx})$
5. A point P moves such that PA = PB where A = (−3, 2) and B = (0, 4) . Find the equation to the locus of P.
6. Transform the line equation of the line $x + y + 2 = 0$ into
   (i) slope – intercept form (ii) intercept from (iii) normal form.
7. The three consecutive vertices of a parallelogram are given as (2,4, –1), (3,6,–1), (4,5,1). Find the fourth vertex.
8. Simplify: $\sin 330^0 . \cos 120^0 + \cos 210^0 . \sin 300^0$.
9. Simplify $\frac{3cos\theta + cos\ 3\theta}{\sin\theta - \sin 3\theta}$
10. If sinh $x = 3/4$, find cosh(2 x)

Section B **3x15 =45**

Note:

i) Answer any **3** questions

ii) Each question carries **15** marks

11 I(a) . Prove by Mathematical Induction $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

I(b). If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ find $(A^T)^{-1}$

OR

II (a). Prove $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ = (a-b)(b-c)(c-a)

II (b) If $a = (1, -2, 1)$, $b = (2, 1, 1)$, $c = (1, 2, -1)$ then find $|(\bar{a} x \bar{b}) x \bar{c}|$ and $|\bar{a} x (\overline{b x c})|$.

12.I(a) Evaluate $\lim_{x \to 0} \dfrac{\sin(a+bx) - \sin(a-bx)}{x}$

I(b) Find the derivative of $x^2 + 2x$ from first principles

OR

II(a) Show that $f(x) = \dfrac{\cos ax - \cos bx}{x^2}$ $for\ x \neq 0$

$= \dfrac{b^2 - a^2}{2}$ $for\ x = 0$

where a and b are real constants, is continuous at $x = 0$

II(b) Find the equations of tangent and normals to the curve

$y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point (0, 5).

13 I(a) Find the foot of the perpendicular drawn from the point (3,0) upon the straight line

5x+ 12y- 41=0.

I (b) Find the equation to the straight line which passes through (0, 0) and also the point of intersection of the lines $x + y + 1 = 0$ and $2y - y + 5 = 0$.

OR

II(a) When the axes are rotated through on angle π , find the transformed equation of

$$3x^2 + 10xy + 3y^2 = 4.$$

II(b). If (6,10,10), (1, 0, −5), (6, −10, 0) are the vertices of a triangle, find the direction ratios of its sides. Also, show that it is a right angled triangle.

14 I(a). If Sin (A+B) = $\frac{24}{25}$ and cos(A-B) = $\frac{4}{5}$ where $0<A<B<\frac{\pi}{4}$ then find the value of Sin2A.

I(b). Solve: $\sin^2\theta - \cos\theta = \frac{1}{4}$

OR

II(a). If $A + B + C = 180^0$ then prove that $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$.

II(b). Solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

15 I(a). Solve $2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$ by matrix inversion method.

I(b) .Find the equations of tangent and normal to the curve of

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5 \text{ at } (0, 5) .$$

OR

II(a). Show that the equation $2x^2 - 13xy - 7y^2 + x + 23y - 6 = 0$ represents a pair of straight lines and also find the angle between and the co-ordinates of the point of intersection of lines.

II(b). Prove that $\cos 2\frac{\pi}{10} + \cos^2\frac{2\pi}{5} + \cos^2\frac{3\pi}{5} + \cos^2\frac{9\pi}{10} = 2.$

<u>**VOCATIONAL BRIDGE COURSE**</u>

First Year - Paper – I (w.e.f. 2018-19)

MATHEMATICS SCHEME OF EXIMATION (WEIGHTAGE)

Total Questions : 15

Time: 3 Hours **Max.Marks: 75**

Note: In section A – Answer all Questions
In section B – Answer any three Questions

Section – A 10x3=30

Note:

i) *Answer all the questions*

*ii) Each question carries **3** marks.*

1. From Algebra
2. From Algebra
3. From Calculus
4. From Calculus
5. From Co-ordinate Geometry
6. From Co-ordinate Geometry
7. From Co-ordinate Geometry
8. From Trigonometry
9. From Trigonometry
10. From Trigonometry

Section – B 3x15=45

Note:

i) *Answer any **3** questions*

*ii) Each question carries **15** marks.*

11. From Algebra with internal choice
12. From Calculus with internal choice
13. From Co-ordinate Geometry with internal choice
14. From Trigonometry with internal choice
15. I(a) – From Algebra
I(b) – From
Calculus

OR

II(a) – from Co-ordinate Geometry
II(b) – from Trigonometry

VOCATIONAL BRIDGE COURSE

MATHEMATICS – First Year

WEIGHTAGE OF MARKS

S.No. Chapter number	Chapters	Number of periods	Weightage
1	FUNCTIONS	12Hours	28
2	MATHEMATICAL INDUCTION	5 Hours	
3	MATRICES	8 Hours	
4	ADITION OF VECTORS	8 Hours	
5	PRODUCT OF VECTORS	8 Hours	
6	TRIGONOMETRIC RATIOS UP TO TRANSFORMATIONS	14 Hours	32
7	TRIGONOMETRIC EQUATIONS	5 Hours	
8	HYPERBOLIC FUNCTIONS	3 Hours	
9	LIMITS AND CONTINUITY	8 Hours	29
10	DIFFERENTIATION	12 Hours	
11	APPLICATIONS OF DERIVATIVES	15 Hours	
12	LOCUS	4 Hours	31
13	TRANSFORMATION OF AXES	4 Hours	
14	THE STRAIGHT LINE	14 Hours	
15	PAIR OF STRAIGHT LINES	18 Hours	
16	THREE DIMENSIONAL CO-ORDINATES	6 Hours	
17	DIRECTION COSINES AND DIRECTION RATIOS	6 Hours	
	TOTAL	150Hours	120 Marks